Chapter 9

Rational Expressions in Open Sentences

Open Sentences Involving Fractions or Percents

OBJECTIVES for Sections 9-1 and 9-2:
1. To solve open sentences with whole-number denominators.
2. To solve problems involving percents.

9–1 Open Sentences Involving Whole-Number Denominators

Open sentences often contain one or more fractions with whole-number denominators. For example, consider the following open sentences.

$$\frac{1}{3}a + \frac{1}{6}a = 1 \qquad \frac{b}{2} - \frac{b-2}{4} \geq 1$$

$$\frac{3c+7}{12} - \frac{c-1}{3} = \frac{1}{2} \qquad \frac{3e+1}{15} + \frac{e-1}{12} < \frac{5}{6}$$

A convenient first step in solving such an open sentence is to transform the given sentence into an equivalent sentence that contains no fractions. If the open sentence is an equation, you can use the multiplication property of equality and multiply both sides of the equation by the LCD of all the fractions appearing in the given equation.

EXAMPLE 1 Solve $\frac{3c + 7}{12} - \frac{c - 1}{3} = \frac{1}{2}$.

SOLUTION

$$\frac{3c + 7}{12} - \frac{c - 1}{3} = \frac{1}{2}$$

$$12\left(\frac{3c + 7}{12} - \frac{c - 1}{3}\right) = 12\left(\frac{1}{2}\right) \longleftarrow \begin{cases}\text{Multiply both sides by 12,}\\ \text{the LCD of the fractions.}\end{cases}$$

$$3c + 7 - 4(c - 1) = 6$$

$$3c + 7 - 4c + 4 = 6$$

$$-c + 11 = 6$$

$$-c = -5$$

$$c = 5$$

Check:

$$\frac{3c + 7}{12} - \frac{c - 1}{3} = \frac{1}{2}$$

$$\frac{3(5) + 7}{12} - \frac{5 - 1}{3} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{22}{12} - \frac{16}{12} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2} \quad \checkmark$$

$\therefore$ the solution set is $\{5\}$.

In Example 1 both sides of the original equation were multiplied by 12, the LCD of all the fractions in the equation. Notice that it was not necessary to use the LCD. Instead, both sides of the equation could have been multiplied by 24, 36, or any other common multiple of the denominators of all the fractions.

A similar method can be used to solve inequalities involving fractions. Using the multiplication property of order stated on page 159, multiply both sides of the inequality by the LCD of the fractions.

EXAMPLE 2 Solve $\frac{3e + 1}{15} + \frac{e - 1}{12} < \frac{5}{6}$.

SOLUTION

$$\frac{3e + 1}{15} + \frac{e - 1}{12} < \frac{5}{6}$$

$$60\left(\frac{3e + 1}{15} + \frac{e - 1}{12}\right) < 60\left(\frac{5}{6}\right) \longleftarrow \begin{cases}\text{Multiply both sides by 60,}\\ \text{the LCD of the fractions.}\end{cases}$$

$$4(3e + 1) + 5(e - 1) < 50$$

$$17e - 1 < 50$$

$$17e < 51$$

$$e < 3$$

$\therefore$ the solution set is $\{e: e < 3\}$.

Oral Exercises

State the LCD of the terms of each open sentence. Then give the equivalent sentence formed by multiplying both sides by the LCD.

1. $\frac{z}{4} + \frac{z}{8} = 3$
2. $\frac{x}{3} + \frac{3x}{2} > \frac{11}{2}$
3. $\frac{3w}{4} - 2 \geq \frac{1}{4}$
4. $\frac{3t + 1}{4} = \frac{1}{2}$
5. $\frac{x + 1}{4} + \frac{2x - 1}{6} = \frac{5}{4}$
6. $\frac{a}{6} - \frac{2a + 3}{8} \leq 0$

Written Exercises

Solve.

A

1. $\frac{8u}{15} - \frac{u}{5} = 2$
2. $\frac{2m}{3} - \frac{m}{6} = -1$
3. $\frac{a}{5} - 1 = \frac{a}{30}$
4. $2v - \frac{v}{5} = \frac{3}{2}$
5. $\frac{f}{4} \leq \frac{3}{2} - \frac{f}{5}$
6. $\frac{t}{3} > 3 - \frac{t}{6}$
7. $\frac{5n - 4}{12} < 3$
8. $\frac{4b + 5}{3} > -5$
9. $\frac{q - 7}{7} = \frac{7 - q}{3}$
10. $\frac{d - 6}{3} = \frac{22 - d}{5}$
11. $\frac{p}{2} - 1 \leq \frac{4}{5} + \frac{3p}{10}$
12. $3c - 1 \geq \frac{c + 3}{2}$
13. $\frac{3s}{5} - \frac{s}{10} > 5$
14. $\frac{3p}{8} - p < \frac{5}{6}$
15. $\frac{2s + 1}{2} + \frac{11}{10} > \frac{s}{5}$
16. $\frac{r - 6}{6} < \frac{1}{4} - \frac{r}{12}$
17. $\frac{5t}{9} - \frac{2t - 3}{6} = 0$
18. $\frac{3b}{14} - \frac{5 - b}{21} = \frac{1}{7}$
19. $\frac{45k + 43}{100} + \frac{k}{5} = 1 - \frac{5k + 6}{50}$
20. $\frac{2h + 1}{2} + \frac{3h - 9}{4} > \frac{1}{8} - 2h$
21. $\frac{3t - 4}{5} - \frac{2t + 1}{4} = -\frac{1}{2}$
22. $\frac{b - 1}{16} - \frac{1 - b}{12} = \frac{7b}{8}$
23. $\frac{a - 2}{6} - \frac{2a + 1}{10} < \frac{1}{15}$
24. $\frac{w + 8}{12} - \frac{3w - 5}{15} < \frac{w}{20}$

B

25. $2.2(i - 3) - 3.5(i - 2) = 0.1(13 - 63i)$
26. $0.25(3x - 5) + 0.07(5x + 3) + 0.06(15x + 14) = 0$
27. $\frac{2}{3}\left(c - \frac{1}{4}\right) - \frac{1}{6}(c + 2) = \frac{3}{2}$
28. $\frac{3}{8}(2 - d) - \frac{5}{6}\left(3d - \frac{1}{4}\right) = 1 - 3d$

In Exercises 29–32 find the slope of the graph of the given equation by first expressing the equation in the form $y = mx + b$.

29. $\frac{2x - 1}{3} + \frac{2y + 1}{2} = \frac{17}{36}$

30. $\frac{2x + 3}{5} + \frac{4y - 9}{8} = \frac{3}{40}$

31. $\frac{14x - 15}{4} - 2y = -\frac{45}{12}$

32. $\frac{4x - 3}{4} - \frac{2y + 5}{5} = \frac{1}{4}$

Solve each system of equations.

C 33. $\frac{y}{2} = \frac{x}{3} + 2$

$\frac{y}{4} = x - \frac{23}{2}$

34. $\frac{9y}{7} = \frac{10x}{7} - 11$

$\frac{y}{2} = \frac{5x}{6} - \frac{17}{3}$

35. $\frac{8y + 5}{4} = \frac{4x - 3}{3} - \frac{23}{12}$

$\frac{y - 5}{10} = \frac{7x - 4}{15} - \frac{7}{12}$

36. $\frac{y - 2}{6} = \frac{3x + 2}{5} + \frac{7}{15}$

$\frac{5y - 9}{6} = \frac{7x - 8}{10} - \frac{2}{15}$

Solve.

37. $\frac{2t^3 - t}{8} - \frac{t^2 - 4t}{6} = \frac{8t - 1}{12}$

38. $\frac{8 - 2k}{7} - \frac{k^2 - 11k}{14} - \frac{7 - k}{2} = 0$

Computer Exercises

For students with computer experience

Write a program that will solve an equation of the form

$$\frac{x}{a} + \frac{x}{b} = \frac{c}{d}$$

when you input values of a, b, c, and d such that a, b, and d are nonzero and $a \neq -b$. RUN the program to solve each of the following.

1. $\frac{x}{2} + \frac{x}{3} = \frac{5}{6}$

2. $\frac{z}{4} - \frac{z}{8} = \frac{1}{4}$

3. $\frac{1}{3}y - \frac{1}{4}y = \frac{2}{3}$

4. $\frac{w}{4} + w = \frac{5}{2}$

Modify the program that you wrote for Exercises 1–4 to solve an equation of the form

$$\frac{ax}{b} + \frac{cx}{d} = \frac{e}{f}$$

when you input values for a, b, c, d, e, and f such that b, d, and f are nonzero and $ad \neq -bc$. RUN the program to solve each of the following.

5. $\frac{3x}{4} + \frac{2x}{3} = \frac{17}{4}$

6. $\frac{3v}{5} = \frac{2v}{3} - \frac{2}{15}$

7. $\frac{12}{5} = \frac{2}{5}m - \frac{3}{2}m$

8. $\frac{3}{2}n + 1 = \frac{n}{6}$

9–2 Percent Problems

The word **percent** (often denoted by %) means "per 100" or "divided by 100." Thus, 7% is another way of writing $\frac{7}{100}$ or 0.07. Any percent can be expressed as a fraction or decimal. For example:

$$25\% = \frac{25}{100} = \frac{1}{4} \qquad \frac{2}{3}\% = \frac{2}{3} \div 100 = \frac{1}{150}$$

$$150\% = \frac{150}{100} = 1.5 \qquad 0.2\% = \frac{0.2}{100} = 0.002$$

Any rational number can be expressed as a percent. For example:

$$\frac{1}{2} = \frac{50}{100} = 50\% \qquad \frac{6}{5} = \frac{120}{100} = 120\%$$

$$3.4 = \frac{340}{100} = 340\% \qquad \frac{2}{3} = \frac{66\frac{2}{3}}{100} = 66\tfrac{2}{3}\%$$

When you multiply a number called the *base* (b) by a percent, or *rate* (r), the result is the *percentage* (p). This can be expressed as

$$p = br.$$

Generally, in percent problems you are asked to find one of the three quantities p, b, or r given in the formula $p = br$.

EXAMPLE 1 **a.** What is 9% of 52? **b.** What percent of 185 is 148? **c.** 525 is 175% of what number?

SOLUTION

a.
$$\begin{aligned} p &= br \\ p &= 52\,(9\%) \\ &= 52(0.09) \\ &= 4.68 \end{aligned}$$

b.
$$\begin{aligned} p &= br \\ 148 &= 185r \\ r &= \frac{148}{185} \\ &= 0.8 \\ &= 80\% \end{aligned}$$

c.
$$\begin{aligned} p &= br \\ 525 &= b(175\%) \\ 525 &= b\left(\frac{7}{4}\right) \\ 4(525) &= 4\left(\frac{7b}{4}\right) \\ 2100 &= 7b \\ 300 &= b \end{aligned}$$

Percents are often used to describe changes in quantities. For example:

A sweater was marked down 20%.
The population increased by 15%.

In each of these cases, the percent change is based on the original quantity or amount. The formula $p = br$ can be used to describe each situation where b is the original amount, r is the rate or percent change, and p is the change in the amount. This can be expressed as follows.

change in amount = original amount × % change

EXAMPLE 2 Blair bought a sweater at a 20% discount. If she paid \$36 for the sweater, what was the original price?

SOLUTION

Step 1 The problem asks for the original price of the sweater.

Step 2 Let x = the original price.
Then $x - 36$ = the amount of the discount.

Step 3

$$\underbrace{\text{Amount of the discount}}_{x - 36} = \underbrace{\text{Original amount}}_{x} \times \underbrace{\text{\% discount}}_{20\%}$$

Step 4

$$x - 36 = x(20\%)$$
$$x - 36 = x\left(\frac{1}{5}\right)$$
$$5(x - 36) = 5\left(\frac{x}{5}\right)$$
$$5x - 180 = x$$
$$4x = 180$$
$$x = 45$$

Step 5 Is \$36 twenty percent less than \$45?
Yes, since

$$20\% \text{ of } \$45 = \frac{1}{5} \times \$45 = \$9$$

and

$$\$45 - \$9 = \$36.$$

$\therefore$ the original price was \$45.

When money is invested, the rate of interest is expressed as a percent of the amount of money invested. The *simple interest* earned annually can be found by multiplying the *principal* (the amount of money invested) by the *annual interest rate*. This can be expressed as follows.

simple annual interest = principal × annual interest rate

This formula can be written as

$$i = pr,$$

where i is the simple annual interest, p is the principal, and r is the annual rate of interest. Thus, the simple annual interest, or income, from \$2000 invested at an annual rate of interest of 5% is

$$i = \$2000 \times 5\% = \$2000 \times 0.05 = \$100.$$

EXAMPLE 3 Murray invests part of \$9000 in bank accounts that pay 6% simple annual interest and the rest in bonds that pay 11% simple annual interest. How much money is invested in each way if his total annual income from these investments is \$890?

SOLUTION

Step 1 The problem asks for the amounts of money invested in bank accounts and in bonds.

Step 2 Let x = amount invested in bank accounts.
Then $9000 - x$ = amount invested in bonds.

Step 3

$$\underbrace{\text{Total interest}}_{890} = \underbrace{\text{Interest from bank accounts}}_{x(6\%)} + \underbrace{\text{Interest from bonds}}_{(9000 - x)(11\%)}$$

Step 4

$$890 = x(6\%) + (9000 - x)(11\%)$$
$$890 = x(0.06) + (9000 - x)(0.11)$$
$$890 = 0.06x + 990 - 0.11x$$
$$0.05x = 100$$
$$x = 2000$$

Thus, $x = 2000$ and $9000 - x = 7000$.

Step 5 Checking the results is left to you.

$\therefore$ Murray invested \$2000 in bank accounts and \$7000 in bonds.

Chemists, druggists, and others frequently are confronted with situations where they find it necessary to mix ingredients. As you saw in Section 4-9, percent is commonly used in such problems to describe the composition of the mixture. Additional problems of this type are found in the following set of problems.

Oral Exercises

Replace each __?__ with a real number to make a true statement.

1. 16% of 150 = __?__
2. $10\frac{1}{2}$% of 400 = __?__
3. __?__ % of 64 = 16
4. __?__ % of 300 = 225
5. 25% of __?__ = 13
6. 120% of __?__ = 60
7. __?__ % of 175 = 350
8. 250% of __?__ = 40
9. 0.7% of 1000 = __?__
10. $133\frac{1}{3}$% of 150 = __?__
11. __?__ % of 90 = 60
12. $\frac{2}{3}$% of __?__ = 30

Problems

Solve.

A

1. Linda received 54% of the votes cast for president of the student council of Fisher High School. If Linda received 594 votes, how many votes were cast?
2. Del paid a 6% sales tax on his new car. What was the price of the car if the sales tax was \$705?

3. A survey found that 1950 people out of 3000 people polled read at least one newspaper per day. What percent of the people polled did not read at least one newspaper per day?

4. A dress was on sale for $52. If the original selling price was $78, what was the percent discount?

5. How many liters of pure acid must be added to 5 L of a solution that is 20% acid to make a solution that is 60% acid?

6. How many liters of water must be evaporated from 220 L of a solution that is 5% salt, to leave a solution that is $5\frac{1}{2}$% salt?

7. Computer Universe stock lost $12\frac{1}{2}$% of its value in one day's trading in the stock market. If the stock sold for $28 per share at the end of the day, what was its price at the start of the day?

8. A buyer for a computer store paid $194 each for some printers. What should be the selling price of each printer, if the markup is to be 25% of the selling price?

9. Liz invested $3000, part at an annual interest rate of 5% and the rest at an annual interest rate of 12%. How much did she invest at each rate if her total income on the investment for one year was $220?

10. Andrew invested a total of $15,000 in two local businesses. He received 6% per year on one investment and 8% per year on the other. If the total income from these two investments for one year was $1090, how much did Andrew invest in each business?

B

11. The bowling team of Franco's Bakery has a record of 8 wins and 12 losses. What is the least number of the remaining 35 games the team must win if they are to finish the season winning at least 60% of all the games played?

12. Aubrey has decided to invest a total of $10,000, some at an annual interest rate of $5\frac{3}{4}$% and the rest at an annual interest rate of $9\frac{1}{2}$%. What is the most he can invest at $5\frac{3}{4}$% if he wants to earn at least $920 interest on the investments during the year?

13. Tom bought several appliances and a new car. He paid a sales tax of $7\frac{1}{2}$% on the appliances and an excise tax of $6\frac{1}{2}$% on the car. Before these taxes, the appliances and car together cost $15,200. If he paid a total of $1015 in taxes, how much did the car cost?

14. When a rubber ball is dropped, it rebounds to a height that is 80% of that from which it is dropped. From what height was it dropped if it has traveled a total of 306.5 cm at the instant it strikes the ground for the fourth time?

15. Michelle has some money invested at an annual interest rate of 9%. She has three fourths as much money invested at an annual interest rate of 8% as she does at 9%. She also has $300 less invested at an annual interest rate of $6\frac{1}{2}$% than she does at 8%. If Michelle's income for one year from these three investments is $378, how much is invested at each rate?

16. Juanita purchased vacuum cleaners at a cost of $171 each to sell in her store. During a summer sale, she marked them down 5%. If this discount price gave her a 25% profit on the cost price, what was the selling price of each vacuum cleaner before the sale?

C 17. Donna invested $2400 in bonds that earn 8% per year. She also invested money in a bank account that earns $6\frac{1}{2}\%$ per year. Her yearly return on the two investments was the same as if both sums had been invested at $7\frac{1}{6}\%$ per year. Find the amount of money invested in the bank account.

18. Jon invested part of a $200,000 fund at 12% per year and the rest at 8% per year. If he had invested twice as much of the fund at 12% and the rest at 8%, he would have increased his annual income from the fund by $2400. How much did he invest at 8%?

19. A chemist has a can full of paint thinner that is 70% alcohol. After replacing 7 L of the solution with 7 L of pure alcohol, the resulting solution is $87\frac{1}{2}\%$ alcohol. How many liters does the can hold?

20. A car radiator that can hold 20 L is full of a solution that is 16% antifreeze. How many liters of the solution should be replaced with pure antifreeze in order to have the radiator full of a solution that is 37% antifreeze?

Computer Exercises For students with computer experience

1. Write a program that will allow you to input the price of an item, a percent, and a code that indicates whether the percent is to be used for discount or markup. The program should then compute the new price of the item after the given percent discount or markup.

2. Write a program that will allow you to input two positive numbers and will compute the percent that the first number is of the second. A sample output would be

   60 IS 25% OF 240.

3. Modify the program that you wrote for Exercise 2 so that you can input two positive numbers and the program will compute the percent increase or decrease of the second number from the first. A sample output would be

   60 IS A 20% INCREASE FROM 50.

4. In baseball, a pitcher's *earned run average*, or *ERA*, is the average number of runs that the pitcher allows the opposing team per nine innings pitched. To find a pitcher's ERA, you divide the number of innings pitched by nine, then divide the number of earned runs that the pitcher has allowed by this result. Write a program that will compute a pitcher's ERA when you input a number of innings pitched and the number of earned runs allowed.

Self-Test 1

VOCABULARY percent (p. 431)

Solve.

1. $\frac{a}{4} = 20 - \frac{a}{6}$

2. $\frac{t+3}{4} + \frac{t-1}{3} = 1$ *Obj. 1, p. 427*

3. $b - \frac{5b}{2} \leq 6$

4. $\frac{5x-1}{3} - \frac{3x+1}{2} > \frac{5}{6}$

5. Bill purchased a sweater that had been marked down 20%. If he paid $35.92 for the sweater, what was the selling price of the sweater before it was marked down? *Obj. 2, p. 427*

6. Doris invested $5000, part at 8% per year and the rest at $10\frac{1}{2}$% per year. How much did she invest at each rate if her total income from the investments was $467.50 for one year?

Check your answers with those at the back of the book.

Charlotte Angas Scott

1858–1931

In 1876 Charlotte Angas Scott graduated from Cambridge University in England with high honors, but she did not receive a degree. At that time, although women were allowed to take courses and examinations at Cambridge, they were not degree candidates. The University of London, however, awarded her the bachelor of science degree in 1882 and the doctorate of science in 1885.

Dr. Scott then came to the United States to teach at Bryn Mawr, a newly founded women's college in Pennsylvania. She directed undergraduate and graduate mathematical studies at the college for forty years until her retirement in 1925. In addition, she served on the council of the American Mathematical Society and wrote many articles for mathematical journals. Her major theoretical contribution to the field of mathematics was in the study of algebraic geometry.

Rational Expressions in Equations and Problems

OBJECTIVES for Sections 9-3 through 9-6:
1. *To solve fractional equations.*
2. *To use fractional equations to solve number problems, work problems, and motion problems.*

9–3 Fractional Equations

An equation in which a variable appears in the denominator of one or more terms is called a **fractional equation.** The method used to solve a fractional equation is similar to the one used in Section 9-1 to solve an equation in which the fractions have whole-number denominators. Note, however, that when you multiply each side of a fractional equation by the LCD of the terms, you may not obtain an equivalent equation.

EXAMPLE Solve $3 + \frac{16}{x^2 - 1} = \frac{8}{x - 1}$.

SOLUTION

$$3 + \frac{16}{x^2 - 1} = \frac{8}{x - 1}$$

$$(x^2 - 1)\left(3 + \frac{16}{x^2 - 1}\right) = (x^2 - 1)\left(\frac{8}{x - 1}\right) \longleftarrow \begin{cases}\text{Multiply each} \\ \text{side by } x^2 - 1, \\ \text{the LCD of the} \\ \text{fractions.}\end{cases}$$

$$3(x^2 - 1) + 16 = 8(x + 1)$$

$$3x^2 + 13 = 8x + 8$$

$$3x^2 - 8x + 5 = 0$$

$$(3x - 5)(x - 1) = 0$$

$$x = \tfrac{5}{3} \quad \text{or} \quad x = 1$$

Notice that $x \neq 1$ is a restriction on the variable in the original equation. If $x = 1$, the denominators $x - 1$ and $x^2 - 1$ would equal zero. Thus, 1 is not a solution of the original equation. Check $\frac{5}{3}$ in the original equation.

$$3 + \frac{16}{x^2 - 1} = \frac{8}{x - 1}$$

$$3 + \frac{16}{\left(\frac{5}{3}\right)^2 - 1} \stackrel{?}{=} \frac{8}{\frac{5}{3} - 1}$$

$$3 + \frac{16}{\frac{16}{9}} \stackrel{?}{=} \frac{8}{\frac{2}{3}}$$

$$3 + 9 \stackrel{?}{=} 12$$

$$12 = 12 \quad \checkmark$$

$\therefore$ the solution set is $\{\frac{5}{3}\}$.

The equation obtained by multiplying each side of the original equation by $x^2 - 1$ has the extra, or *extraneous*, root 1, a number for which the multiplier $x^2 - 1$ represents 0. Whenever you multiply an equation in a variable by a polynomial in that variable, the solution set of the resulting equation always contains all the roots of the original equation. But it sometimes also contains numbers that are *not* roots of the original equation. Therefore, *you must check each root of the resulting equation to see that it satisfies the original equation.*

Oral Exercises

For each equation:

a. State all restrictions on the value of the variable.

b. State the LCD of the fractions.

c. State the equation that results when both sides are multiplied by the LCD.

d. Solve.

1. $\frac{7}{m + 5} = \frac{3}{5}$

2. $\frac{6}{y + 2} - \frac{3}{y} = -1$

3. $\frac{4}{p^2} + \frac{1}{p - 3} = \frac{1}{p}$

4. $\frac{2x}{x - 1} = x + \frac{2}{x - 1}$

5. $\frac{e + 2}{e - 2} - \frac{e - 2}{e + 2} = 0$

6. $\frac{5}{t^2 - 9} = \frac{1}{t + 3}$

Written Exercises

Solve.

A

1. $\frac{b}{b + 2} = \frac{7}{8}$

2. $\frac{5}{m - 3} = 5$

3. $\frac{9}{z} - \frac{3}{z} = \frac{1}{2}$

4. $\frac{7}{c} = 2 + \frac{3}{c}$

5. $\frac{9}{2y} + \frac{3}{y} = \frac{8}{y} - \frac{1}{16}$

6. $\frac{6}{n} - \frac{1}{2} = \frac{9}{2n} - \frac{1}{n}$

7. $\frac{7}{3d} = \frac{5}{6d} + \frac{d + 1}{2d}$

8. $\frac{1}{2} + \frac{2x - 5}{2x} = \frac{14}{x} - \frac{1}{3}$

9. $\frac{3}{y - 3} + 9 = \frac{y}{y - 3}$

10. $1 + \frac{4}{q + 2} = \frac{q + 6}{q + 2}$

11. $\frac{2}{e} - \frac{2e + 1}{6e} = \frac{2e - 1}{18}$

12. $\frac{5}{2} - \frac{3z - 6}{3z} = \frac{z - 8}{2z}$

13. $f = \frac{f + 5}{f - 2} - 5$

14. $4x = \frac{14 - x}{x - 1} + 14$

15. $\frac{20}{p} - 3 = \frac{22}{3p - 3}$

16. $\frac{10 - z}{4z} = \frac{1}{z - 1}$

17. $\frac{5b - 4}{3b + 1} = \frac{1}{14}(2b - 5) - \frac{1}{7}b$

18. $\frac{5c^2 + 4c - 7}{5c + 8} = \frac{1}{15}(15c - 13)$

19. $\frac{2}{r^2 - 2r} - \frac{1}{r} = \frac{1}{3}$

20. $\frac{4}{s - 1} = \frac{s^2 + 4}{s^2 - 1}$

21. $\frac{2t}{t - 1} = \frac{t + 6}{1 - t}$

22. $\frac{5 - x}{x - 3} - \frac{1}{3 - x} = 0$

23. $\frac{a}{a - 1} - \frac{a - 1}{a} = \frac{3}{2}$

24. $\frac{2b^2 + 3b}{2b + 1} + \frac{1}{3b} = b + 1$

25. $\frac{9}{t + 5} - \frac{1}{t - 5} = \frac{3t}{t^2 - 25}$

26. $\frac{10w}{w + 2} - \frac{2w - 3}{w - 2} = \frac{2w^2 - 3}{w^2 - 4}$

27. $\frac{4t - 36}{t^2 - 9} + \frac{11}{3 - t} = \frac{11}{t + 3}$

28. $\frac{5}{3 - x} = \frac{10}{x + 3} - \frac{7x + 1}{x^2 - 9}$

Solve for *x*.

B 29. $y = \frac{x}{x + b}$

30. $t = \frac{x - \pi r^2}{\pi x}$

31. $P = \frac{n}{x} - \frac{m}{cx}$

32. $R = \frac{x}{a + \frac{x}{t}}$

Solve.

33. $\frac{5}{u + 1} + \frac{1}{u - 1} - \frac{7}{3u - 5} = 0$

34. $\frac{1}{c - 3} - \frac{2}{2c - 1} = \frac{5}{3c + 3}$

35. $\frac{2}{y - 2} - \frac{1}{y + 2} = \frac{4}{y^2 - 2y}$

36. $\frac{b + 1}{b^2 - b} - \frac{b}{b^2 - 1} = \frac{b - 1}{b^2 + b}$

37. $\frac{5}{x - 2} - \frac{x}{x + 5} = \frac{x^2 - 4}{10 - 3x - x^2}$

38. $\frac{a - 1}{3a + 2} + \frac{3a + 4}{1 - 2a} = \frac{3a^2 - 5}{6a^2 + a - 2}$

39. a. Are $\frac{x^2 - 1}{x^2 - 1} = 1$ and $x^2 - 1 = x^2 - 1$ equivalent equations over $\mathcal{R}$? Explain.

b. Are $\frac{x^2 + 1}{x^2 + 1} = 1$ and $x^2 + 1 = x^2 + 1$ equivalent equations over $\mathcal{R}$? Explain.

C 40. For what value of *k* will the solution set of $\frac{7x + 4}{k} = x - 13$ be $\{-6\}$?

41. For what value of *k* will the solution set of $\frac{4x - k}{x - 5} = 3$ be the empty set?

Solve. (*Hint*: If $\frac{a}{b} > 0$, then either (1) $a > 0$ and $b > 0$ or (2) $a < 0$ and $b < 0$.)

42. $\frac{x + 1}{x - 2} > 0$

43. $\frac{x - 4}{x - 1} < 0$

44. $\frac{x + 3}{x - 1} < -1$

45. $\frac{8 - x}{6 + x} > 1$

9–4 Number Problems

Fractional equations are often used in solving number problems.

EXAMPLE 1 The numerator of a fraction is 15 less than the denominator, and the fraction is equal to $\frac{4}{7}$. Find the fraction.

SOLUTION

Step 1 The problem asks for a fraction whose numerator is 15 less than the denominator.

Step 2 Let $x =$ the denominator of the fraction.
Then $x - 15 =$ the numerator of the fraction.

Step 3
$$\frac{x - 15}{x} = \frac{4}{7}$$

Step 4
$$7x\left(\frac{x - 15}{x}\right) = 7x\left(\frac{4}{7}\right)$$
$$7(x - 15) = 4x$$
$$7x - 105 = 4x$$
$$3x = 105$$
$$x = 35, \text{ and } x - 15 = 20$$

Step 5 Checking the results is left to you.

$\therefore$ the fraction is $\frac{20}{35}$.

The following theorem is often helpful in solving problems involving the difference of two reciprocals. It is proved in Exercise 23 on page 443.

> ***Theorem.*** For all positive real numbers a and b,
>
> if $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

EXAMPLE 2 The difference of the reciprocals of two positive numbers is $\frac{1}{6}$, and one of the numbers is 3 times the other. Find the numbers.

SOLUTION

Step 1 The problem asks for two positive numbers, one of which is 3 times the other.

Step 2 Let $x =$ the lesser number.
Then $3x =$ the greater number.
The reciprocals are $\frac{1}{x}$ and $\frac{1}{3x}$.

Step 3 Since $x > 0$, $3x > x$. Thus, $\frac{1}{3x} < \frac{1}{x}$.

$$\underbrace{\text{Greater reciprocal}}_{\frac{1}{x}} - \underbrace{\text{Lesser reciprocal}}_{\frac{1}{3x}} = \underbrace{\text{Difference}}_{\frac{1}{6}}$$

Step 4

$$\frac{1}{x} - \frac{1}{3x} = \frac{1}{6}$$

$$6x\left(\frac{1}{x} - \frac{1}{3x}\right) = 6x\left(\frac{1}{6}\right)$$

$$6 - 2 = x$$

$$4 = x$$

Thus $x = 4$, and $3x = 12$.

Step 5 Checking the results is left to you.

$\therefore$ the numbers are 4 and 12.

In the next example, a problem that is almost identical to the problem solved in Example 2 is considered. The only difference is that you are not told that the two numbers are positive.

EXAMPLE 3 The difference of the reciprocals of two numbers is $\frac{1}{6}$, and one of the numbers is 3 times the other. Find the numbers.

SOLUTION

Step 1 The problem asks for two numbers, one of which is 3 times the other.

Step 2 Let x = one of the numbers.

Then $3x$ = the other number.

The reciprocals are $\frac{1}{x}$ and $\frac{1}{3x}$.

Step 3 If $\frac{1}{3x} < \frac{1}{x}$, then $\frac{1}{x} - \frac{1}{3x} = \frac{1}{6}$.

If $\frac{1}{3x} > \frac{1}{x}$, then $\frac{1}{3x} - \frac{1}{x} = \frac{1}{6}$. That is, $\frac{1}{x} - \frac{1}{3x} = -\frac{1}{6}$.

The disjunction

$$\frac{1}{x} - \frac{1}{3x} = \frac{1}{6} \quad \text{or} \quad \frac{1}{x} - \frac{1}{3x} = -\frac{1}{6}$$

can be represented by the following equation involving the absolute value of the difference.

$$\left|\frac{1}{x} - \frac{1}{3x}\right| = \frac{1}{6}$$

Steps 4 and 5 By solving the disjunction, you will find that there are two possible solutions, 4 and 12 or -4 and -12.

Problems

Solve.

A

1. The sum of two numbers is 30 and their quotient is $\frac{2}{3}$. Find the numbers.
2. The difference of two positive numbers is 8 and their quotient is $\frac{5}{7}$. Find the two numbers.
3. Five sixths of a number is 14 more than half of the number. Find the number.
4. What number added to both the numerator and denominator of the fraction $\frac{3}{8}$ results in a fraction equal to $\frac{2}{3}$?
5. The sum of the reciprocals of two consecutive positive integers is $\frac{17}{72}$. Find the integers.
6. Find two consecutive even integers such that 4 times the reciprocal of the lesser integer is equal to 5 times the reciprocal of the greater integer.
7. One number is 14 more than another number. Five ninths of the lesser number is equal to five sixteenths of the greater number. Find the numbers.
8. The denominator of a fraction is 2 greater than 3 times the numerator, and the fraction is equal to $\frac{5}{16}$. Find the fraction.
9. The numerator of a fraction is 11 more than twice the denominator, and the fraction is equal to $\frac{7}{3}$. Find the fraction.
10. When a number is added to the numerator of $\frac{5}{16}$ and twice the same number is subtracted from the denominator, the result is a fraction equal to 6. Find the number.
11. Find two numbers whose sum is 14 and the sum of whose reciprocals is $\frac{7}{24}$.
12. The sum of a number and its reciprocal is $\frac{34}{15}$. Find the two numbers.

B

13. The difference of the reciprocals of two numbers is $\frac{1}{4}$ and one of the numbers is 4 times the other. Find the numbers.
14. One number is 3 times another. If the difference of the reciprocals of the two numbers is 12, find the numbers.
15. The difference between two numbers is 4. When the reciprocal of the lesser number is subtracted from the reciprocal of the greater number, the result is $-\frac{1}{24}$. Find the numbers.
16. Find the least positive integer for which the sum of its reciprocal and $\frac{9}{10}$ is greater than 4 times the reciprocal.
17. The difference of a number and its reciprocal is $\frac{9}{20}$. Find the two numbers.
18. The difference of two numbers is 10 and their quotient is $\frac{3}{8}$. Find the two numbers.

19. One positive number is 4 times another. If 240 is divided by each number, the greater quotient exceeds the lesser by 15. Find the two numbers.

20. Find two consecutive integers such that twice the reciprocal of the lesser, increased by the reciprocal of the greater, is equal to 11 times the reciprocal of the product of the integers.

C 21. The sum of two numbers is 100. When the greater number is divided by the lesser, the partial quotient is 7 and the remainder is 4. Find the numbers.

22. An integer is to be subtracted from both the numerator and the denominator of $\frac{7}{12}$ to yield a fraction whose value is greater than $\frac{1}{3}$. Find the greatest value of such an integer less than 10 and the least value greater than 10.

Write a direct proof of each theorem.

23. If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.

24. a. If $a < b < 0$, then $\frac{a}{b} > 1$. b. If $0 < a < b$, then $\frac{a}{b} < 1$.

9–5 Work Problems

It is often useful to be able to solve problems that involve finding how long it takes to accomplish a task when a steady rate of work is assumed. If the *work rate* is the fraction of the whole job that can be done per unit of time, then

$$\text{work rate} \times \text{time} = \text{work done.}$$

EXAMPLE 1 Molly can paint a house in four days. Tess could paint the same house in six days. How long would it take them to paint the house if they worked together?

SOLUTION

Step 1 The problem asks for the number of days it would take to paint the house if they worked together.

Step 2 Let x = number of days required to do the job together.

Since Molly can do the whole job in 4 days, she can do $\frac{1}{4}$ of the job per day. In x days, she could do $\frac{1}{4} \cdot x$ of the job, or $\frac{x}{4}$ of the job. Similarly, in x days Tess could do $\frac{x}{6}$ of the job.

Solution continued on following page.

Step 3 $\underbrace{\text{Part of job done by Molly}}_{\frac{x}{4}} + \underbrace{\text{Part of job done by Tess}}_{\frac{x}{6}} = \underbrace{\text{Whole job}}_{1}$

Step 4
$$\frac{x}{4} + \frac{x}{6} = 1$$
$$3x + 2x = 12$$
$$5x = 12$$
$$x = \frac{12}{5}, \text{ or } 2\tfrac{2}{5}$$

Step 5 In $2\frac{2}{5}$ days, Molly would paint $\frac{1}{4} \times \frac{12}{5} = \frac{3}{5}$ of the house and Tess would paint $\frac{1}{6} \times \frac{12}{5} = \frac{2}{5}$ of the house. Since $\frac{3}{5} + \frac{2}{5} = 1$, the whole house would be painted in $2\frac{2}{5}$ days.

$\therefore$ it would take $2\frac{2}{5}$ days to paint the house if they worked together.

EXAMPLE 2 There are two intake pipes to a large storage tank. Using the smaller pipe alone, it takes twice as long to fill the tank as it does using the larger pipe alone. The tank can be filled in 8 min if both pipes are used. How long would it take using only the smaller pipe?

SOLUTION

Step 1 The problem asks for the time it would take to fill the tank using the smaller pipe alone.

Step 2 Let x = time in minutes to fill tank using large pipe alone.
Then $2x$ = time in minutes to fill tank using smaller pipe alone.

	Work rate	*Time*	*Work done*
Larger pipe	$\frac{1}{x}$	8	$\frac{8}{x}$
Smaller pipe	$\frac{1}{2x}$	8	$\frac{8}{2x}$

Step 3 $\underbrace{\text{Part of tank filled by larger pipe}}_{\frac{8}{x}} + \underbrace{\text{Part of tank filled by smaller pipe}}_{\frac{8}{2x}} = \underbrace{\text{Whole tank}}_{1}$

Step 4
$$\frac{8}{x} + \frac{8}{2x} = 1$$
$$16 + 8 = 2x$$
$$24 = 2x, \text{ and } x = 12$$

Step 5 In 8 min, the larger pipe would fill $\frac{1}{12} \times 8 = \frac{2}{3}$ of the tank and the smaller pipe would fill $\frac{1}{24} \times 8 = \frac{1}{3}$ of the tank. Since $\frac{2}{3} + \frac{1}{3} = 1$, the whole tank would be filled in 8 min.

$\therefore$ it would take the smaller pipe 24 min to fill the whole tank.

Problems

Solve.

A

1. Jack can enter the payroll data into the computer in 6 h; Amanda requires 8 h to complete the job. How long would it take both people to enter the data if they work together?
2. It takes one crew of cleaners 12 h to wash and wax the floors in the Euclid Office Building. Another crew does the same job in 15 h. How long would it take both crews working together to do the job?
3. One outlet of a grain elevator will empty the elevator in 1.5 h; a second outlet will empty the same elevator in 4 h. How long would it take both outlets working together to discharge the stored grain from the elevator?
4. A computer can process a company's invoices in 2 h. A newer computer can process the same number of invoices in 1.2 h. How long would it take to process the invoices using both computers?
5. To reduce the oil level in a fuel storage tank, two pumps are used to transfer the oil into fuel trucks. If one pump can lower the oil level 1 m in 2 h and the other pump can lower the oil level 1 m in 5 h, how long must both pumps operate together to lower the surface of the oil 14 m?
6. The Warren County Pool uses its own pump along with a pump on the local fire truck to fill the swimming pool. If one pump alone takes 8 h to fill the pool and the other pump alone takes 12 h to fill the pool, how long would it take both pumps working together to fill the pool?
7. One optical scanner can read and grade a set of standard answer sheets in $\frac{1}{3}$ the time it takes another scanner to do the same job. Together they can read and grade the sheets in 12 min. How long would it take each scanner alone to do the job?
8. It takes Gary twice as long to mow his lawn as it takes his father to do the same job. If they work together they can complete the job in $1\frac{1}{2}$ h. How long would it take Gary to do the job alone?

B

9. Michele can type the school newspaper in $1\frac{1}{2}$ h. Working together with Paul, the job is completed in $1\frac{1}{8}$ h. How long would it take Paul alone to do the job?

10. Sandie can complete her paper route in 45 min. When her sister Christina helps her, it takes them 18 min to complete the route. How long would it take Christina alone to complete the route?

11. It takes 6 min to fill a certain pool and 18 min to drain the same pool when it is full. With the drain open and the pool empty, how long would it take to fill the pool?

12. An aquarium can be filled by two inlets in 0.4 h and 0.25 h, respectively, and emptied by an outlet in 0.5 h. How long would it take to fill the aquarium if the inlets and outlet were operating simultaneously?

C 13. Diann can complete a roofing job in 4 h. Sue can complete the same job in 5 h. After working together on the job for 1.25 h, Diann leaves. How long will it take Sue to complete the work?

14. Tony can build a fence in 8 h, while his brother Sam can do it in 6 h. If Tony works alone for 5 h and then lets Sam finish the job, how long will it take Sam working alone to finish the fence?

15. Ruth and Terry planted tulip and crocus bulbs for 6 h. Ruth planted 5 bulbs in the time it took Terry to plant 3 bulbs. How long would it take Terry alone to complete the job?

16. In 2 h, Kelly laid new floor tile over $\frac{1}{5}$ of the room. When she was joined by Annette, the rest of the work was completed in 3 h. How long would it have taken Annette to do the entire job alone?

9–6 Motion Problems

In Sections 4-8 and 6-6 you solved problems involving uniform motion using the following formula.

$$\text{distance} = \text{rate} \times \text{time}$$
$$d = rt$$

Now that you know how to solve fractional equations, you may sometimes find it convenient to use the formula for uniform motion rewritten as

$$t = \frac{d}{r} \quad \text{or} \quad r = \frac{d}{t}.$$

EXAMPLE Lorraine can ride 17 km on her bicycle in the same time that it takes her to walk 9 km. If her riding speed is 4 km/h faster than her walking speed, how fast does she walk?

SOLUTION

Step 1 The problem asks for Lorraine's speed walking.

Step 2 Let x = speed walking in km/h.
Then $x + 4$ = speed riding in km/h.

	rate (km/h)	distance (km)	time (h)
Riding	$x + 4$	17	$\frac{17}{x+4}$
Walking	x	9	$\frac{9}{x}$

Step 3

$$\underbrace{\text{Time walking}} = \underbrace{\text{Time riding}}$$
$$\frac{9}{x} = \frac{17}{x+4}$$

Step 4

$$\frac{9}{x} = \frac{17}{x+4}$$
$$9(x+4) = 17x$$
$$9x + 36 = 17x$$
$$36 = 8x$$
$$4.5 = x$$

Step 5 Can Lorraine ride 17 km in the same time that she can walk 9 km? Yes, since

$$\frac{17}{x+4} = \frac{17}{4.5+4} = \frac{17}{8.5} = 2 \quad \text{and} \quad \frac{9}{x} = \frac{9}{4.5} = 2.$$

$\therefore$ Lorraine walks at a speed of 4.5 km/h.

Problems

Solve.

A 1. A freight train travels 280 km in the same time an express train travels 420 km. If the speed of the express train is 40 km/h greater than that of the freight train, find the speed of each train.

2. Joe drove 480 km in the same amount of time it took Marie, traveling 10 km/h faster, to travel 540 km. How fast was Joe traveling?

3. An airplane whose speed in still air is 370 km/h can travel 1000 km with the wind in the same time it travels 850 km against the wind. Find the speed of the wind.

4. Mike can row in still water at a speed twice that of the current in a certain river. It takes Mike 2 h more to row 10 km up the river than it takes him to row 15 km down the river. What is the speed of the current in the river?

5. A train required 2 h longer to travel 500 km than a plane required to travel 1200 km. Find the speed of the train, if the plane travels 4 times as fast as the train.

6. The Bedford family made a 43.25 km trip in 5.5 h. On the first part of the trip they crossed a lake by boat traveling at 12 km/h. On the rest of the trip they walked along a scenic trail. If their average walking speed was 5 km/h, how far did they walk?

7. Flying at a steady speed, a pilot flew 765 km due south. He then changed course and flew west at the same speed for 918 km. If the second part of the trip took 36 min more than the first part, what was the plane's air speed?

B 8. Lynn drove 225 km to visit a college campus. On the trip home, she averaged 15 km/h more than on the trip to the college. If her total travel time was $6\frac{3}{4}$ h, what was her average speed on the trip home?

9. In still water, Sarah's boat can travel 15 km/h. If it takes her a total of $4\frac{1}{2}$ h to travel 30 km up a river and then to return by the same route, what is the speed of the current in the river?

10. A bus trip of 200 km to the marching band competition would have taken five eighths as long if the average speed that the bus traveled had been increased by 30 km/h. Find the speed at which the bus traveled.

11. A racer on a motorcycle required 1 h longer to travel 180 km than a bicyclist required to travel 30 km. If the speed of the motorcycle was $2\frac{1}{2}$ times that of the bicycle, find both speeds.

12. After driving at a steady speed for 30 km, Roberto reached the expressway, where he increased his speed by 45 km/h. If after 120 km of travel on the expressway his total travel time had been 2 h, what was his speed before reaching the expressway?

13. Maurice left his home at 1:30 P.M. and drove to the airport at an average speed of 45 km/h. After a 40 min wait, he took off on a flight with an average speed of 350 km/h. He reached his destination at 4:20 P.M. If the total distance that Maurice traveled by car and by plane was 555 km, how far was Maurice's home from the airport?

C 14. Tony and Jason are test driving two cars on a 30 km track. Just as Jason starts his car, Tony passes him traveling at 120 km/h. After 2 min, Jason is traveling at 130 km/h and has traveled 2.5 km. If Tony and Jason maintain these speeds, how far will Jason travel before he passes Tony?

Self-Test 2

VOCABULARY fractional equation (p. 437)

Solve.

1. $5 - \frac{3}{a} = 4$ **2.** $\frac{m - 1}{m + 1} = \frac{6}{7}$ *Obj. 1, p. 437*

3. $\frac{7}{z - 3} = \frac{14}{z + 1}$ **4.** $\frac{2}{b - 1} - \frac{1}{b^2 - 1} = \frac{1}{b + 1}$

5. Find two numbers whose sum is 96 and whose quotient is $\frac{1}{5}$. *Obj. 2, p. 437*
6. One pipe can fill a tank in 5 h. A second pipe can fill the same tank in 10 h. How long would it take both pipes together to fill the tank?
7. The Driscoll family went on a weekend trip to their cottage, which was 90 km from their home. Their average speed going to the cottage was 20 km/h less than their average speed returning home. If the traveling time for the round trip was $3\frac{3}{4}$ h, what was the family's average speed on the way to the cottage?

Check your answers with those at the back of the book.

Ratio, Proportion, and Variation

OBJECTIVES for Sections 9-7 through 9-10:
1. To use the concepts of ratio and proportion to solve problems.
2. To solve problems involving direct, inverse, joint, and combined variation.

9–7 Ratio and Proportion

Quotients of real numbers can be used to describe many situations. For example, in the packages of tulip bulbs that Larry sells, there are 2 red bulbs for every 3 yellow bulbs. You can describe this situation mathematically as follows.

$$\frac{\text{number of red tulip bulbs}}{\text{number of yellow tulip bulbs}} = \frac{2}{3}$$

You can describe this relationship by stating that the number of red bulbs and the number of yellow bulbs are in the *ratio* 2 to 3, or, in symbols, 2 : 3. The **ratio** of two numbers is the quotient of one number divided by a second number, provided that the second number is not zero.

If you know that two numbers are in the ratio 4 to 5, you can represent one number as $4x$ and the other as $5x$, since

$$\frac{4x}{5x} = \frac{4}{5}, \qquad \text{or} \qquad 4x:5x = 4:5.$$

This fact is useful when solving problems involving ratios.

EXAMPLE 1 If $2700 is divided between George and Mary in the ratio 4 to 5, respectively, how much does each receive?

SOLUTION

Step 1 The problem asks how much money George and Mary each receive.

Step 2 Let $4x$ = the amount of money in dollars that George receives.
Then $5x$ = the amount of money in dollars that Mary receives.

Step 3

$$\underbrace{\text{Amount of money George receives}}_{4x} + \underbrace{\text{Amount of money Mary receives}}_{5x} = \underbrace{\text{Total amount of money}}_{2700}$$

Step 4

$$4x + 5x = 2700$$
$$9x = 2700$$
$$x = 300$$

Thus $4x = 1200$, and $5x = 1500$.

Step 5 Are 1200 and 1500 in the ratio 4:5?

$$\frac{1200}{1500} = \frac{4}{5} \quad \checkmark$$

Does the money total $2700?

$$1200 + 1500 \stackrel{?}{=} 2700$$
$$2700 = 2700 \quad \checkmark$$

$\therefore$ George receives $1200 and Mary receives $1500.

Often the idea of ratio is extended to include three or more numbers. For instance, you say that a, b, and c are in the ratio 2:3:7 if the ratio of a to b is $\frac{2}{3}$, the ratio of a to c is $\frac{2}{7}$, and the ratio of b to c is $\frac{3}{7}$. If this situation occurred in a problem, you could represent a by $2x$, b by $3x$, and c by $7x$.

An equation which states that two ratios are equal is called a **proportion.** An example of a proportion is

$$\frac{a}{b} = \frac{c}{d}.$$

This proportion can also be written as

$$a:b = c:d$$

and can be read "a is to b as c is to d." In this proportion, a and d are called the **extremes,** and b and c are called the **means.**

The next theorem is proved in Exercise 39 on page 454.

> ***Theorem.*** For all real numbers a and c and all nonzero real numbers b and d,
>
> $$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

Thus, in any proportion, *the product of the extremes equals the product of the means*. In Example 2, this property is used in solving fractional equations.

EXAMPLE 2 Solve.

a. $\frac{2}{x} = \frac{4}{9}$ **b.** $\frac{6x-2}{7} = \frac{5x+7}{8}$

SOLUTION

a.

$$\frac{2}{x} = \frac{4}{9}$$
$$18 = 4x$$
$$\frac{9}{2} = x$$

Check:
$$\frac{2}{x} = \frac{4}{9}$$
$$\frac{2}{\frac{9}{2}} \stackrel{?}{=} \frac{4}{9}$$
$$\frac{4}{9} = \frac{4}{9} \quad \checkmark$$

$\therefore$ the solution set is $\left\{\frac{9}{2}\right\}$.

b.

$$\frac{6x-2}{7} = \frac{5x+7}{8}$$
$$8(6x-2) = 7(5x+7)$$
$$48x - 16 = 35x + 49$$
$$13x = 65$$
$$x = 5$$

Check:
$$\frac{6x-2}{7} = \frac{5x+7}{8}$$
$$\frac{6(5)-2}{7} \stackrel{?}{=} \frac{5(5)+7}{8}$$
$$\frac{28}{7} \stackrel{?}{=} \frac{32}{8}$$
$$4 = 4 \quad \checkmark$$

$\therefore$ the solution set is $\{5\}$.

EXAMPLE 3 Find the ratio of x to y, given that

$$\frac{3x+10y}{y} = 5.$$

SOLUTION

$$\frac{3x+10y}{y} = \frac{5}{1}$$
$$3x + 10y = 5y$$
$$3x = -5y$$
$$\frac{x}{y} = \frac{-5}{3}$$

$\therefore$ the ratio of x to y is -5 to 3.

EXAMPLE 4 If a train can travel 1600 mi in 30 h, how long will the train take to travel 1800 mi if it maintains the same average speed?

SOLUTION

Step 1 The problem asks how many hours the train will take to travel 1800 mi.

Step 2 Let $x =$ the number of hours.

Step 3 The train travels 1600 mi in 30 h and 1800 mi in x h.

Since the average speed $\left(r = \frac{d}{t}\right)$ is the same,

$$\frac{1600}{30} = \frac{1800}{x}.$$

Step 4

$$\frac{1600}{30} = \frac{1800}{x}$$
$$1600x = 54{,}000$$
$$x = 33\frac{3}{4}$$

Step 5 Will the train travel 1800 miles in $33\frac{3}{4}$ h?

The train's average speed is $\frac{1600}{30} = \frac{160}{3}$ mi/h.

The distance in miles the train will travel in $33\frac{3}{4}$ h is

$$d = r \cdot t = \frac{160}{3} \cdot 33\frac{3}{4} = \frac{160}{3} \cdot \frac{135}{4} = 1800.$$

$\therefore$ it takes the train $33\frac{3}{4}$ h to travel 1800 mi.

Oral Exercises

Simplify each ratio.

1. $\frac{6}{8}$ **2.** $16:24$ **3.** $\frac{3c}{7c}$ **4.** 10 km to 8 km

Replace each _?_ with the number that makes a true statement.

5. If the ratio of profit to selling price is $3:11$, then out of every 11 cents in sales, _?_ cents are profit.

6. If the ratio of new homes to old ones in a city is $2:5$, then there are 4 new homes for every _?_ old ones.

7. If the ratio of iron to oxygen, by mass, in ferric oxide is $7:3$, then 10 grams of this compound contains _?_ grams of iron and _?_ grams of oxygen.

8. If 4 out of every 9 foundry workers are 40 years old or older, then the ratio of the number of workers 40 or more years old to the number under 40 is _?_.

State the equation that results when you equate the product of the extremes and the product of the means in the following proportions.

9. $\frac{-5}{6} = \frac{x}{4}$
10. $\frac{4}{3b} = -\frac{6}{5}$
11. $\frac{a-2}{3} = 2a$
12. $\frac{5}{x+1} = \frac{3}{x-2}$

Written Exercises

Throughout this set of exercises, assume that the variables represent real numbers that do not result in division by zero.

Simplify each ratio.

A

1. Men to women in a college with 2400 men and 2000 women.
2. Women to men in a college with 10,000 students, of whom 7500 are women.
3. Profit to selling price if the profit is $2 and the selling price $9.
4. Profit to selling price if the cost is $8 and the selling price is $10.
5. The perimeter of a rectangle that measures 6 cm by 8 cm to the perimeter of a rectangle that measures 5 cm by 12 cm.
6. The circumference of a circle with radius 3 cm to the circumference of a circle with radius 5 cm.
7. The area of a rectangle that measures 7 cm by 9 cm to the area of a rectangle that measures 6 cm by 7 cm.
8. The area of a circle with radius 5 cm to the area of a circle with radius 6 cm.

Solve.

9. $\frac{5}{x} = \frac{3}{2}$
10. $\frac{4}{a} = \frac{-2}{3}$
11. $\frac{n}{-5} = \frac{-2n}{5}$
12. $-2 = \frac{1}{y}$
13. $\frac{x-3}{x+5} = \frac{11}{15}$
14. $\frac{-3}{5} = \frac{5-4c}{3c+10}$
15. $\frac{a+9}{a+3} = 2$
16. $\frac{3b-1}{4} = -\frac{3}{2}$
17. $\frac{-4}{2r-9} = \frac{-16}{3r+14}$
18. $\frac{2}{10+3e} = \frac{10}{28-7e}$
19. $\frac{u-7}{-12} = -\frac{2u+6}{72}$
20. $-\frac{2+w}{8} = -\frac{w-4}{24}$

Find the ratio of *a* to *b*.

21. $a = 6b$
22. $a = b$
23. $4a + 2b = 0$
24. $15b - 10a = 0$

Find the ratio of a to b.

25. $\frac{3}{a} = \frac{4}{b}$

26. $\frac{b}{3} = \frac{a}{7}$

B

27. $\frac{a}{c} = \frac{b}{d}$

28. $\frac{d}{b} = \frac{c}{a}$

29. $\frac{4a + 3b}{3b} = 7$

30. $\frac{5a - b}{5a} = -\frac{1}{5}$

31. $\frac{2a + b}{a - b} = \frac{2}{3}$

32. $\frac{3a - 2b}{6} = \frac{a + b}{5}$

Solve for x.

33. $R = \frac{1}{x} + \frac{1}{R}$

34. $M = \frac{1}{M + \frac{1}{x}}$

35. $\frac{x - 4y}{2y} = \frac{5y}{x - y}$

36. $\frac{x^2 - 5y^2}{y^2} = \frac{2(3x - 7y)}{y}$

C

37. Find b, given that

$$\frac{a + b}{b} = \frac{c + 18}{18} \quad \text{and} \quad \frac{a - 12}{12} = \frac{c - 36}{36}.$$

38. Find z, given that

$$\frac{6y + z}{z} = \frac{2 - 8x}{2} \quad \text{and} \quad \frac{2x - y}{6} = \frac{y - x}{3}.$$

Write a direct proof of each theorem.

39. If $\frac{a}{b} = \frac{c}{d}$, and if b and d are nonzero, then $ad = bc$.

40. If $\frac{a}{b} = \frac{c}{d}$, and if b, c, and d are nonzero, then $\frac{a}{c} = \frac{b}{d}$.

41. If $\frac{a}{b} = \frac{c}{d}$, and if b and d are nonzero, then $\frac{a + b}{b} = \frac{c + d}{d}$.

42. If $\frac{a}{b} = \frac{c}{d}$, and if a, b, c, and d are nonzero, then $\frac{b}{a} = \frac{d}{c}$.

Problems

Solve.

A

1. If 540 acres of land are divided between two people in the ratio 4 to 5, how many acres does each receive?
2. Two numbers are in the ratio 2 : 3. The lesser number is 164. What is the other number?

3. The measures of two complementary angles are in the ratio 7:8. Find the measure of each angle in degrees.
4. The measures of two supplementary angles are in the ratio 3:1. Find the measure of each angle in degrees.
5. The lengths of the sides of a triangle are in the ratio 3:3:5. The perimeter of the triangle is 44 cm. Find the lengths of the sides.
6. The lengths of the sides of a quadrilateral are in the ratio 2:4:5:9. The perimeter of the quadrilateral is 460 cm. Find the lengths of the sides.
7. A certain type of rain gutter has a mass of 4 kg for every 3 m of length. What is the mass of a 7.5 m length?
8. On an average, fifteen loblolly pine trees provide enough wood pulp for 12,000 paper grocery bags. How many bags could be made from the pulp of 1250 trees?
9. Each cube of chicken bouillon is used to make $1\frac{1}{2}$ c of broth. How many cubes are needed to make 12 c of broth?
10. A person walks 6 km in three fourths of an hour. How long would it take the person to walk 10 km at the same speed?
11. Ron earned $27.30 working 6.5 h. How much would he earn working 20 h?
12. Aquarium gravel can be purchased at $3.65 for 3 kg. What would be the cost of 9 kg?
13. A certain kind of carpet sells for $16 per square meter. What area can be covered for $312?
14. A certain fabric costs $2.50 per yard. How many yards of fabric can be bought for $17?

B 15. In a collection of dimes and quarters worth $9.00, the ratio of the number of dimes to the number of quarters is 5:7. How many coins are there in the collection?
16. Pam earns $10.50 per hour and Jane earns $11.25 per hour. In one week, the ratio of the number of hours Pam worked to the number of hours Jane worked was 6:5. Together they earned $715.50. How much will Pam earn the following week if she works 4 h more than she did the week before?
17. Show that a profit to selling-price ratio of $\frac{1}{4}$ is equivalent to a profit to cost ratio of $\frac{1}{3}$.
18. Show that a profit to cost ratio of $\frac{1}{4}$ is equivalent to a profit to selling-price ratio of $\frac{1}{5}$.
19. A segment n units long is to be separated into two parts whose lengths have the ratio $a : b$. Find the length of each part in terms of a, b, and n.
20. A polygon whose area is n square units is to be divided into three parts whose areas are in the ratio $h : j : k$. Find the area of each part in terms of h, j, k, and n.

C **21.** The ratio of Jane's speed paddling upstream to her speed paddling downstream is 5:6. She can paddle 12 km upstream and then return to her starting point in a total of 8.8 h. Paddling at the same speed, how long would it take her to travel 5.5 km in still water?

22. George decided to invest a certain amount of money, some at an annual interest rate of 5% and the remainder at an annual interest rate of 8%. The ratio of the amount of money invested at 5% to the amount invested at 8% was 3:5. In one year, he earned $200 more from this investment than if he had invested all the money in a fund earning $6\frac{1}{4}$% per year. How much would his income from the investment have been if he had invested all the money in a fund earning 10% per year?

23. In calcium sulfate the ratio, by mass, of calcium to sulfur is 5:4 and of calcium to oxygen is 5:8. How many grams of each are there in 34 g of calcium sulfate?

24. A man wills $18,000 to his three sons. He specifies that the first and second sons are to receive amounts in the ratio 4:3, and that the first and third sons are to receive amounts in the ratio 2:1. Find the amount each son gets.

25. A metallurgist decides to experiment with a new alloy by using iron and nickel in the ratio 21:5 by mass and nickel and copper in the ratio 4:3 by mass. How many grams of each metal does the metallurgist use to make 238 g of the alloy?

26. A woman offers to give n dollars to three university departments. She specifies that the mathematics and German departments receive sums in the ratio $a:b$, and that the chemistry and German departments receive sums in the ratio $j:k$. How much does the mathematics department receive in terms of n, a, b, j, and k?

Computer Exercises

For students with computer experience

Write a program that will compute the fourth term of a proportion when you input nonzero values for the other three terms. The program should first display the general form

A/B = C/D,

then ask which of the four terms is to be found. The values of the other three terms should then be input in order. RUN the program to compute the fourth term of each of the following proportions.

1. $\frac{a}{9} = \frac{7}{3}$
2. $\frac{2}{3} = \frac{14}{d}$
3. $\frac{9}{-4} = \frac{c}{5}$
4. $\frac{3}{b} = \frac{-2}{9}$
5. $\frac{4}{x} = \frac{18}{12}$
6. $\frac{z}{-15} = \frac{-3}{50}$
7. $\frac{2}{25} = \frac{-1}{y}$
8. $\frac{-3}{8} = \frac{w}{-11}$

Careers

Architecture

Architects design buildings and supervise their construction. An architect is responsible for making sure that the structure meets the client's requirements and is safe, attractive, and compatible in design with other buildings in the area.

There are many steps in the creation of a design for a structure. The architect makes preliminary drawings of the floor plan and the exterior and interior details of the building before meeting with the client to decide on the final design. This final design is used by consulting engineers to prepare detailed working drawings of the plumbing, electrical connections, and climate-control systems.

In addition to planning the design of a building, the architect advises and represents the client in dealing with the building contractor. Periodic visits to the construction site ensure that the design and specifications are being followed.

EXAMPLE An architect is designing a house that will be 15 m long and 8 m wide. A scale drawing of the floor plan is 45 cm long.

a. How wide is the floor plan?

b. If the living room is 18 cm by 13.5 cm on the floor plan, what will be the actual dimensions?

SOLUTION **a.** Let P = the width in centimeters of the floor plan.

Since

$$\frac{\text{true length in m}}{\text{plan length in cm}} = \frac{15}{45},$$

$$\frac{\text{true width in m}}{\text{plan width in cm}} = \frac{8}{P} = \frac{15}{45}.$$

Solving, you find that the width is 24 cm.

b. Let l and w be the actual dimensions in meters of the living room.

$$\frac{l}{18} = \frac{15}{45} \qquad\qquad \frac{w}{13.5} = \frac{15}{45}$$

$$45l = 270 \qquad\qquad 45w = 202.5$$

$$l = 6 \qquad\qquad w = 4.5$$

$\therefore$ the living room will be 6 m by 4.5 m.

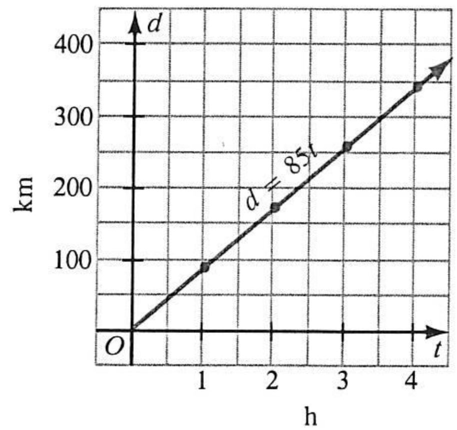
9–8 Direct Variation

The following table shows the distance, d, in kilometers traveled by a car in t h at a speed of 85 km/h. You can see that

$$\frac{d}{t} = 85, \quad \text{or} \quad d = 85t.$$

The graph of the linear equation $d = 85t$ is shown at the right of the table. Note that this equation defines a linear function.

t (h)	d (km)
1	85
2	170
3	255
4	340



Notice from the table that, when the time is doubled, the distance is doubled; when the time is tripled, the distance is tripled; and so on. You say that the distance *varies directly* as the time. This function is an example of a *direct variation.*

Any function defined by an equation of the form

$$y = kx, \quad \text{where } k \text{ is a nonzero constant,}$$

is a **direct variation.** You say that *y varies directly as x*, or that *y is directly proportional to x*. The constant k is called the **constant of variation,** or the **constant of proportionality.**

When the domain is the set of real numbers, the graph of a direct variation defined by the equation $y = kx$ is a line with slope k that passes through the origin.

EXAMPLE 1 If r varies directly as s, and if $r = 2$ when $s = 10$, determine the following.

a. the constant of variation

b. the value of r when $s = 30$

SOLUTION Let $r = ks$, where k is the constant of variation.

a. Since $r = 2$ when $s = 10$,

$$k = \frac{r}{s} = \frac{2}{10} = \frac{1}{5}.$$

$\therefore$ the constant of variation is $\frac{1}{5}$.

b. Since $k = \frac{1}{5}$ and $s = 30$,

$$r = ks = \frac{1}{5} \cdot 30 = 6.$$

$\therefore$ the value of r when $s = 30$ is 6.

Consider any two ordered pairs (x_1, y_1) and (x_2, y_2) in the direct variation specified by $y = kx$. Then

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2.$$

Hence, if $x_1 \neq 0$ and $x_2 \neq 0$, then

$$\frac{y_1}{x_1} = k \quad \text{and} \quad \frac{y_2}{x_2} = k.$$

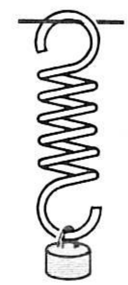
Therefore,

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Notice that this proportion does not involve the constant of variation k.

The next example illustrates two methods for solving a problem involving direct variation. In the first solution, the constant of variation is determined. In the second solution, a proportion is used and the constant of variation is not determined.

EXAMPLE 2 The distance that a spring is stretched is directly proportional to the mass of the object that is stretching the spring. If a mass of 15 g stretches a spring 6 cm, how far will an 8 g mass stretch the spring?



SOLUTION 1

Step 1 The problem asks how far an 8 g object will stretch the spring.

Step 2 Let d = the distance in centimeters that the spring is stretched,
m = the mass in grams of the object stretching the spring,
and k = the constant of variation.

Step 3 $d = km$

Step 4 First find k.

Since $d = 6$ when $m = 15$,

$$k = \frac{d}{m} = \frac{6}{15} = 0.4.$$

Then find d when $m = 8$.

Since $k = 0.4$ and $m = 8$,

$$d = km = 0.4 \times 8 = 3.2.$$

Step 5 Checking the results is left to you.

$\therefore$ the spring is stretched 3.2 cm by an 8 g object.

SOLUTION 2

Step 1 The problem asks how far the 8 g object will stretch the spring.

Step 2 Let d = the distance in centimeters that the spring is stretched, and m = the mass in grams of the object stretching the spring.

Then $d_1 = 6$ cm, $m_1 = 15$ g, and $m_2 = 8$ g.

Step 3 $$\frac{d_1}{m_1} = \frac{d_2}{m_2}$$

Step 4 $$\frac{6}{15} = \frac{d_2}{8}$$

$$48 = 15d_2$$

$$d_2 = \frac{48}{15} = 3.2$$

Step 5 Does $\frac{d_1}{m_1} = \frac{d_2}{m_2}$?

$$\frac{6}{15} \stackrel{?}{=} \frac{3.2}{8}$$

$$6 \times 8 \stackrel{?}{=} 15 \times 3.2$$

$$48 = 48 \quad \checkmark$$

$\therefore$ the spring is stretched 3.2 cm by an 8 g object.

There are many situations in which one quantity *varies directly as the square* of another quantity. For example, the area of a circle varies directly as the square of the radius of the circle. In general, you can say that *y varies directly as the square of x* if

$$y = kx^2, \quad \text{where } k \text{ is a nonzero constant.}$$

You can also say that *y varies directly as*, or *is directly proportional to, the nth power of x* if

$$y = kx^n, \quad \text{where } n > 0, \text{ and } k \text{ is a nonzero constant.}$$

Oral Exercises

State whether the set of ordered pairs (x, y) satisfying the given equation, or specified by the given table, is a direct variation. For each direct variation, state the constant of variation.

1. $y = -8x$ **2.** $y = \frac{3}{4}x$ **3.** $xy = 5$ **4.** $y = \frac{3}{x}$ **5.** $y + 4x = 0$ **6.** $y - 3x = 1$

7.

x	1	2	3	4
y	−5	−10	−15	−20

8.

x	2	4	6	8
y	−1	2	−3	4

9.

x	-2	0	2	4
y	-4	0	4	8

10.

x	2	4	6	8
y	10	8	6	4

In Exercises 11 and 12, solve using the following two methods.

a. Determine the constant of variation. Use it to complete the solution.

b. Use a proportion. Do not determine the constant of variation.

11. If y varies directly as x, and if $y = 24$ when $x = 8$, find y when $x = 50$.
12. If p is directly proportional to t, and if $p = 2$ when $t = 10$, find p when $t = -1$.

Problems

Solve.

A

1. If d varies directly as t, and if $d = 4$ when $t = 9$, find d when $t = 21$.
2. If v is directly proportional to w, and if $v = -6$ when $w = 15$, find w when $v = 3$.
3. If x is directly proportional to y, and if $x = a$ when $y = b$, find x when $y = 3b$.
4. If a varies directly as b, and if $a = 5$ when $b = 6$, find a when $b - 1 = 11$.
5. A fish with a mass of 3 kg causes a fishing pole to bend 9 cm. If the amount of bending varies directly as the mass, how much will the pole bend for a 2 kg fish?
6. The mass of a uniform copper bar varies directly as its length. If a bar 40 cm long has a mass of approximately 420 g, find the mass of a bar 136 cm long.
7. A tropical fern 90 cm tall has fronds that are 3 cm long. A fossil of the same kind of fern was found. If the height of this type of fern varies directly as the length of its fronds, what must the height of the fossilized fern have been if one of its fronds was 4.8 cm long?
8. The 220 km bus trip to the math league competition cost each of the students from Canton $26.40. Using the same bus company, the students from Hampton paid $42 each. If the bus fare is directly proportional to the number of kilometers traveled, how far was the Hampton students' trip?

B

9. If c varies directly as $d - 3$, and if $c = 16$ when $d = 7$, find c when $d = 10$.
10. If r varies directly as $s + 5$, and if $r = 1$ when $s = -2$, find r when $s = -8$.

11. If e varies directly as f^3, and if $e = 32$ when $f = \frac{2}{5}$, write a formula for e in terms of f.

12. If s varies directly as v^4, and if $s = 8$ when $v = -\frac{2}{3}$, write a formula for s in terms of v.

13. The surface area of a cube varies directly as the square of the length of one edge. If the surface area of a cube is 150 cm^2 when the length of an edge is 5 cm, what is the surface area of a cube with an edge of length 8 cm?

14. The surface area of a sphere varies directly as the square of the radius. If the surface area of a sphere is 36π m^2 when the radius is 3 m, what is the surface area of a sphere with a radius of 6 m?

15. A certain mineral's price varies directly as the square of its mass. If a sample with mass 4.2 g is worth $61.74, what will be the value of a sample with mass 100 g?

16. A plastic rod changes length in direct proportion to the change in temperature. At 22°C it is 500 mm long. At 42°C it has increased by 0.05 mm. What is the length if the temperature drops to 18°C?

C 17. Is every linear function a direct variation? Explain.

18. If y varies directly as x, must y increase when x increases? Explain.

19. If r varies directly as t, show that r is doubled when t is doubled.

20. If b varies directly as c, and c varies directly as d, show that b varies directly as d.

21. If u varies directly as v, and u increases by 18 when v is increased by 3, express u in terms of v.

22. If p varies directly as the square of q, and q varies directly as the square of r, what effect will doubling r have on the value of p?

9–9 Inverse Variation

The table at the right shows the time, t, that it takes a car traveling at a speed of r km/h to cover a distance of 90 km. You can see that

$$rt = 90 \qquad \text{or} \qquad t = \frac{90}{r}.$$

r (km/h)	t (h)
45	2
50	1.8
60	1.5
75	1.2
90	1

This relationship is an example of an *inverse variation.*

Any function defined by an equation of the form

$$xy = k, \qquad \text{or} \qquad y = \frac{k}{x},$$

where k is a nonzero constant, is an **inverse variation.** You say that *y is inversely proportional to x*, or that *y varies inversely as x*. As with direct variation, the nonzero constant k is called the *constant of variation*, or the *constant of proportionality*.

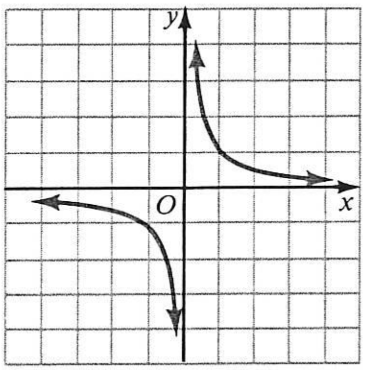
The graph of an inverse variation is not a line, since an equation of the form

$$y = \frac{k}{x}, \qquad \text{or} \qquad xy = k,$$

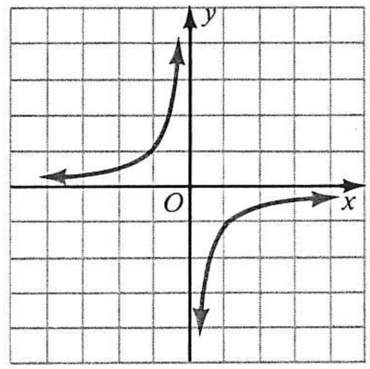
is not linear. In a linear equation no term can be of degree greater than 1; the term xy is of degree 2.

Each of the following diagrams illustrates the graph of an inverse variation

$$y = \frac{k}{x}, \text{ where } k \text{ is a nonzero constant.}$$



$y = \frac{k}{x}, k > 0$



$y = \frac{k}{x}, k < 0$

In the diagram at the left, $k > 0$. In the diagram at the right, $k < 0$. Note that, in each case, the graph does not intersect the x-axis or the y-axis since neither x nor y can have the value 0.

The graph of an inverse variation is an example of a curve called a *hyperbola*. If the constant of variation is positive, the branches of the hyperbola will be in Quadrants I and III. If the constant of variation is negative, the branches of the hyperbola will be in Quadrants II and IV.

EXAMPLE If y varies inversely as x, and if $y = 4$ when $x = 72$, find the following.

a. the constant of variation

b. the value of y when $x = 9$

SOLUTION **a.** Let $xy = k$, where k is the constant of variation.
Since $y = 4$ when $x = 72$,

$$k = xy = 72 \cdot 4 = 288.$$

$\therefore$ the constant of variation is 288.

b. Since $k = 288$ and $x = 9$,

$$y = \frac{k}{x} = \frac{288}{9} = 32.$$

$\therefore$ when $x = 9$, the value of y is 32.

Consider any two ordered pairs of nonzero real numbers (x_1, y_1) and (x_2, y_2) in the inverse variation specified by $y = \frac{k}{x}$, $k \neq 0$. Then

$$y_1 = \frac{k}{x_1} \quad \text{and} \quad y_2 = \frac{k}{x_2}.$$

Since $x_1y_1 = k$ and $x_2y_2 = k$, you have

$$x_1y_1 = x_2y_2.$$

Thus if three of these values are known, you can find the fourth value without first finding the constant of variation. For example, using the information given in the example on page 463, you can let $x_1 = 72$, $y_1 = 4$, $x_2 = 9$, and solve for y_2.

You say that *y varies inversely as the square of x* if

$$y = \frac{k}{x^2}, \quad \text{where } k \text{ is a nonzero constant.}$$

In general, you say that *y varies inversely as*, or *is inversely proportional to, the nth power of x* if

$$y = \frac{k}{x^n}, \quad \text{where } n > 0 \text{ and } k \text{ is a nonzero constant.}$$

Oral Exercises

State the relationship between the given variables as an equation, using k for the constant of variation.

1. The volume V of a gas at a fixed temperature varies inversely as the pressure P.
2. The current I in an electrical circuit of fixed voltage varies inversely as the resistance R.
3. The height h of a cylinder of fixed volume varies inversely as the area A of the base.
4. The frequency f of an electromagnetic wave is inversely proportional to the length l of the wave.

In Exercises 5 and 6, solve using the following two methods.

a. Determine the constant of variation. Use it to complete the solution.

b. Solve without first finding the constant of variation.

5. If y varies inversely as x, and if $y = 9$ when $x = 2$, find y when $x = 3$.
6. If u is inversely proportional to v, and if $u = 12$ when $v = 3$, find u when $v = 9$.

Problems

Solve.

A

1. If a varies inversely as b, and if $a = 4$ when $b = 25$, find a when $b = 5$.
2. If r varies inversely as s, and if $r = 25$ when $s = 10$, find s when $r = 50$.
3. If c is inversely proportional to d, and if $c = 18$ when $d = \frac{2}{3}$, find d when $c = \frac{6}{7}$.
4. If m is inversely proportional to n, and if $m = 0.02$ when $n = 5$, find m when $n = 0.2$.
5. The frequency of a radio wave is inversely proportional to the length of the wave. If a wave of length 500 m has a frequency of 600 kHz (kilohertz), find the frequency of a wave whose length is 750 m.
6. The time needed to travel from one place to another is inversely proportional to the speed. A person traveling 72 km/h can go from New Glasgow to Pictou in 10 h. How fast must the person travel to make the trip in 9 h?
7. The force needed to pry up a rock varies inversely as the length of the crowbar used. Using a crowbar 100 cm long, a force of 150 N (newtons) is needed to pry up a certain rock. How long a crowbar is needed to pry up the same rock when the force applied is 120 N?
8. The number of hours needed to clear the trees from some land is inversely proportional to the number of people who are working. If it would take 4 people 15 h to do the job, how many people would be needed to complete the job in 6 h?
9. A pulley revolves at a speed that is inversely proportional to its diameter. A pulley with a diameter of 12 cm is belted to a pulley with a diameter of 8 cm. If the smaller pulley is revolving at a rate of 96 rpm (revolutions per minute), how fast is the larger pulley revolving?
10. If the rotational speed of a gear wheel is inversely proportional to the number of teeth on the wheel, how fast is a gear wheel with 25 teeth revolving if it is meshed with a gear wheel with 40 teeth that is revolving at 150 rpm?

B

11. If a varies inversely as $b + 2$, and if $a = 8$ when $b = 1.5$, find a when $b = 3$.
12. If x varies inversely as $y - 4$, and if $x = -5$ when $y = \frac{1}{2}$, find x when $y = -1$.
13. If a varies inversely as t^2, and if $a = 2$ when $t = 0.3$, write a formula for a in terms of t.
14. If f varies inversely as m^3, and if $f = 0.5$ when $m = 2$, write a formula for f in terms of m.

15. The interest rate required to yield a given income is inversely proportional to the amount of money invested. Chris receives income from \$16,000 that she has invested at an annual interest rate of 8%. How much money should she invest to receive the same income if the annual interest rate increases to 10%?

16. The height of a cylinder of fixed volume varies inversely as the area of the base. Ken uses a cylindrical vat that has a base area of 6 m^2 and a height of 1.5 m for mixing paint. He wants to replace this vat with a new one having the same volume but with a height of 2 m. What will be the base area of the new vat?

C

17. If y is inversely proportional to x, how does y change when x is doubled?

18. If y is inversely proportional to x, and if x is inversely proportional to t, what is the relationship of y to t? Explain.

19. If a is inversely proportional to b, and if b is inversely proportional to c^2, what effect will doubling c have on a?

20. Assume that a is inversely proportional to b, b is inversely proportional to c, and c is inversely proportional to d^2. What effect will doubling d have on a?

PROGRAMMING IN BASIC

The program on page 247 graphs open sentences in two variables only for integral values of X and Y. In the following graphing program, the replacement set of X is restricted to {−10, −9, −8, . . . , 10}, but the corresponding values of Y may be fractional. In such cases, the TAB function *approximates* the value of Y by assigning to it the value of the greatest integer less than or equal to it. Since Y is associated with TAB, the program plots values of Y horizontally and values of X vertically.

```
10   PRINT "TO GRAPH THE FUNCTION"
20   PRINT "DEFINED IN LINE 70:"
30   PRINT "INPUT K (> 0)";
40   INPUT K
50   PRINT
60   FOR X = -10 TO 10
70   LET Y = K*X
80   IF ABS(Y) > 10 THEN 200
90   IF INT(Y) = 0 THEN 150
100  IF Y < 0 THEN 130
110  PRINT TAB(11);"!";TAB(Y + 11);"*";
120  GOTO 160
130  PRINT TAB(Y + 11);"*";TAB(11);"!";
140  GOTO 160
```

```
150   PRINT TAB(11);"*";
160   IF X = 0 THEN 190
170   PRINT
180   GOTO 200
190   PRINT TAB(22);"----Y"
200   NEXT X
210   PRINT TAB(11);"X"
220   END
```

Exercises

Type in the program as given. Note that the equation in line 70, $y = kx$, represents a *direct* variation. RUN the program for the following values of k.

1. 1 **2.** 2 **3.** 0.5

Change line 70 to represent a variation directly as the *square* of x, that is, $y = kx^2$. RUN the program for the following values of k.

4. 0.25 **5.** 0.5 **6.** 1

To graph that portion of an *inverse* variation that is in the first quadrant, first delete lines 80, 180, and 190, then type in these lines.

```
55   PRINT TAB(11);"!";TAB(K + 12);"----Y"
60   FOR X = 1 TO K
70   LET Y = K/X
160   GOTO 170
```

RUN the revised program for the following values of k.

7. 12 **8.** 15 **9.** 18

9–10 Joint and Combined Variation

If one variable varies directly as the product of two or more other variables, the resulting relationship is called a **joint variation.** For example, the distance traveled by a car starting from rest is directly proportional to the product of its acceleration, a, and the square of the time, t, it has been traveling. This joint variation is defined by the equation

$$d = \frac{1}{2}at^2,$$

where the constant of variation is $\frac{1}{2}$. You can say that the distance *varies jointly as* the acceleration and the square of the time.

If you know that z varies jointly as x and y, you can express the relationship as

$$z = kxy, \text{ where } k \text{ is a nonzero constant.}$$

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two ordered triples that satisfy the equation $z = kxy$, and if x_1, x_2, y_1, and y_2 are nonzero, then

$$\frac{z_1}{x_1 y_1} = k \quad \text{and} \quad \frac{z_2}{x_2 y_2} = k,$$

and so

$$\frac{z_1}{x_1 y_1} = \frac{z_2}{x_2 y_2}.$$

EXAMPLE 1 If r varies jointly as s and the square of t, and if $r = 144$ when $s = 2$ and $t = 3$, find r when $s = 5$ and $t = 4$.

SOLUTION Let $r_1 = 144$, $s_1 = 2$, $t_1 = 3$, $s_2 = 5$, $t_2 = 4$, and use the proportion

$$\frac{r_1}{s_1(t_1)^2} = \frac{r_2}{s_2(t_2)^2}$$

$$\frac{144}{2(3)^2} = \frac{r_2}{5(4)^2}$$

$$\frac{144}{18} = \frac{r_2}{80}$$

$$144(80) = 18r_2$$

$$640 = r_2$$

Notice in Example 1 that the required value of r could also have been found by first determining the constant of variation.

If a variable *varies directly as* one variable (or a power of the variable) and *inversely as* another variable (or a power of the variable), the resulting relationship is called a **combined variation.** For example, if you know that z varies directly as x and inversely as y, you can express the relationship as

$$z = k\left(\frac{x}{y}\right), \text{ where } k \text{ is a nonzero constant.}$$

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two ordered triples that satisfy the equation $z = k\left(\frac{x}{y}\right)$, and if x_1 and x_2 are nonzero, then

$$\frac{z_1 y_1}{x_1} = k \quad \text{and} \quad \frac{z_2 y_2}{x_2} = k,$$

and so

$$\frac{z_1 y_1}{x_1} = \frac{z_2 y_2}{x_2}.$$

EXAMPLE 2 The pressure required to force water through a pipe varies directly as the square of the speed of the water and inversely as the diameter of the pipe. If it requires 80 N/m² (newtons per square meter) pressure to drive water at 40 km/h through a pipe with a 2 cm diameter, what would be the pressure required to drive water at 30 km/h through a pipe with a diameter of 1.5 cm?

SOLUTION

Step 1 The problem asks for the pressure needed to drive water at 30 km/h through a pipe with a 1.5 cm diameter.

Step 2 Let p = the pressure,
s = the speed of the water,
d = the diameter of the pipe,
and k = the constant of variation.

Step 3 $p = k\left(\frac{s^2}{d}\right)$

Step 4 Since $p = 80$ when $s = 40$ and $d = 2$,

$$80 = k\left(\frac{40^2}{2}\right),$$

and

$$k = \frac{80 \cdot 2}{40^2} = \frac{1}{10}.$$

Since $k = \frac{1}{10}$, $s = 30$, and $d = 1.5$,

$$p = k\left(\frac{s^2}{d}\right) = \frac{1}{10}\left(\frac{30^2}{1.5}\right) = 60.$$

Step 5 Checking the results is left to you.
∴ the required pressure is 60 N/m².

Notice in Example 2 that the required pressure could also have been found by using the proportion

$$\frac{p_1 d_1}{(s_1)^2} = \frac{p_2 d_2}{(s_2)^2},$$

where $p_1 = 80$, $d_1 = 2$, $s_1 = 40$, $d_2 = 1.5$, and $s_2 = 30$.

Oral Exercises

Express the relationship in words, assuming that k is the constant of variation.

1. $c = kef$ **2.** $r = ks^2t^2$ **3.** $e = \frac{kf^2}{g}$ **4.** $m = \frac{kp^3}{q^2}$

Express the relationship in words, assuming that k is the constant of variation.

5. $u = kv^2\sqrt{w}$

6. $g = \frac{k}{hi}$

7. $t = \frac{kc^2}{de^3}$

8. $h = \frac{ke_1e_2^{\,2}}{e_3^{\,3}}$

In Exercises 9 and 10, solve using the following two methods.

a. Determine the constant of variation. Use it to complete the solution.

b. Use a proportion. Do not determine the constant of variation.

9. If b varies jointly as g and h, and if $b = 12$ when $g = 10$ and $h = 6$, find b when $g = 15$ and $h = 3$.
10. If n varies directly as s and inversely as r, and if $n = 1$ when $s = 8$ and $r = 12$, find n when $s = 10$ and $r = 15$.

Problems

Solve.

A

1. If d varies jointly as p and q, and if $d = 20$ when $p = 8$ and $q = 15$, find d when $p = 9$ and $q = 16$.
2. If q varies directly as b and inversely as c, and if $q = 0.2$ when $b = 16$ and $c = 8$, find q when $b = 2$ and $c = 5$.
3. If z varies directly as x^2 and inversely as y, and if $z = 8$ when $x = 4$ and $y = 6$, find z when $x = 0.1$ and $y = 0.2$.
4. If Q varies jointly as f^2 and g^3, and if $Q = 24$ when $f = 4$ and $g = 0.5$, find Q when $f = 0.25$ and $g = 10$.
5. If z varies jointly as u and v^2, and if $z = 12$ when $u = 6$ and $v = 4$, find u when $z = 63$ and $v = 6$.
6. If A varies jointly as m^2 and n, and if $A = 15$ when $m = 3$ and $n = 10$, find n when $A = 0.4$ and $m = 0.5$.
7. If T varies directly as s and inversely as v^2, and if $T = 0.1$ when $s = 72$ and $v = 6$, find s when $T = 5$ and $v = 0.2$.
8. If H varies directly as c^2 and inversely as d, and if $H = \frac{5}{3}$ when $c = 2$ and $d = 15$, find d when $H = \frac{1}{8}$ and $c = \frac{1}{2}$.

In Exercises 9–12 assume that Q varies directly as x and inversely as y.

9. If x is doubled and y is doubled, what happens to Q?
10. If x is tripled and y is doubled, what happens to Q?
11. If x is halved and y is doubled, what happens to Q?
12. If x is doubled and y is halved, what happens to Q?

Solve.

B 13. The volume of a cylinder varies jointly as its height and the square of its radius. When the height is 3 cm and the radius is 4 cm, the volume is 48π cm^3. Find the volume in terms of π when the height is 3 cm and the radius is 5 cm.

14. If a wire carries an electric current for a given time, the heat developed varies jointly as the resistance and the square of the current. If a current of 4 A (amperes) produces 225 J (joules) of heat in a wire having a resistance of 15 Ω (ohms), find the heat produced by a current of 3 A in a wire having a resistance of 12 Ω.

15. The number of persons needed to do a job varies directly as the amount of work to be done and inversely as the time in which the job must be done. If 2 people can cut 6 trees into fireplace logs in 8 h, how long will it take 6 people to cut 12 trees into logs?

16. The heat loss through a window pane varies jointly as the difference of the inside and outside temperatures and the window area, and inversely as the thickness of the pane. In 1 h, 189 J (joules) are lost through a pane measuring 45 cm by 70 cm that is 1 cm thick, when the temperature difference is 5°C. How many joules are lost in 1 h through a pane measuring 30 cm by 60 cm that is 0.8 cm thick when the temperature difference is 12°C?

17. The volume of a pyramid varies jointly as its height and the area of its base. A pyramid whose base is a square 5 cm on each side and whose height is 12 cm has a volume of 100 cm^3. Find the volume of a pyramid whose base is a square 6 cm on each side and whose height is 8 cm.

18. The centrifugal force of an object moving in a circle varies jointly with the object's mass and the radius of the circular path, and inversely with the square of the time it takes to complete one full circle. The centrifugal force of a 6 g object moving in a circle with radius 100 cm and completing each revolution in 2 s is 6000 dynes. What is the centrifugal force of an 18 g object moving in a circle with radius 200 cm and completing each revolution in 3 s?

C 19. In the formula $x = \frac{y^2zw}{4}$, z remains constant. If y is doubled and w is tripled, how is x changed?

20. In the formula $F = \frac{m\pi l^2}{p}$, m and π remain constant, l is halved, and p is doubled. How does F change?

21. The heat generated by a stove element varies directly as the square of the voltage and inversely as the resistance. If the voltage is constant, what could be done to double the heat generated?

22. The power in an electric circuit varies jointly as the resistance and the square of the current. If the current is halved, what other change needs to be made if the power is to remain the same?

Self-Test 3

VOCABULARY ratio (p. 449)
proportion (p. 450)
extremes (p. 450)
means (p. 450)
direct variation (p. 458)
constant of variation (p. 458)
constant of proportionality (p. 458)
inverse variation (p. 462)
joint variation (p. 467)
combined variation (p. 468)

1. A rope that is 8.4 m long is to be cut into two pieces in the ratio of 5:7. How long will each piece be? *Obj. 1, p. 449*
2. Property tax varies directly as assessed value. The tax on a certain house assessed at \$60,000 is \$1410. What is the tax on a house in the same community that is assessed at \$100,000? *Obj. 2, p. 449*
3. If c varies inversely as d, and if $c = 4$ when $d = 9$, find c when $d = 6$.
4. The volume of a cone varies jointly as its height and the square of its radius. A cone whose height is 3 cm and whose radius is 7 cm has a volume of 154 cm^3. What is the volume of a cone whose height is 7 cm and whose radius is 6 cm?
5. If r varies directly as s and inversely as t, and if $r = 4$ when $s = 16$ and $t = 2$, find r when $s = 36$ and $t = 9$.

Check your answers with those at the back of the book.

Chapter Summary

1. If an open sentence contains fractions with whole-number denominators, the fractions may be eliminated by multiplying both sides by their LCD.
2. Percent means "per 100." Percents are commonly used in problems involving rates and mixtures. When solving an equation that contains a percent, it is convenient to first express the percent as a fraction or as a decimal.
3. An equation with variables in the denominator of one or more terms is called a *fractional equation*. Fractions can be eliminated in a fractional equation by multiplying both sides by the LCD of the terms. This may introduce *extraneous roots*, and so each root of the new equation must be checked to determine whether it satisfies the original equation.
4. Fractional equations are often used in solving number problems, work problems, and motion problems.

5. The *ratio* 3 to 4 can be written as $\frac{3}{4}$ or $3:4$. In solving problems, if the ratio of two numbers is given as $3:4$, the numbers can be represented as $3x$ and $4x$.

6. A *proportion* is an equation stating that two ratios are equal. In a proportion, the product of the *extremes* equals the product of the *means*. Thus,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \quad \text{then } ad = bc.$$

7. Many relationships can be described in terms of *direct, inverse, joint,* or *combined variation.* The following is a summary of these types of variations. In each case, k is a nonzero constant called the *constant of variation.*

   a. A *direct variation* is a function defined by a linear equation of the form $y = kx$. You say that y varies directly as x, or that y is directly proportional to x. The graph of an equation of the form $y = kx$ is a line.

   b. An *inverse variation* is a function defined by an equation of the form $xy = k$, or $y = \frac{k}{x}$. You say that y varies inversely as x, or that y is inversely proportional to x. The graph of an inverse variation is a hyperbola.

   c. An equation of the form $z = kxy$ defines a *joint variation.* You say that z varies directly as the product xy, or that z varies jointly as x and y.

   d. An equation of the form $z = k\left(\frac{x}{y}\right)$ defines a *combined variation.* You say that z varies directly as x and inversely as y.

8. In solving problems involving variations, it is helpful to be familiar with the two methods illustrated in Example 2 on pages 459 and 460. In the first method, the constant of variation is determined. In the second method, the constant of variation is not determined.

Chapter Review

Write the letter of the correct answer.

1. Solve $\frac{x - 4}{6} - \frac{20 - 2x}{8} = 1.$ 9-1

   a. $\{12\}$ b. $\{10\}$ c. $\{-50\}$ d. $\{8\}$

2. Solve $\frac{3a - 14}{6} \leq \frac{2a}{9} + \frac{a}{4}.$

   a. $\{a: a \leq 84\}$ b. $\{a: a \geq 84\}$ c. $\{a: a \leq -84\}$ d. $\{a: a \geq -84\}$

3. In 1980 the population of a certain town was 20,000. If the population of the same town in 1985 was 25,000, what was the percent increase from the 1980 population? 9-2

a. 75% b. 20% c. 25% d. 125%

4. Sharon invested \$8000, part at an annual interest rate of 6% and the rest at an annual interest rate of 8%. How much did she invest at 8% if her total income for one year from these investments was \$540?

a. \$2000 b. \$3000 c. \$5000 d. \$6000

5. Solve $\frac{4}{3e} + \frac{3}{3e + 1} = -2$. 9-3

a. $\left\{\frac{1}{6}, \frac{4}{3}\right\}$ b. $\left\{6, \frac{3}{4}\right\}$ c. $\left\{-\frac{4}{3}, -\frac{1}{6}\right\}$ d. $\left\{-\frac{1}{2}, -\frac{2}{3}\right\}$

6. Solve $\frac{30}{d^2 - 9} + 2 = \frac{5}{d - 3}$.

a. $\left\{-\frac{1}{2}, 3\right\}$ b. $\left\{-\frac{1}{2}\right\}$ c. $\left\{-\frac{1}{2}, -3\right\}$ d. $\left\{\frac{1}{2}, 3\right\}$

7. One number is 40 less than another number. Two thirds of the lesser number is equal to one fourth of the greater number. Find the greater number. 9-4

a. 24 b. 16 c. 64 d. 42

8. The denominator of a fraction is 8 less than twice the numerator, and the fraction is equal to $\frac{17}{32}$. Find the fraction.

a. $\frac{68}{128}$ b. $\frac{34}{64}$ c. $\frac{51}{96}$ d. $\frac{85}{160}$

9. One pipe can fill a tank in 12 h and another pipe can fill the same tank in 8 h. How long will it take both pipes together to fill the tank? 9-5

a. 4 h b. $4\frac{4}{5}$ h c. 5 h d. $3\frac{1}{2}$ h

10. It takes John twice as long to clear out his garage as it would take Mark to do the same job. If they work together, they can complete the job in $1\frac{1}{3}$ h. How long would it take John to do the job alone?

a. 4 h b. 2 h c. 1 h d. 6 h

11. Sam drove his car during part of his trip at an average rate of 75 km/h. He then took a bus that averaged 90 km/h during the remainder of his trip. If he traveled by bus 55 km longer than he had traveled by car, and his total travel time was $5\frac{1}{2}$ h, how far did he travel by car? 9-6

a. 130 km b. 145 km c. 255 km d. 200 km

12. Jill travels 24 km up a river in the same amount of time it takes her to go 36 km down the river. If the current is flowing at a speed of 3 km/h, what is Jill's speed in still water?

a. 10 km/h **b.** 12 km/h **c.** 15 km/h **d.** 18 km/h

13. Solve $\frac{3w}{w + 2} = \frac{5}{2}$. *9-7*

a. $\{2\}$ **b.** $\{10\}$ **c.** $\{5\}$ **d.** $\{\frac{1}{5}\}$

14. The number of votes received by the winner and loser in a county election was in the ratio 8:5. If the winner received 11,880 votes, how many votes did the loser receive?

a. 8125 **b.** 7425 **c.** 7050 **d.** 6995

15. If w is directly proportional to z, and if $w = 8$ when $z = 24$, find w when $z = 12$. *9-8*

a. -4 **b.** $\frac{1}{3}$ **c.** 36 **d.** 4

16. The distance that a free falling body travels varies directly as the square of the time it has been falling. If an object falls 320 m in 8 s, how far will it fall in 3 s?

a. 5 m **b.** 15 m **c.** 120 m **d.** 45 m

17. If x varies inversely as y, and if $x = 18$ when $y = 52$, find x when $y = 234$. *9-9*

a. 4 **b.** 20 **c.** 14 **d.** 24

18. The base of a rectangle of fixed area varies inversely as its height. The height of a certain rectangle is 12 cm and its base is 42 cm. Find the base of another rectangle of equal area whose height is 14 cm.

a. 15 cm **b.** 30 cm **c.** 36 cm **d.** 12 cm

19. If e varies directly as g and inversely as h, and if $e = 6$ when $g = 18$ and $h = 9$, find g when $e = 12$ and $h = 3$. *9-10*

a. 12 **b.** 3 **c.** 4 **d.** 8

20. If A varies jointly as c and d^3, and if $A = 16$ when $c = 4$ and $d = 2$, find A when $c = 6$ and $d = 3$.

a. 108 **b.** 18 **c.** 81 **d.** 36

Chapter Test

1. Solve $\frac{r + 1}{4} - \frac{3}{2} \leq \frac{2r - 9}{10}$. *9-1*

2. Solve $\frac{2x + 5}{8} - \frac{3x + 1}{7} = \frac{3 - 2x}{4}$.

3. A sports coat was on sale for \$96. If the original selling price was \$120, what was the percent discount? 9-2

4. Carl has some money invested at an annual rate of 6%. He has twice as much money invested at an annual rate of 9%. How much has he invested at each rate if his total income for one year from these two investments is \$168?

5. Solve $\frac{1}{k^2 - k} = \frac{3}{k} - 1$. 9-3

6. Solve $\frac{12}{a^2 - 4} + 1 = \frac{2a - 1}{a - 2}$.

7. Three eighths of a number is 2 less than one half the number. Find the number. 9-4

8. The difference of two numbers is 3, and the sum of their reciprocals is $\frac{1}{2}$. Find the numbers.

9. Working alone, Tim can paint his house in 3 days. It takes his daughter 5 days to paint the house if she works alone. How long will it take them to paint the house if they work together? 9-5

10. Sylvia rows 4 km up a stream in twice the time it takes her to row 12 km down the stream. If the current in the river flows at a speed of 6 km/h, how fast does Sylvia row in still water? 9-6

11. Solve $\frac{9}{n} = \frac{7}{n - 4}$. 9-7

12. Find the ratio of a to b when $\frac{2a - b}{a + 4b} = \frac{1}{2}$.

13. If m varies directly as w^2, and if $m = 1600$ when $w = 2$, find m when $w = \frac{3}{2}$. 9-8

14. If x is inversely proportional to y, and if $x = 60$ when $y = 80$, find y when $x = 240$. 9-9

15. The frequency of a radio wave is inversely proportional to the wave length. If the frequency is 60 Mc/s (megacycles per second) for a wave 5 m long, what is the frequency for a wave 3 m long?

16. If a varies jointly as b and c, and if $a = 6$ when $b = 0.5$ and $c = 0.3$, find a when $b = 0.75$ and $c = 8$. 9-10

17. The height of a square pyramid varies directly with the volume of the pyramid and inversely with the square of the length of one side of its base. A pyramid whose volume is 100 m^3 and whose base is 5 m by 5 m has a height of 12 m. What is the height of a pyramid whose volume is 80 m^3 and whose base is 4 m by 4 m?

Cumulative Review

Chapter 4

Solve each open sentence and graph its solution set.

1. $-6 < 8 - 2x \leq 4$
2. $4 - (3t + 1) < 5t + 2(t - 6)$
3. $5 + 2a < 3a$ or $2 - 4a > 6$
4. $x + 2 > 6x - 3$ and $7 - 2x \leq 5x + 14$
5. $|6p + 12| \leq 6$
6. $15 + |2x - 1| > 20$
7. The measure of an angle is 20° less than three times the measure of its supplement. Find the measure of the angle.
8. The lengths of two sides of a rectangle are consecutive even integers. The perimeter of the rectangle is 196 cm. Find the area.
9. Margaret rode her bike to school one day, but walked home along the same route after school. If her average speed was 7 km/h while riding and 4 km/h while walking, and her total traveling time that day was 1.1 h, find the distance from Margaret's house to the school.
10. An 80 g solution is 16% salt. How much salt must be added to produce a solution that is 36% salt?

Chapter 5

Tell whether each statement is true or false.

11. The graph of the point $(-4, 0)$ lies in the third quadrant.
12. The relation $\{(-4, 4), (-5, 4), (-6, 2)\}$ is a function.
13. The ordered pair $(-2, -2)$ is a solution of $3x - y < -4$.
14. The points $(1, 8)$, $(2, 9)$, and $(3, 10)$ are collinear.

Solve each open sentence if $x \in \{-1, 0, 1\}$ and $y \in \{-2, -1, 2\}$.

15. $y = x$
16. $3x - y = 3$
17. $x + y > 0$
18. $y - 3 \leq 2x$

Given the function $f: x \longrightarrow 2x^2 - 1$, find the following values of f.

19. $f(3)$
20. $f(4)$
21. $f(-2)$
22. $f(-5)$

Graph each of the following on a coordinate plane.

23. $y = 4x + 1$
24. $3x - 2y = 10$
25. $y < \frac{2}{3}x + 2$
26. $5x - 4y \leq 8$

Determine an equation of the line that satisfies the given requirements.

27. has slope $-\frac{3}{2}$ and y-intercept 1
28. has slope 0 and y-intercept 4
29. has slope -2 and passes through the point $(-3, 0)$
30. passes through the points $(5, 3)$ and $(9, 2)$

Chapter 7

Simplify.

31. $3x^4 + 5x^3 - 6 + (-2x^4 - 4x^3 + 3x)$
32. $7rst^2 - 3r^2st - 2(rst^2 - rs^2t)$
33. $(3y^3)(-8y^2)(-y^4)$
34. $(-4e^2f)^2(5ef^2)(-2ef)$
35. $-x^3y^2(x^3 - 4x^2y^2 + 2y^3)$
36. $2a^3(a^2 - 3b) + 4a^2b(a - 2b^2)$
37. $(2x - 3)(x + 1)$
38. $(10x - 2y)(10x + 2y)$
39. $(4c + 3)^2$
40. $(x - 5y)^2$
41. $(2x + 1)^3$
42. $(3s - t)(s^2 - 5st + t^2)$

Factor completely.

43. $16x^2 - 24x + 9$
44. $6m^2n^2 - 8m^3n^2 + 10m^2n$
45. $10n^3 + 10$
46. $6x^2 - 11x - 10$
47. $9x^2 - 25y^2$
48. $x^2y - 4z - 4y + x^2z$
49. $12x^3 + 32x^2 - 12x$
50. $105 + 14x - 7x^2$

Solve.

51. $2x(x - 3)(3x + 1) = 0$
52. $(6x - 1)(4x + 3) = (3x - 4)(8x + 7)$
53. $2a^2 + 5a = -3$
54. $4y^2 = 5 - 8y$
55. Find two consecutive negative odd integers such that the square of the lesser integer is 40 more than the square of the greater integer.

Chapter 8

Simplify using only positive exponents.

56. $\dfrac{(-3x^3y)(3xy^2)^2}{y(6x^3)^2}$
57. $\dfrac{e^{-2}h^4k^{-6}}{e^{-10}h^0k^{-4}}$
58. $\dfrac{z^2 - 36}{2z - 12}$
59. $\dfrac{25m^5 + 35m^3 - 45m}{-5m}$
60. $\dfrac{5a^3 + 3a^2b^2 - 2ab}{ab^2}$
61. $\dfrac{12m^2 + 5m - 3}{6m^2 - 17m + 5}$
62. $\dfrac{t^2 - 3t - 4}{3t^2 + 10t - 8} \cdot \dfrac{6t^2 - t - 2}{2t^2 + 3t + 1}$
63. $\dfrac{7c - 2}{5c^2 + 2c - 3} - \dfrac{4c - 5}{5c^2 + 2c - 3}$
64. $\dfrac{a^2 + 3ab}{a^2 + 3ab + 2b^2} + \dfrac{a}{a + 2b}$
65. $\dfrac{16x^2 - 8xy + y^2}{6x - 12y} \div \dfrac{16x^2 - y^2}{x^2 - xy - 2y^2}$
66. $\dfrac{1 + a^{-1}}{1 - a^{-2}}$
67. $\dfrac{\frac{x}{2y} - \frac{y}{2x}}{\frac{2}{y} + \frac{2}{x}}$

Divide the first polynomial by the second.

68. $x^2 + 6x - 7;\ x - 1$
69. $2a^2 + 11a - 18;\ 2a - 3$

APPLICATION

Gravity

No matter how high a person throws a javelin, it always falls back to the ground. As the javelin falls, its velocity keeps increasing; that is, the javelin accelerates. A force is needed to cause an object to accelerate. The acceleration depends on the object's mass and the force acting on it. The force that pulls a javelin back to Earth is the **force of gravity,** or **gravitational force.** Gravitational force exists between any two objects in the universe. The equation for this force, F, is

$$F = G\frac{m_1 m_2}{d^2},$$

where G is the gravitational constant. The masses of the two objects are m_1 and m_2. The distance between their centers of mass is d. When F is in newtons, m_1 and m_2 are in kilograms, and d is in meters, G is equal to 0.667×10^{-10} m^3/(kg · s^2). (Newtons are expressed as kg · m/s^2).

Suppose that m_1 is the mass of an object on Earth's surface. The distance d between its center of mass and Earth's is Earth's radius. Since Earth's mass and radius are constant, $\frac{F}{m_1}$ is also constant.

$$\frac{F}{m_1} = G\frac{m_2}{d^2}$$

The ratio $\frac{F}{m_1}$ is the *acceleration due to gravity.* This acceleration, represented by the symbol g, does not depend on m_1. If air resistance is small, all objects fall to Earth at the same rate.

Exercises

What gravitational force, in newtons, acts between the two objects in each pair? (Earth: mass = 5.98×10^{24} kg; radius = 6.38×10^6 m. Moon: mass = 7.34×10^{22} kg; radius = 1.74×10^6 m.)

1. Earth and a 1-kg object at its surface
2. the moon and a 1-kg object at its surface
3. two 1-kg spheres, each with radius 0.333 m and center of mass at the center of the sphere, when the spheres are just touching
4. From your answer to Exercise 1, find g.