Chapter 8

Polynomials and Rational Expressions

Division of Polynomials

OBJECTIVES for Sections 8-1 through 8-3:
1. To divide monomials.
2. To divide a polynomial by a monomial.
3. To divide polynomials.

8–1 Dividing Monomials

Numerical computations often reveal number properties that are useful in algebraic work. For example, notice that

$$\frac{12 \cdot 10}{3 \cdot 2} = \frac{120}{6} = 20 \qquad \text{and} \qquad \frac{12}{3} \cdot \frac{10}{2} = 4 \cdot 5 = 20.$$

Thus, by the transitive axiom of equality,

$$\frac{12 \cdot 10}{3 \cdot 2} = \frac{12}{3} \cdot \frac{10}{2}.$$

This result suggests the following *basic property of quotients*.

Basic Property of Quotients

For all real numbers r and s and all nonzero real numbers t and u,

$$\frac{rs}{tu} = \frac{r}{t} \cdot \frac{s}{u}.$$

The basic property of quotients is a theorem that can be proved as follows.

PROOF

Statements	Reasons
1. $\frac{rs}{tu} = (rs) \cdot \left(\frac{1}{tu}\right)$	Definition of division
2. $= (rs) \cdot \left(\frac{1}{t} \cdot \frac{1}{u}\right)$	Property of the reciprocal of a product
3. $= \left(r \cdot \frac{1}{t}\right) \cdot \left(s \cdot \frac{1}{u}\right)$	Commutative and associative axioms for multiplication
4. $= \frac{r}{t} \cdot \frac{s}{u}$	Definition of division
5. $\therefore \frac{rs}{tu} = \frac{r}{t} \cdot \frac{s}{u}$	Transitive axiom of equality

The following useful results are obtained by substituting 1 for t or 1 for r in the basic property of quotients just proved.

> **1.** For all real numbers r and s and all nonzero real numbers u,
>
> $$\frac{rs}{u} = r \cdot \frac{s}{u}.$$
>
> **2.** For all real numbers s and all nonzero real numbers t and u,
>
> $$\frac{s}{tu} = \frac{1}{t} \cdot \frac{s}{u}.$$

The basic property of quotients and the results just discussed can be used along with the laws of exponents for multiplication to simplify a quotient of monomials.

EXAMPLE 1 Simplify. Assume that $a \neq 0$.

a. $\frac{a^3}{a^3}$ **b.** $\frac{a^7}{a^4}$ **c.** $\frac{a^2}{a^6}$

SOLUTION **a.** $\frac{a^3}{a^3} = a^3 \cdot \frac{1}{a^3} = 1$

b. $\frac{a^7}{a^4} = \frac{a^4 \cdot a^3}{a^4} = \frac{a^4}{a^4} \cdot a^3 = 1 \cdot a^3 = a^3$

c. $\frac{a^2}{a^6} = \frac{a^2}{a^4 \cdot a^2} = \frac{1}{a^4} \cdot \frac{a^2}{a^2} = \frac{1}{a^4} \cdot 1 = \frac{1}{a^4}$

The preceding example suggests a pattern for simplifying any quotient of powers of the form $\frac{a^m}{a^n}$, where a is a nonzero real number and m and n are positive integers.

$$\text{If } m = n, \text{ then } \frac{a^m}{a^n} = \frac{a^n}{a^n} = a^n \cdot \frac{1}{a^n} = 1.$$

$$\text{If } m > n, \text{ then } \frac{a^m}{a^n} = \frac{a^{m-n} \cdot a^n}{a^n} = a^{m-n} \cdot \frac{a^n}{a^n} = a^{m-n} \cdot 1 = a^{m-n}.$$

$$\text{If } m < n, \text{ then } \frac{a^m}{a^n} = \frac{a^m}{a^{n-m} \cdot a^m} = \frac{1}{a^{n-m}} \cdot \frac{a^m}{a^m} = \frac{1}{a^{n-m}} \cdot 1 = \frac{1}{a^{n-m}}.$$

These results can be summarized as the following *laws of exponents* for division.

Laws of Exponents for Division

For all nonzero real numbers a and all positive integers m and n:

1. If $m = n$, then $\frac{a^m}{a^n} = 1$.
2. If $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$.
3. If $m < n$, then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.

EXAMPLE 2 Simplify. Assume that no denominator equals zero.

a. $\frac{12x^4}{-3x^6}$ b. $\frac{27x^3y^5}{18x^4y^2}$ c. $\frac{-5xy^5}{-x^3y^5z}$

SOLUTION

a. $\frac{12x^4}{-3x^6} = \frac{12}{-3} \cdot \frac{x^4}{x^6} = -4 \cdot \frac{1}{x^2} = -\frac{4}{x^2}$

b. $\frac{27x^3y^5}{18x^4y^2} = \frac{27}{18} \cdot \frac{x^3}{x^4} \cdot \frac{y^5}{y^2} = \frac{3}{2} \cdot \frac{1}{x} \cdot y^3 = \frac{3y^3}{2x}$

c. $\frac{-5xy^5}{-x^3y^5z} = \frac{-5}{-1} \cdot \frac{x}{x^3} \cdot \frac{y^5}{y^5} \cdot \frac{1}{z} = 5 \cdot \frac{1}{x^2} \cdot 1 \cdot \frac{1}{z} = \frac{5}{x^2z}$

CONDENSED SOLUTION

a. $\frac{12x^4}{-3x^6} = -\frac{4}{x^2}$ b. $\frac{27x^3y^5}{18x^4y^2} = \frac{3y^3}{2x}$ c. $\frac{-5xy^5}{-x^3y^5z} = \frac{5}{x^2z}$

In simplifying a quotient, it is sometimes more convenient to first use the laws of exponents for multiplication to simplify either the numerator or the denominator, or both.

EXAMPLE 3 Simplify $\frac{(3x^2y)(2xy^5)^2}{(-4x^3y)(3y^6)}$, $x \neq 0$, $y \neq 0$.

SOLUTION $\frac{(3x^2y)(2xy^5)^2}{(-4x^3y)(3y^6)} = \frac{(3x^2y)(4x^2y^{10})}{(-4x^3y)(3y^6)} = \frac{12x^4y^{11}}{-12x^3y^7} = -xy^4$

Oral Exercises

Replace each _?_ with the factor that will make a true statement. Assume that no denominator equals zero.

1. $\frac{a^7}{a^2} = \frac{? \cdot a^2}{a^2}$
2. $\frac{b^6}{b^{10}} = \frac{b^6}{b^6 \cdot ?}$
3. $\frac{x^9}{x^8} = \frac{? \cdot x^8}{x^8}$
4. $\frac{t}{t^{12}} = \frac{t}{t \cdot ?}$
5. $\frac{10^5}{10^2} = \frac{10^2 \cdot ?}{10^2}$
6. $\frac{2^8}{2^{10}} = \frac{2^8}{2^8 \cdot ?}$

Simplify. Assume that $a \neq 0$.

7. $\frac{a^5}{a^2}$
8. $\frac{a^2}{a^5}$
9. $\frac{a^5}{a^5}$
10. $\frac{a^5}{a^4}$
11. $\frac{5a^2}{-a^3}$
12. $\frac{2a^2}{6a^2}$

Written Exercises

Simplify. Assume that no denominator equals zero.

A

1. $\frac{x^7}{x^3}$
2. $\frac{y^5}{y^8}$
3. $\frac{b}{b^5}$
4. $\frac{a^6}{a}$
5. $\frac{10c^3}{5c^2}$
6. $\frac{3y^4}{4y^5}$
7. $\frac{2n}{-22n^3}$
8. $\frac{-6r^2}{24r^8}$
9. $\frac{16e^5f^2}{2ef}$
10. $\frac{-8a^4b}{16ab^4}$
11. $\frac{5v^2w^5}{15v^3w^5}$
12. $\frac{20c^3d}{5c^2d}$
13. $\frac{3a^2b^3c}{-2a^2c^2}$
14. $\frac{-24x^2y^{10}}{-12x^2z^{10}}$
15. $\frac{2^6 \cdot p^6}{2^5 \cdot p^5}$
16. $\frac{10^3 \cdot t^2}{10 \cdot t^2}$
17. $\frac{0.8a^3b^2c}{0.2ab^4c^3}$
18. $\frac{5x^7y^4z^3}{0.2y^3z}$
19. $\frac{10m^3n^2}{0.5mn^4}$
20. $\frac{0.24r^5s^2t^2}{0.8r^3s^5t}$

B

21. $\frac{(xy)^3}{x^2y}$
22. $\frac{(a^3)^2}{(a^2)^3}$
23. $\frac{(-a^4)^2}{-a^4}$
24. $\frac{(-2x)^3}{-2x^3}$
25. $\frac{(t^6)^2}{t^6 \cdot t^2}$
26. $\frac{(3s^2)^3 \cdot (-2s)^2}{-12(-s)^2}$
27. $\frac{(6cd^2)(2c^2d)}{(-3c^2d^3)(4cd)}$
28. $\frac{(3s^2t)(-2st^2)^2}{(-4st)^3(-2st)}$
29. $\frac{(-3a^2)(-3a)^2}{-(3a^2)^2(3a^3)^2}$

30. $\dfrac{(-5e)^2(5e)^2}{(10e)^2(-2e)^2}$ 31. $\dfrac{s^2(3t-1)^5}{s^3(3t-1)^3}$ 32. $\dfrac{v^5(w-3)^5}{(-v)^7(w-3)^7}$

In Exercises 33–46, assume that variable expressions appearing as exponents denote positive integers and that no denominator equals zero.

Simplify. Express the answer so that all exponents are positive integers.

33. $\dfrac{d^{2n}}{d^n}$ 34. $\dfrac{w^{2m}}{w^{3m}}$ 35. $\dfrac{2p}{6p^{3n}}$ 36. $\dfrac{6e^{5m}}{3e}$

37. $\dfrac{f^{n-1}}{f^{n-3}}$ 38. $\dfrac{2g^{2n-1}}{6g^{2n-5}}$ 39. $\dfrac{u^{2m+1}}{u^{m-2}}$ 40. $\dfrac{k^{3m-1}}{k^{4m+1}}$

Given that all exponents are positive integers, specify the set of all possible values for *m*.

C 41. a. $\dfrac{q^{m-3}}{q^2} = q^{m-5}$ b. $\dfrac{q^{m-3}}{q^2} = \dfrac{1}{q^{5-m}}$ c. $\dfrac{q^{m-3}}{q^2} = 1$

42. a. $\dfrac{t^5}{t^{8-m}} = t^{m-3}$ b. $\dfrac{t^5}{t^{8-m}} = \dfrac{1}{t^{3-m}}$ c. $\dfrac{t^5}{t^{8-m}} = 1$

Simplify. Express the answer so that all exponents are positive integers.

43. $\dfrac{x^{4-m}}{x^3}$ 44. $\dfrac{w^6}{w^{n-2}}$ 45. $\dfrac{a^{2-t}}{a^{t+3}}$ 46. $\dfrac{b^{v+4}}{b^{8-v}}$

8–2 Dividing a Polynomial by a Monomial

In determining a method for dividing a polynomial by a monomial, it is again helpful to look at a numerical computation. For example,

$$\frac{56+14}{7} = \frac{70}{7} = 10 \quad \text{and} \quad \frac{56}{7} + \frac{14}{7} = 8 + 2 = 10.$$

Thus, by the transitive axiom of equality,

$$\frac{56+14}{7} = \frac{56}{7} + \frac{14}{7}.$$

This result illustrates the following theorem, discussed in Section 2-11.

> ***Theorem.*** For all real numbers a, b, and c such that $c \neq 0$,
>
> $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad \text{and} \quad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}.$$

EXAMPLE Express each quotient as a sum. Assume that no denominator equals zero.

a. $\frac{8w^3 - 6w^2 + 2w}{2w}$ b. $\frac{ax^2 - 3ax + a^2}{ax}$

SOLUTION a. $\frac{8w^3 - 6w^2 + 2w}{2w} = \frac{8w^3}{2w} - \frac{6w^2}{2w} + \frac{2w}{2w} = 4w^2 - 3w + 1$

b. $\frac{ax^2 - 3ax + a^2}{ax} = \frac{ax^2}{ax} - \frac{3ax}{ax} + \frac{a^2}{ax} = x - 3 + \frac{a}{x}$

The Example above illustrates the following rule.

> To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, then add all the quotients.

One polynomial is said to be **divisible** by another polynomial if the quotient is also a polynomial. The preceding Example shows that the polynomial $8w^3 - 6w^2 + 2w$ is divisible by $2w$ since the quotient, $4w^2 - 3w + 1$, is a polynomial. The polynomial $ax^2 - 3ax + a^2$ is *not* divisible by ax since the quotient, $x - 3 + \frac{a}{x}$, is not a polynomial.

Oral Exercises

Replace each ___?___ with a monomial to make a true statement. Assume that no denominator equals zero.

1. $\frac{20 + 36}{4} = \frac{?}{4} + \frac{?}{4}$
2. $\frac{18 - 12 + 30}{6} = \frac{?}{6} - \frac{?}{6} + \frac{?}{6}$
3. $\frac{5m^2 + 2n}{3n} = \frac{?}{3n} + \frac{?}{3n}$
4. $\frac{7b^3 - 9c^2}{2b} = \frac{?}{2b} - \frac{?}{2b}$
5. $\frac{a^3 + b^2 + c}{5a} = \frac{?}{5a} + \frac{?}{5a} + \frac{?}{5a}$
6. $\frac{3x^2y - 4xy - 5}{8xy} = \frac{?}{8xy} - \frac{?}{8xy} - \frac{?}{8xy}$

Written Exercises

Express each quotient as a sum. Assume that no denominator equals zero.

A

1. $\frac{6m + 9n}{3}$
2. $\frac{8b - 12c}{2}$
3. $\frac{15e^2 - 5e}{5e}$
4. $\frac{8z^3 + 12z}{4z}$
5. $\frac{7h + 3}{2h^3}$
6. $\frac{5a^2 - 6}{3a^5}$
7. $\frac{2a^2 - 3b^3}{4a^3b^2}$
8. $\frac{5x^2 + 6y^2}{6xy^2}$
9. $\frac{3t^2 - 6t^5}{-12t^4}$
10. $\frac{8w^5 + 3w^2}{-6w^8}$
11. $\frac{9v^4 + 27v^8}{-9v^4}$
12. $\frac{3s^3 - 7s^5}{-5s^4}$

13. $\frac{6r^3 + 8r^2 - 14r}{2r}$

14. $\frac{12e^4 - 6e^2 + 4e}{-2e}$

15. $\frac{12z^4 - 6z^3 + 16z^2}{-2z^2}$

16. $\frac{24f^5 + 16f^3 - 12f}{4f}$

17. $\frac{15u^2v^3 + 6uv^2 - 12u^3v}{9uv}$

18. $\frac{12a^2b^2 + 16ab - 32b}{-16ab}$

19. $\frac{20e^4 - 10e^2t^2 + 8t^4}{-4e^2t^2}$

20. $\frac{6a^4b^4 - 2b^2c^2 + 3ac}{12a^2b^2c^2}$

For the functions P and Q defined in Exercises 21–24, express $\frac{P(x)}{Q(x)}$ as a sum. Assume that $Q(x) \neq 0$.

B 21. $P(x) = 12x^5 - 9x^3 + 15x; \quad Q(x) = 3x$

22. $P(x) = 40x^4 + 15x^3 - 20x^2 + 10; \quad Q(x) = -5x$

23. $P(x) = 32x^6 - 8x^4 + 24x^2 - 1; \quad Q(x) = 8x^2$

24. $P(x) = 36x^8 - 72x^6 - 12x^4 + 48x^2; \quad Q(x) = -12x^4$

Simplify. Assume that variable expressions appearing as exponents denote positive integers and that no denominator equals zero.

25. $\frac{b^{3n} - b^{2n} + b^{n+1}}{b^n}$

26. $\frac{6k^{4n-1} - 3k^{3n+1} + 9k^{2n} - 12k^{n+1}}{3k^{n-1}}$

27. $\frac{a^{3n}b^{n+1} + 2a^{2n}b^2 - 3ab^n}{-a^nb}$

28. $\frac{t^{5n+2}v^{2n} - (t^2v)^n - t(v)^{2n+1}}{(tv)^{2n}}$

C 29. Evaluate $\frac{x}{y}$ when $\frac{x + y}{y} = 35$.

30. Evaluate $\frac{x}{y}$ when $\frac{x - y}{y} = 29$.

31. Find the positive value of $\frac{a}{x}$ that is a solution of $\frac{a^2 - x^2}{x^2} = -0.64$.

32. Solve $\frac{5r^2 + 20r}{5r} = r + 4$.

33. a. Are $\frac{2x^2 + 12x}{2x} = 8$ and $x + 6 = 8$ equivalent open sentences? Explain.

b. Are $\frac{2x^2 + 12x}{2x} = 6$ and $x + 6 = 6$ equivalent open sentences? Explain.

34. a. Are $\frac{6m^2 - 20m}{2m} < 2$ and $3m - 10 < 2$ equivalent open sentences? Explain.

b. Are $\frac{6m^2 - 20m}{2m} > 2$ and $3m - 10 > 2$ equivalent open sentences? Explain.

8–3 Dividing Polynomials: Rational Expressions

In order to divide one polynomial by another, you can use a division algorithm very similar to the one used for integers. For example, first consider the following division of integers.

Step 1

$$\begin{array}{r} 1 \\ 28\overline{)\,354} \\ \underline{28} \\ 74 \end{array}$$

Step 2

$$\begin{array}{r} 12 \leftarrow \text{(partial quotient)} \\ 28\overline{)\,354} \\ \underline{28} \\ 74 \\ \underline{56} \\ 18 \end{array}$$

Check:

$$354 \stackrel{?}{=} 12 \cdot 28 + 18$$
$$354 \stackrel{?}{=} 336 + 18$$
$$354 = 354 \quad \checkmark$$

$$\therefore \frac{354}{28} = 12\frac{18}{28}, \text{ or } 12\frac{9}{14}$$

Now consider the following division of polynomials.

Step 1

$$\begin{array}{rll} & 3x & \\ 2x - 3\,\overline{)\,6x^2 - 11x + 8} & & \\ \underline{6x^2 - 9x} & \longleftarrow & \{\text{Multiply } 2x - 3 \text{ by } 3x \\ -2x + 8 & \longleftarrow & \{\text{Subtract} \end{array}$$

Step 2

$$\begin{array}{rll} 3x - 1 & \longleftarrow & \text{(partial quotient)} \\ 2x - 3\,\overline{)\,6x^2 - 11x + 8} & & \\ \underline{6x^2 - 9x} & & \\ -2x + 8 & & \\ \underline{-2x + 3} & \longleftarrow & \{\text{Multiply } 2x - 3 \text{ by } -1 \\ 5 & \longleftarrow & \{\text{Subtract} \end{array}$$

Check:

$$6x^2 - 11x + 8 \stackrel{?}{=} (3x - 1)(2x - 3) + 5$$
$$6x^2 - 11x + 8 \stackrel{?}{=} (6x^2 - 11x + 3) + 5$$
$$6x^2 - 11x + 8 = 6x^2 - 11x + 8 \quad \checkmark$$

$$\therefore \frac{6x^2 - 11x + 8}{2x - 3} = 3x - 1 + \frac{5}{2x - 3}$$

Notice that, in both divisions just shown, the answer was expressed in the form

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Unless the remainder is zero, the quotient in this equation is the *partial quotient* indicated in Step 2. By transforming this equation, you obtain the relationship that was used in checking each of the divisions:

$$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}$$

A **rational expression** is any expression that can be written as the quotient of two polynomials, provided the denominator is not zero. Using division, a rational expression can also be written as the sum of a polynomial and a rational expression. When dividing one polynomial by another, it is important that the terms in both dividend and divisor are arranged in order of descending degree in one variable.

EXAMPLE 1 Divide $4y^2 + 15y^4 + 9y^3 + 1$ by $3y^2 - 1$.

SOLUTION Arrange the terms of the dividend in order of descending degree. The dividend has no first-degree term. This "missing" term can be inserted as shown by using zero as its coefficient.

$$\begin{array}{r|l} & 5y^2 + 3y + 3 \\ \hline 3y^2 - 1 & 15y^4 + 9y^3 + 4y^2 + 0y + 1 \\ & \underline{15y^4 \qquad\quad - 5y^2} \\ & \qquad 9y^3 + 9y^2 + 0y + 1 \\ & \qquad \underline{9y^3 \qquad\quad - 3y} \\ & \qquad\qquad 9y^2 + 3y + 1 \\ & \qquad\qquad \underline{9y^2 \qquad\quad - 3} \\ & \qquad\qquad\qquad 3y + 4 \end{array}$$

Check:
$$\begin{aligned} 15y^4 + 9y^3 + 4y^2 + 1 &\stackrel{?}{=} (3y^2 - 1)(5y^2 + 3y + 3) + (3y + 4) \\ &\stackrel{?}{=} 15y^4 + 9y^3 + 4y^2 - 3y - 3 + 3y + 4 \\ &= 15y^4 + 9y^3 + 4y^2 + 1 \quad \checkmark \end{aligned}$$

$\therefore$ the quotient is $5y^2 + 3y + 3 + \dfrac{3y + 4}{3y^2 - 1}$, $3y^2 - 1 \neq 0$.

The division process ends when the remainder is either zero or a polynomial of lesser degree than that of the divisor.

You can use division to determine whether one polynomial is a factor of another polynomial.

EXAMPLE 2 **a.** Is $x^2 - x + 1$ a factor of $4x^4 - 4x^3 + 3x^2 + x - 1$?
b. Factor $4x^4 - 4x^3 + 3x^2 + x - 1$ completely.

SOLUTION **a.**

$$\begin{array}{r|l} & 4x^2 - 1 \\ \hline x^2 - x + 1 & 4x^4 - 4x^3 + 3x^2 + x - 1 \\ & \underline{4x^4 - 4x^3 + 4x^2} \\ & \qquad\qquad -x^2 + x - 1 \\ & \qquad\qquad \underline{-x^2 + x - 1} \\ & \qquad\qquad\qquad\qquad 0 \end{array}$$

Since the remainder is 0,

$$4x^4 - 4x^3 + 3x^2 + x - 1 = (4x^2 - 1)(x^2 - x + 1).$$

$\therefore x^2 - x + 1$ is a factor of $4x^4 - 4x^3 + 3x^2 + x - 1$.

b.
$$\begin{aligned} 4x^4 - 4x^3 + 3x^2 + x - 1 &= (4x^2 - 1)(x^2 - x + 1) \\ &= (2x + 1)(2x - 1)(x^2 - x + 1) \end{aligned}$$

Oral Exercises

In each exercise, tell how you would write the dividend before using the division algorithm.

1. $(9 + 6b + b^2) \div (b + 2)$
2. $(d^2 + 20 - 9d) \div (d - 5)$
3. $(y^2 - 9) \div (y + 3)$
4. $(z^3 - 8) \div (z - 2)$
5. $(2f^2 + 3f^3) \div (3f - 1)$
6. $(w^4 - 16) \div (w^2 + 4)$

Written Exercises

Divide the first polynomial by the second. Assume that no divisor equals zero.

A

1. $m^2 - m - 12; \quad m - 4$
2. $a^2 + 9a + 20; \quad a + 5$
3. $18s^2 + 27s + 10; \quad 6s + 5$
4. $15b^2 + 4b - 3; \quad 3b - 1$
5. $35t^2 - 51t + 16; \quad 7t - 6$
6. $4u^2 + 28u + 14; \quad 2u + 7$
7. $3x^2 - 2x^3 + x - 1; \quad x + 1$
8. $3y + 3y^3 - 2 - 2y^2; \quad 3y - 2$
9. $d^3 + 125; \quad d - 2$
10. $p^4 + 2p + 1; \quad p + 3$
11. $18x^4 + 9x^3 + 3x^2; \quad 3x^2 + 1$
12. $2b^5 - 8b^4 + 2b^3 + b^2; \quad 2b^3 + 1$
13. $t^6 + 1; \quad t^2 - 1$
14. $2w^5 - 3; \quad w^2 + 2$
15. $r^4 + 2r^2 - 1; \quad r^2 + 2r - 3$
16. $h^5 - 1; \quad h^2 - h + 2$

Determine whether or not the first polynomial is a factor of the second.

B

17. $3x + 2; \quad 6x^3 - 11x^2 - 7x + 2$
18. $a^2 + 3a + 1; \quad a^3 + 2a^2 - 2a + 4$
19. $v^2 - 1; \quad v^6 - 2v^4 + v^2 - 2$
20. $b^2 + 1; \quad 3b^7 + 3b^5 - b^2 - 1$

In Exercises 21–24, the first polynomial is a factor of the second polynomial. Factor the second polynomial completely.

21. $k - 3; \quad k^3 - k^2 - 21k + 45$
22. $2r + 5; \quad 4r^3 + 28r^2 + 9r - 90$
23. $w^2 - 1; \quad w^5 - w^3 + 8w^2 - 8$
24. $p^4 - 16; \quad p^6 - 5p^5 + 4p^4 - 16p^2 + 80p - 64$

C

25. Determine p so that $4q + 3$ is a factor of $20q^3 + 23q^2 - 10q + p$.
26. Determine p so that $(6b^2 + pb + 36) \div (2b + 7) = 3b + 5 + \frac{1}{2b + 7}$.
27. a. Divide $a^{100} - 1$ by $a - 1$.
    b. Divide $a^n - 1$ by $a - 1$, where n is a positive integer.
28. a. Divide $a^{100} + 1$ by $a + 1$.
    b. Divide $a^{101} + 1$ by $a + 1$.
    c. Divide $a^n + 1$ by $a + 1$, where n is a positive integer.
29. Is the quotient of two polynomials always a polynomial? Explain.
30. Is the quotient of two polynomials, if the quotient exists, always a rational expression? Explain.

Self-Test 1

VOCABULARY basic property of quotients (p. 381)
laws of exponents for division (p. 383)
divisible (p. 386)
rational expression (p. 390)

Simplify. Assume that no denominator equals zero.

1. $\frac{28m^3t}{4mt}$ 2. $\frac{-50r^3s^2}{-5r^3s^3}$ *Obj. 1, p. 381*

Express each quotient as a sum. Assume that no denominator equals zero.

3. $\frac{12b^3 - 16b^2 + 4b}{4b}$ 4. $\frac{10e^4f^3 - 15e^3f^2 + 20f}{5ef^2}$ *Obj. 2, p. 381*

Divide as indicated. Assume that no divisor equals zero.

5. $(c^2 + 2c - 35) \div (c - 5)$ 6. $2r + 3\overline{)6r^2 - r - 10}$ *Obj. 3, p. 381*

Check your answers with those at the back of the book.

Simplifying Rational Expressions

OBJECTIVES for Sections 8-4 and 8-5:
1. To simplify fractions.
2. To simplify expressions that contain zero or negative integral exponents.
3. To use scientific notation.

8–4 Simplifying Rational Expressions

Using the basic property of quotients given in Section 8-1, you can prove the following theorem, which is useful in simplifying rational expressions. (See Exercises 35 and 36 on page 396.)

Theorem. For all real numbers r and all nonzero real numbers s and t,

$$\frac{r}{s} = \frac{r \cdot t}{s \cdot t} \quad \text{and} \quad \frac{r}{s} = \frac{r \div t}{s \div t}.$$

The preceding theorem states that dividing or multiplying the numerator and denominator of a fraction by the same nonzero number produces a fraction equivalent to the given fraction. For example,

$$\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20} \quad \text{and} \quad \frac{24}{30} = \frac{24 \div 6}{30 \div 6} = \frac{4}{5}.$$

Any fraction whose numerator and denominator are integers or polynomials with integral coefficients is in **simplest form** if the greatest common factor (GCF) of the numerator and denominator is 1. Thus, to simplify a fraction you divide the numerator and denominator by their GCF.

EXAMPLE 1 Simplify $\frac{18x^4}{24x}$.

SOLUTION Divide the numerator and denominator by their GCF.

$$\frac{18x^4}{24x} = \frac{18x^4 \div 6x}{24x \div 6x} = \frac{3x^3}{4}, \; x \neq 0$$

In finding the simplest form of some rational expressions, you may first need to factor either the numerator or denominator, or both.

EXAMPLE 2 Simplify $\frac{5b - 15}{2b^2 - 18}$.

SOLUTION Factor the numerator and the denominator. Then divide the numerator and denominator by their GCF.

$$\frac{5b - 15}{2b^2 - 18} = \frac{5(b - 3)}{2(b + 3)(b - 3)} = \frac{5}{2(b + 3)}, \; b \neq 3, -3$$

Another way to write the answer to Example 2 is $\frac{5}{2b + 6}$. However, *unless you are otherwise directed, you may leave an answer in its factored form,* as was done in Example 2.

Sometimes factors of the numerator and denominator are opposites of one another, as in the next example.

EXAMPLE 3 Simplify $\frac{6 - 3m}{m^2 + m - 6}$.

SOLUTION

$$\begin{aligned} \frac{6 - 3m}{m^2 + m - 6} &= \frac{3(2 - m)}{(m + 3)(m - 2)} \\ &= \frac{3(-1)(m - 2)}{(m + 3)(m - 2)} \\ &= \frac{-3}{m + 3} \\ &= -\frac{3}{m + 3}, \; m \neq -3, 2 \end{aligned}$$

Notice that in Examples 1, 2, and 3, the *restrictions* on the values of the variables are stated. This is done since the original fraction and the fraction in lowest terms are equivalent only for those values of the variable for which neither denominator is zero.

Throughout the rest of this book, it will be assumed that the replacement sets of the variables in a fraction include no numbers for which the denominator is zero. If you use a rational expression in the process of solving an equation, however, you must be aware of restrictions on the values of the variables.

EXAMPLE 4 Solve the following equation for x. Restrict a so that no division by zero results.

$$ax - a^2 = 4x - 8a + 16$$

SOLUTION

$$ax - a^2 = 4x - 8a + 16$$
$$ax - 4x = a^2 - 8a + 16$$
$$x(a - 4) = a^2 - 8a + 16$$

If $a \neq 4$, then

$$x = \frac{a^2 - 8a + 16}{a - 4} = \frac{(a - 4)^2}{a - 4} = a - 4.$$

$\therefore$ if $a \neq 4$, then $x = a - 4$.

Notice in Example 4 that, if $a = 4$, the original equation is true for *all* real values of x.

Oral Exercises

State all restrictions on the values of the variables in the given rational expression.

1. $\frac{5}{x}$
2. $\frac{4}{a - 2}$
3. $\frac{2n - 8}{3}$
4. $\frac{y - 5}{y - 5}$
5. $\frac{4}{x^2 + 1}$
6. $\frac{3}{(n + 1)(n + 2)}$
7. $\frac{a - 2}{a^2 + 3a}$
8. $\frac{x - 3}{(x - 2)(x + 4)}$

Name the greatest common factor of the numerator and denominator of each rational expression.

9. $\frac{15}{25}$
10. $\frac{9b}{6b}$
11. $\frac{2m^5}{2m^2}$
12. $\frac{a + 5}{a - 5}$
13. $\frac{7(3m - 5)}{8(3m - 5)}$
14. $\frac{r + 4}{(r + 3)(r + 4)}$
15. $\frac{2c - 6}{8c - 24}$
16. $\frac{x + 2}{x^2 - 4}$
17. Do the open sentences $\frac{x}{x} = 1$ and $x = x$ have the same solution set over $\mathcal{R}$? Explain.

Written Exercises

Simplify.

A

1. $\frac{25a^2b^2}{35ab}$
2. $\frac{-9st^2}{15s^2t}$
3. $\frac{a(b-4)^2}{3a(b-4)}$
4. $\frac{6(a-1)}{18(a-1)^2}$
5. $\frac{(3w+2)(w-2)}{(w-2)(3w-2)}$
6. $\frac{(a-b)^2(2a+b)}{2(a+b)(a-b)}$
7. $\frac{6b^3-3b^2}{5-10b}$
8. $\frac{y^2-5y}{15-3y}$
9. $\frac{2f+6}{f^2-2f-15}$
10. $\frac{6u-4}{3u^2-10u+8}$
11. $\frac{c^2+c-6}{8-2c^2}$
12. $\frac{5w-4v}{16v^2-25w^2}$
13. $\frac{3d^2+13d+12}{9d^2+24d+16}$
14. $\frac{8u^2+6u-9}{6-5u-4u^2}$
15. $\frac{a^2+ab-20b^2}{a^2+2ab-15b^2}$
16. $\frac{5w^2-21wq+4q^2}{5w^2-19wq-4q^2}$
17. $\frac{18e^2+21e-15}{12e^2+32e+20}$
18. $\frac{15t^2-9t+18}{20t^2-12t+24}$

Solve each equation for x in terms of a. Restrict a so that no division by zero results.

B

19. $2ax - 5a = 2a^2 + 3x - 12$
20. $ax + 4a - 4 = a^2 + 2x$
21. $2a(x - a) = 5(x - 3) + a$
22. $4x - 10 = 9(3a^3 - 1) - 3x(a - 1)$

Simplify.

23. $\frac{3a^3-24}{a^2-4a+4}$
24. $\frac{125-8x^3}{4x^2-20x+25}$
25. $\frac{a^6+1}{a^4-1}$
26. $\frac{b^{12}-1}{(b^4-1)(b^6-1)}$
27. $\frac{3(x-2)^2+4(x-2)-32}{x^2-4}$
28. $\frac{9a^2-4a-5}{3(3a+2)^2+5(3a+2)-2}$
29. $\frac{4(2a^2+2b^2)+b(18a+b)}{(2a+3b)^2-4a^2}$
30. $\frac{(2b-1)^2-4}{(2b-1)^3-8}$

Determine all values of x for which the given fraction has a value of zero.

C

• 31. $\frac{28x^2+5x-12}{16x^3-9x}$
32. $\frac{2x^2+9x}{16x^3+68x^2-18x}$

Solve the following equations for x in terms of a.

a. If $\mathcal{R}$ is the solution set, what is the value of a?

b. If $\emptyset$ is the solution set, what is the value of a?

c. Determine the solution set for all values of a except those found in (a) and (b).

33. $a^2(x - 1) = 4(x - 1) + 5(a + 2)$
34. $2a^2(x - 1) - 3(x + 1) - a(5x - 7) = 0$

Write a direct proof of each theorem.

35. For all real numbers r, and all nonzero real numbers s and t, $\frac{r}{s} = \frac{r \cdot t}{s \cdot t}$.

36. For all real numbers r, and all nonzero real numbers s and t, $\frac{r}{s} = \frac{r \div t}{s \div t}$.

Computer Exercises For students with computer experience

1. Write a program that will compute the greatest common factor (GCF) of any two integers that you input. (*Hint*: Start by using the absolute value of either integer as a "trial factor," and test to determine if it is a factor of *both* integers. If it is, then it is the GCF. If it is not, decrease the trial factor by one and test again. Continue this process down to the number one, if necessary, until you find the GCF.)
2. Incorporate the program that you wrote for Exercise 1 into a program that will compute the simplest form of any fraction that you input. The numerator and denominator of the fraction should be input separately, and the output should be in the form of a fraction, such as 7/9, not a decimal.

Julio Rey Pastor

1888–1962

A young poet turned scientist, Julio Rey Pastor published an article on number theory when he was only seventeen. At twenty-three he was named professor of mathematical analysis at the University of Oviedo in his native Spain. Five years later, he wrote a book that discussed the synthetic geometry of space in n dimensions, and he founded a mathematics laboratory in Madrid. In 1917 he published his classic work, *Elementos de análisis algebraico (Elements of Algebraic Analysis).*

Also in 1917 Rey Pastor began dividing his professional life between two continents, teaching half the year in Spain and the other half in Buenos Aires, Argentina. It was during this time that he established the journal *Revista matemática hispanoamericana (Spanish American Mathematical Review).* Julio Rey Pastor was also noted as an historian of Spanish cartography.

8–5 Zero and Negative Exponents

From your work in Section 1-6, you are already familiar with expressions that involve *positive* integral exponents, such as 3^5, 2^3, and $(-5)^2$. In this section, you will learn to assign a meaning to expressions that involve *zero* or *negative* integral exponents, such as 2^0 and 2^{-5}.

In extending the meaning of a power to include any integer as an exponent, it is convenient to make definitions so that operations with powers continue to obey the laws already established for positive integral exponents. For example, the first law of exponents for division in Section 8-1 implies that

$$\frac{2^7}{2^7} = 1.$$

If the second law were to hold for $m = n$, then this quotient could also be expressed as

$$\frac{2^7}{2^7} = 2^{7-7} = 2^0.$$

This suggests that it is appropriate to define 2^0 to be 1, and, in general, to make the following definition.

> For every nonzero real number a,
>
> $$a^0 = 1.$$

Notice that no meaning is assigned to the expression 0^0.

The third law of exponents for division implies that

$$\frac{2^3}{2^8} = \frac{1}{2^{8-3}} = \frac{1}{2^5}.$$

If the second law of exponents for division were to hold for $m < n$, then this quotient could be expressed as

$$\frac{2^3}{2^8} = 2^{3-8} = 2^{-5}.$$

This suggests that it is appropriate to define 2^{-5} to be $\frac{1}{2^5}$ and, in general, to make the following definition.

> For every nonzero real number a and every integer n,
>
> $$a^{-n} = \frac{1}{a^n}.$$

Thus, for every *nonzero* real number a, a^{-n} is the reciprocal of a^n.

With these definitions, the laws of exponents that you have learned will apply to expressions involving negative or zero exponents if the *base* of the expression is not zero. For example, if $a \neq 0$, you can use the law of exponents for multiplication given in Section 7-2 to simplify the following expression.

$$a^{-4} \cdot a^{-3} = a^{-4+(-3)} = a^{-7}$$

You can justify this method by showing that the result is consistent with the result you would obtain using the laws of exponents for positive integral exponents together with the definition just given for negative exponents.

$$a^{-4} \cdot a^{-3} = \frac{1}{a^4} \cdot \frac{1}{a^3} = \frac{1 \cdot 1}{a^4 \cdot a^3} = \frac{1}{a^{4+3}} = \frac{1}{a^7} = a^{-7}$$

Similarly, you can justify the following statements for every nonzero value of each variable.

$$\frac{a^{-3}}{a^{-5}} = a^{-3-(-5)} = a^2$$

$$(3a^{-2})^4 = 3^4 \cdot a^{-2 \cdot 4} = 81a^{-8}$$

EXAMPLE 1 Simplify each expression. Express the answer using only positive exponents. Assume that no variable has zero as a value.

a. $a^2 \cdot a^{-2}$ **b.** $\dfrac{(a^0 b^{-2})^5}{2a^{-1}}$

SOLUTION **a.** $a^2 \cdot a^{-2} = a^{2+(-2)} = a^0 = 1$

b. $\dfrac{(a^0 b^{-2})^5}{2a^{-1}} = \dfrac{b^{-10}}{2a^{-1}} = \dfrac{a}{2b^{10}}$

The next example illustrates some other useful results that can be obtained for expressions with zero and negative exponents.

EXAMPLE 2 Show that each of the following open sentences is true if $a \neq 0$ and $b \neq 0$.

a. $\left(\dfrac{a}{b}\right)^0 = 1$ **b.** $\left(\dfrac{1}{b}\right)^{-1} = b$ **c.** $\left(\dfrac{a}{b}\right)^{-1} = \dfrac{b}{a}$

SOLUTION **a.** $\left(\dfrac{a}{b}\right)^0 = 1$ by definition since $\frac{a}{b}$ is a nonzero real number.

b. $\left(\dfrac{1}{b}\right)^{-1} = (1 \cdot b^{-1})^{-1} = b$

c. $\left(\dfrac{a}{b}\right)^{-1} = (a \cdot b^{-1})^{-1} = a^{-1} \cdot b = \dfrac{b}{a}$

Using the laws of exponents, you can express any positive number in the form

$$k \times 10^n, \text{ where } 1 \le k < 10, \text{ and } n \text{ is an integer.}$$

For example:

$$52{,}000 = 5.2 \times 10{,}000 = 5.2 \times 10^4$$

$$0.006 = \frac{6}{1000} = \frac{6}{10^3} = 6 \times 10^{-3}$$

Numbers written in this form are said to be written in **scientific notation.** Scientific notation provides an efficient way to read and write the very large and very small numbers that frequently occur in scientific measurements. For example:

Earth is about 150,000,000 km from the sun.
This distance can be written as 1.5×10^8 km.

One of the nearest stars, Alpha Centauri, is about 39,000,000,000,000 km from Earth.
This distance can be written as 3.9×10^{13} km.

The diameter of a silver atom is about 0.00000000025 m.
This measurement can be written as 2.5×10^{-10} m.

Scientific notation is also useful in some numerical computations, since numbers written in scientific notation can be multiplied and divided easily using the laws of exponents.

EXAMPLE 3 Simplify.

a. 220×4000 **b.** $\frac{0.072}{6000}$ **c.** $(15{,}000)^2$

SOLUTION

a.
$$\begin{aligned} 220 \times 4000 &= (2.2 \times 10^2)(4.0 \times 10^3) \\ &= (2.2 \times 4) \times (10^2 \times 10^3) \\ &= 8.8 \times 10^5 \\ &= 880{,}000 \end{aligned}$$

b.
$$\begin{aligned} \frac{0.072}{6000} &= \frac{7.2 \times 10^{-2}}{6.0 \times 10^3} \\ &= \frac{7.2}{6} \times \frac{10^{-2}}{10^3} \\ &= 1.2 \times 10^{-5} \\ &= 0.000012 \end{aligned}$$

c.
$$\begin{aligned} (15{,}000)^2 &= (1.5 \times 10^4)^2 \\ &= 1.5^2 \times 10^8 \\ &= 2.25 \times 10^8 \\ &= 225{,}000{,}000 \end{aligned}$$

Oral Exercises

Simplify.

1. 2^{-3}
2. 7^0
3. $(-3)^0$
4. $(-4)^{-1}$
5. $\left(\frac{4}{5}\right)^{-1}$
6. $\left(-\frac{3}{4}\right)^0$
7. $5^2 \cdot 5^{-3}$
8. $\frac{5^2}{5^{-3}}$

Simplify using only positive exponents. Assume that no variable has zero as a value.

9. a^{-3}
10. $\frac{1}{b^{-2}}$
11. $m^{-3} \cdot m^5$
12. $b^{-8} \cdot b^0$
13. $\frac{d^{-2}}{d^{-6}}$
14. $\frac{r^{-3}}{r^2}$
15. $(x^{-2})^{-1}$
16. $\left(\frac{1}{y^{-2}}\right)^2$
17. $(e^0)^{-2}$
18. $6(-c)^0$
19. $(3w^{-2})^0$
20. $\left(\frac{1}{4v}\right)^0$

Express each number in scientific notation.

21. 41,600
22. 2,000,000
23. 0.12
24. 0.00372

Written Exercises

Throughout this set of exercises, assume that no variable has zero as a value.

Simplify using only positive exponents.

A

1. $\frac{5}{2^{-1}}$
2. $\frac{3}{2^{-2}}$
3. $\frac{2^{-3}}{-4}$
4. $\frac{5^0}{-3}$
5. $5z^{-3}$
6. $(4y)^{-2}$
7. $6c^{-3}b^2$
8. $-3j^{-2}k^{-1}$
9. $\frac{n^{-3}}{m^{-3}}$
10. $\frac{2t^{-3}}{x^{-1}}$
11. $a^{-4} \cdot a^{-1}$
12. $2b^{-3} \cdot b$
13. $\frac{5r^3}{10r^{-1}}$
14. $\frac{6p^{-2}}{8p^2}$
15. $(uv^2)^{-2}$
16. $(cd^{-1})^{-1}$
17. $(g^{-3}y)^0$
18. $(a^0b^2)^{-3}$
19. $(e + f)^{-1}$
20. $(-3x^2y^{-1})^0$
21. $\left(\frac{2 + 3t^0}{s}\right)^{-1}$
22. $\left(\frac{3x^2 + 2x^2y^0}{10x^2}\right)^{-2}$
23. $\frac{2^{-3}c^{-1}d^2}{4^{-1}cd^{-1}}$
24. $\frac{r^{-3}s^2t^{-5}}{r^0s^{-1}t^{-7}}$

In Exercises 25–32, first express each number in scientific notation. Then simplify.

25. $500{,}000 \times 0.005$

26. $200 \times 2{,}000{,}000 \times 0.002$

27. $0.00256 \div 160$

28. $81{,}000 \div 0.09$

29. $(0.00012)^2$

30. $(3000)^3$

31. $\dfrac{40{,}000{,}000 \times 0.006}{8000}$

32. $\dfrac{(0.00048)(0.0012)}{(3000)(0.000024)}$

Simplify using only positive exponents.

B 33. $\dfrac{(-2)^{-3}c^{-2}h}{(4c)^{-1}h^{-3}}$

34. $\dfrac{a^0b^{-3}c^2d^{-1}}{(ac)^{-2}b^{-5}d^0}$

35. $\left[\dfrac{(-2)^3}{2^2}\right]^{-1}$

36. $\left[\dfrac{3^{-1}}{(-2)^{-2}}\right]^{-2}$

37. $\left[\dfrac{5^{-1}r}{5s^{-1}}\right]^2$

38. $\left[\dfrac{3^{-3}u^2w^{-5}}{6^{-1}u^{-1}w^{-2}}\right]^{-2}$

Simplify each expression. Express the answer so that no denominator contains a variable.

39. $\dfrac{(3^2)^{-1}m^{-1}n^{-2}}{3^{-3}m^0n^3}$

40. $\dfrac{r^{-3}s^2t^{-4}}{r^0s^{-4}t^{-5}}$

41. $\dfrac{(2^{-3})^{-1}d^{-2}e^0f^5}{(2^{-2})^{-1}d^{-1}e^6f^{-2}}$

42. $\dfrac{10^{-1}k^{-3}t^{-5}m^2}{(2^{-1})^{-2}kt^{-3}m^{-1}}$

43. $\dfrac{(-5)^{-2}x^5(y^{-1}z^{-2})^2}{10^{-1}(x^{-1}y^2)^{-1}z^{-3}}$

44. $\dfrac{(2^{-1})^{-2}(a^{-1}b^2c^{-3})^{-2}}{(6^{-1})^2(a^2b^{-1}c^2)^{-3}}$

In Exercises 45–48, evaluate each expression when $r = -3$ and $s = 2$.

C 45. $\dfrac{5}{r^{-1} + s^{-1}}$

46. $\dfrac{(r + s)^{-1}}{r^{-1} - s^{-1}}$

47. $\dfrac{r^{-2} - s^{-2}}{r^{-2} + s^{-2}}$

48. $\left(\dfrac{r^{-1}}{s^{-1}} + \dfrac{s^{-2}}{r^0}\right)^{-1}$

49. If $a = 2^m$ and $b = 2^{m+1}$, show that

$$\frac{8a^3}{b^2} = 2^{m+1}.$$

50. a. Express each of the following as a power of 2.

$$8, \quad 8^x, \quad 16^{x+3}$$

b. Simplify the following expression by first expressing each factor as a power of 2.

$$\frac{8^{2x-1} \cdot (2^x)^{-1} \cdot 16^{x+3}}{4^{3x+1}}$$

Computer Exercises For students with computer experience

1. Without using the computer's exponentiation operation, write a program that will compute the value of a^n when you input any real value for a and any integral value—positive, negative, or zero—for n.
2. Write a program that will determine whether or not a given integer *greater than one* is a power of a second integer, the *base*. If it is, the program should also determine what power it is. (*Hint*: Test all positive integral powers of the base until one of them is either equal to or greater than the given integer. Do *not* use the computer's exponentiation operation, since this will give inexact results.)
3. Modify the program that you wrote for Exercise 2 so that it will determine whether any real number *greater than zero* and *less than one* is a power of a given integer.
4. Write a program that will express a given real number greater than one in scientific notation. (*Hint*: First find the greatest power of ten that is less than or equal to the given number.)

Self-Test 2

VOCABULARY simplest form of a fraction (p. 393) scientific notation (p. 399)

Simplify.

1. $\frac{-16bc}{24cd}$ 2. $\frac{3s - 6t}{5s - 10t}$ 3. $\frac{6a^2 - 7a - 5}{3a - 5}$ *Obj. 1, p. 392*

Simplify using only positive exponents. Assume that no variable has zero as a value.

4. $c^{-5} \cdot c^2$ 5. $\frac{x^{-1}}{x^{-5}}$ 6. $(3a^2b^{-5})^{-2}$ *Obj. 2, p. 392*

7. $(5s^{-1}t^3)^0$ 8. $\left(\frac{2a^2}{3b}\right)^{-1}$ 9. $\frac{r^4p^{-3}m^0}{r^{-1}p^{-4}m^{-1}}$

Express each number in scientific notation.

10. 270,000 11. 0.0027 *Obj. 3, p. 392*

Check your answers with those at the back of the book.

Operations with Rational Expressions

OBJECTIVES for Sections 8-6 through 8-9:
1. *To simplify products and quotients of rational expressions.*
2. *To simplify sums and differences of rational expressions.*
3. *To simplify complex fractions.*

8–6 Products and Quotients of Rational Expressions

By using the symmetric property of equality, you can rewrite the basic property of quotients given in Section 8-1 as follows.

> ***Theorem.*** For all real numbers r and s and all nonzero real numbers t and u,
>
> $$\frac{r}{t} \cdot \frac{s}{u} = \frac{rs}{tu}.$$

You will notice that this is the rule that is used in multiplying rational numbers. For example,

$$\frac{3}{8} \cdot \frac{5}{2} = \frac{3 \cdot 5}{8 \cdot 2} = \frac{15}{16}.$$

The same rule can be used in multiplying rational expressions.

EXAMPLE 1 Simplify.

a. $\frac{a-2}{a} \cdot \frac{a+3}{a-3}$ **b.** $4t \cdot \frac{5t}{s}$ **c.** $\left(\frac{u}{v}\right)^3$

SOLUTION **a.** $\frac{a-2}{a} \cdot \frac{a+3}{a-3} = \frac{(a-2)(a+3)}{a(a-3)} = \frac{a^2+a-6}{a^2-3a}$

b. $4t \cdot \frac{5t}{s} = \frac{4t}{1} \cdot \frac{5t}{s} = \frac{20t^2}{s}$

c. $\left(\frac{u}{v}\right)^3 = \left(\frac{u}{v}\right)\left(\frac{u}{v}\right)\left(\frac{u}{v}\right) = \frac{u \cdot u \cdot u}{v \cdot v \cdot v} = \frac{u^3}{v^3}$

Part (c) of Example 1 illustrates the first part of the theorem on the following page.

Theorem. Let a and b denote real numbers and let n denote an integer.

1. If $b \neq 0$ and a and n are not both 0, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

2. If $a \neq 0$ and $b \neq 0$, then $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n}$.

3. If $a \neq 0$ and $b \neq 0$, then $\left(\frac{a}{b}\right)^0 = \frac{a^0}{b^0} = 1$.

EXAMPLE 2 Simplify $\left(\frac{3a^2b^{-3}}{2c^{-4}d}\right)^{-2}$.

SOLUTION $$\left(\frac{3a^2b^{-3}}{2c^{-4}d}\right)^{-2} = \frac{(3a^2b^{-3})^{-2}}{(2c^{-4}d)^{-2}} = \frac{3^{-2}a^{-4}b^6}{2^{-2}c^8d^{-2}} = \frac{4b^6d^2}{9a^4c^8}$$

You can often simplify products of rational expressions by factoring numerators and denominators.

EXAMPLE 3 Simplify $\frac{4c + 12}{5c} \cdot \frac{c^2 - 6c + 9}{c^2 - 9}$.

SOLUTION $$\begin{aligned}\frac{4c + 12}{5c} \cdot \frac{c^2 - 6c + 9}{c^2 - 9} &= \frac{4(c + 3)}{5c} \cdot \frac{(c - 3)(c - 3)}{(c + 3)(c - 3)} \\ &= \frac{4(c + 3)(c - 3)(c - 3)}{5c(c + 3)(c - 3)} \\ &= \frac{4(c - 3)}{5c} \\ &= \frac{4c - 12}{5c}\end{aligned}$$

The definition of division given in Section 2-11, together with the fact that the reciprocal of $\frac{s}{u}$ is $\frac{u}{s}$ if $s \neq 0$ and $u \neq 0$, leads to the following theorem.

Theorem. For all real numbers r and all nonzero real numbers s, t, and u,

$$\frac{r}{t} \div \frac{s}{u} = \frac{r}{t} \cdot \frac{u}{s}.$$

You will notice that this is the rule used in dividing rational numbers.

For example,

$$\frac{5}{8} \div \frac{2}{3} = \frac{5}{8} \cdot \frac{3}{2} = \frac{5 \cdot 3}{8 \cdot 2} = \frac{15}{16}.$$

The same rule can be used in dividing rational expressions.

EXAMPLE 4 Simplify. Assume that no variable has a value that results in division by zero.

a. $\frac{3x}{4y} \div \frac{x}{2y}$ b. $\frac{4m^2 - 8m - 5}{6m + 3} \div \frac{2m^2 - 5m}{6m + 12}$

SOLUTION a. $\frac{3x}{4y} \div \frac{x}{2y} = \frac{3x}{4y} \cdot \frac{2y}{x} = \frac{6xy}{4xy} = \frac{3}{2}$

b. $$\frac{4m^2 - 8m - 5}{6m + 3} \div \frac{2m^2 - 5m}{6m + 12} = \frac{4m^2 - 8m - 5}{6m + 3} \cdot \frac{6m + 12}{2m^2 - 5m}$$
$$= \frac{(2m + 1)(2m - 5)}{3(2m + 1)} \cdot \frac{6(m + 2)}{m(2m - 5)}$$
$$= \frac{3 \cdot 2(2m + 1)(2m - 5)(m + 2)}{3m(2m + 1)(2m - 5)}$$
$$= \frac{2(m + 2)}{m}, \text{ or } \frac{2m + 4}{m}$$

Oral Exercises

Throughout this set of exercises, assume that no variable has a value that results in division by zero.

Simplify.

1. $\frac{a}{b} \cdot \frac{c}{d}$
2. $-\frac{3x}{2y} \cdot \frac{w}{z}$
3. $\frac{a^2b}{c} \cdot \frac{c^2}{ab}$
4. $\frac{y^2}{2y} \cdot \frac{8}{2y}$
5. $\frac{a - 2}{a + 1} \cdot \frac{3}{a - 1}$
6. $\frac{x + 5}{x - 2} \cdot \frac{x - 2}{x + 1}$
7. $\left(\frac{2x}{3y}\right)^0$
8. $\left(\frac{-1}{b^2}\right)^{-1}$
9. $\left(\frac{3w^2}{2v^3}\right)^{-1}$
10. $\left(\frac{a^2}{2b}\right)^4$
11. $\left(\frac{x^3}{y^2}\right)^{-2}$
12. $\left(-\frac{3z}{2}\right)^{-3}$

Express each quotient as a product.

13. $\frac{b}{r} \div \frac{c}{q}$
14. $\frac{x}{y} \div 2x$
15. $\frac{3k}{k - 2} \div \frac{2}{k}$
16. $\frac{e}{e - 1} \div \frac{e - 1}{e + 1}$
17. $-\frac{r}{5s} \div \frac{s - 2}{3r}$
18. $\frac{a}{b} \div \left(-\frac{a - 1}{b - 1}\right)$

Written Exercises

Throughout this set of exercises, assume that no variable has a value that results in division by zero.

Simplify.

A

1. $\frac{5m}{2} \cdot \frac{2}{5}$

2. $\frac{12}{7} \cdot \frac{7n}{4}$

3. $\frac{5a^2}{b} \div \frac{2a}{3b}$

4. $\frac{12x^2}{5y} \div \frac{3x}{2y}$

5. $\frac{15k}{-4} \div 3k$

6. $3k \div \frac{15k}{-4}$

7. $\frac{8t^2}{3} \div \left(-\frac{4t}{9}\right)$

8. $\frac{-3r^2}{2s} \div \left(-\frac{2r}{s}\right)$

9. $\frac{16pq}{25} \cdot \frac{5p}{8q^2}$

10. $\frac{-36j^2k^3}{7i} \cdot \frac{14i^2j^2}{9k^2}$

11. $\frac{rs}{4r - 4s} \cdot \frac{r^3 - rs^2}{r^2}$

12. $\frac{2g - 2h}{g + h} \cdot \frac{2g + 2h}{4g}$

13. $\frac{5s + 10t}{10s - 10t} \div \frac{3s + 6t}{6s - 6t}$

14. $\frac{3e - 9f}{2e + 4f} \div \frac{5e - 15f}{7e + 14f}$

15. $\frac{3t + 1}{t - 2} \cdot \frac{t^2 - 4}{9t^2 - 1}$

16. $\frac{4d^2 - 9}{d + 1} \cdot \frac{10d^2 - 10}{10d + 15}$

17. $\frac{3u - 6}{u^2 - 1} \div \frac{4u - 8}{2u^2 - 2}$

18. $\frac{c^2 - d^2}{2c + d} \div \frac{c^2 - cd}{2c^2 + cd}$

19. $\frac{4v - 12}{v^2} \div (2v^2 - 6v)$

20. $\frac{b^2 + 3b}{4b} \div (3b + 9)$

21. $(5w - 20) \div \frac{w^2 - 16}{3w + 12}$

22. $(4a - 24) \div \frac{2a^2 - 72}{3a}$

23. $\frac{1}{3x + 9} \div \frac{10x}{5x + 15}$

24. $\frac{3t^2 - 27}{t - 2} \div \frac{1}{t^2 - 4}$

Simplify using only positive exponents.

25. $\left(\frac{3a^{-2}b^4}{2c^2d^{-2}}\right)^{-3}$

26. $\left(\frac{5x^0y^{-5}}{10xy^{-6}}\right)^{-1}$

27. $\left(\frac{2a^4b^{-3}}{10a^{-2}b}\right)^0$

28. $\frac{3}{4}\left(\frac{3x^4y^{-1}}{-2x^0y}\right)^{-2}$

29. $\left(\frac{a^4b^{-1}}{6c^{-1}}\right)^{-2} \div \frac{ac^2}{-2b}$

30. $\left(\frac{x - 1}{3x}\right)^{-2} \cdot \left(\frac{1 - x}{2}\right)^3$

Simplify.

B

31. $\frac{k^2 + 2k - 3}{k^2 + k - 2} \cdot \frac{3k + 6}{k + 3}$

32. $\frac{z^2 - 2z}{z + 1} \cdot \frac{z^2 - 1}{z^2 - 2z + 1}$

33. $\frac{t^2 + 4t + 4}{t^2 - 3t - 10} \cdot \frac{t^2 - 10t + 25}{t^2 - 4}$

34. $\frac{w^2 + 3w}{w^3 - 6w} \cdot \frac{w^2 - 6}{w^3 - 9w}$

35. $\frac{m^2 - 2m - 8}{m^2 - 5m + 6} \div \frac{m^2 - 3m - 4}{m^2 - 9}$

36. $\frac{v^2 - 2v - 24}{v^2 + 6v + 8} \div \frac{v^2 - 8v + 12}{v^2 + v - 2}$

37. $\left(\frac{x - 2y}{x + 2y}\right)^{-2} \cdot \frac{x^3 - 8y^3}{x^2 + 4xy + 4y^2}$

38. $\left(\frac{3a - 4b}{a - 1}\right)^3 \div \frac{(4b - 3a)^5}{a^2 - 2a + 1}$

39. $\frac{4t^2 + 4t - 3}{9t^2 - 1} \cdot \left(\frac{4t^2 - 8t + 3}{6t^2 - 7t - 3}\right)^{-1}$

40. $\frac{2p^2 - 5p - 3}{p^2 + 7p + 6} \div \left(\frac{3p^2 + 2p - 1}{6p^2 + p - 1}\right)^{-1}$

41. $\left(\frac{6x^2 + 5x - 6}{6x^2 - 4x - 6}\right)^0 \div \left(\frac{9x^2 - 4}{3x^2 + x - 2}\right)^{-1}$

42. $\left(\frac{5a^2 - 4a - 1}{5a - 5}\right)^{-1} \cdot \left(\frac{2a^2 + a - 3}{25a^2 + 10a + 1}\right)^0$

C 43. $\frac{3j + 9}{10j + 10} \cdot \frac{j - 2}{6j + 18} \div \frac{6 - 3j}{5j + 25}$

44. $\frac{z^2 - 1}{z^2 + 4} \cdot \frac{6z + 30}{z^2 + z} \div \left(\frac{z^3 + 4z}{3z - 3}\right)^{-1}$

45. $\frac{6h^2 - 7h - 20}{2 + 5h - 12h^2} \div \frac{15h^2 + 14h - 8}{3h^2 - 2h} \cdot \frac{20h^2 - 3h - 2}{2h^2 + 7h - 30}$

46. $\frac{y^2 - y - 12}{y^2 - 16} \div \frac{20 - y - y^2}{y^2 + 10y + 24} \div \frac{y^2 + 4y - 12}{y^2 - 8y + 16}$

47. Is the product of two rational expressions always a rational expression?

48. Is the quotient of two nonzero rational expressions always a rational expression?

49. Explain why the set of rational numbers is closed with respect to multiplication.

50. Explain why the set of rational numbers is closed with respect to division, excluding division by zero.

Exercises 51–52 refer to the three parts of the theorem stated at the top of page 404.

51. Use the definition of a power (page 25) and the rule for multiplying rational expressions (page 403) to prove the first part of the theorem.

52. Use the first part of the theorem to prove the second and third parts of the theorem.

8–7 Sums and Differences of Rational Expressions with Equal Denominators

By using the symmetric axiom of equality, you can rewrite the theorem given in Section 8-2 as follows.

Theorem. For all real numbers a, b, and c such that $c \neq 0$,

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$

You will notice that this is the rule that is used in adding or subtracting rational numbers with equal denominators, as shown on the following page.

For example,

$$\frac{4}{15} + \frac{3}{15} = \frac{4 + 3}{15} = \frac{7}{15}$$

and

$$\frac{8}{11} - \frac{7}{11} = \frac{8 - 7}{11} = \frac{1}{11}.$$

The same rule can be used in adding and subtracting rational expressions that have equal denominators.

EXAMPLE Simplify.

a. $\frac{2a}{5} + \frac{3a}{5}$ **b.** $\frac{a}{3b} - \frac{5a}{3b}$ **c.** $\frac{a^2}{a^2 - 9} + \frac{2a - 3}{a^2 - 9}$ **d.** $\frac{3a}{a - 1} - \frac{a + 1}{a - 1}$

SOLUTION **a.** $\frac{2a}{5} + \frac{3a}{5} = \frac{2a + 3a}{5} = \frac{5a}{5} = a$

b. $\frac{a}{3b} - \frac{5a}{3b} = \frac{a - 5a}{3b} = \frac{-4a}{3b} = -\frac{4a}{3b}$

c. $\frac{a^2}{a^2 - 9} + \frac{2a - 3}{a^2 - 9} = \frac{a^2 + 2a - 3}{a^2 - 9}$

$$= \frac{(a + 3)(a - 1)}{(a + 3)(a - 3)} = \frac{a - 1}{a - 3}$$

d. $\frac{3a}{a - 1} - \frac{a + 1}{a - 1} = \frac{3a - (a + 1)}{a - 1}$

$$= \frac{3a - a - 1}{a - 1} = \frac{2a - 1}{a - 1}$$

Oral Exercises

Simplify.

1. $\frac{5}{13} + \frac{2}{13}$

2. $\frac{9}{16} - \frac{11}{16}$

3. $\frac{3x}{7} - \frac{x}{7}$

4. $\frac{10a}{7} + \frac{3b}{7}$

5. $\frac{2}{5m} - \frac{n}{5m}$

6. $\frac{5b}{9a} - \frac{4b}{9a}$

7. $\frac{7n}{3n + 1} + \frac{n + 2}{3n + 1}$

8. $\frac{5x}{x^2 + 1} + \frac{2x - 1}{x^2 + 1}$

9. $\frac{8b}{5} - \frac{2b + 1}{5}$

10. $\frac{4t}{9} - \frac{3t - 1}{9}$

11. $\frac{2d - 1}{d - 4} - \frac{d - 1}{d - 4}$

12. $\frac{3j + 4}{j - 7} - \frac{2j + 11}{j - 7}$

Written Exercises

Simplify.

A

1. $\frac{7}{a} + \frac{5}{a} - \frac{3}{a}$

2. $\frac{3}{2b} - \frac{4}{2b} + \frac{7}{2b}$

3. $\frac{2n}{n+1} + \frac{n-1}{n+1}$

4. $\frac{5c-2}{c-1} + \frac{3c-1}{c-1}$

5. $\frac{3t}{4} - \frac{t-1}{4}$

6. $\frac{5x}{3} - \frac{2x+1}{3}$

7. $\frac{4a+1}{a-3} - \frac{3a+4}{a-3}$

8. $\frac{3s-5}{s-2} - \frac{2s-3}{s-2}$

9. $\frac{3a}{2a-6} - \frac{9}{2a-6}$

10. $\frac{3b}{2b+12} + \frac{18}{2b+12}$

11. $\frac{x^2}{x^2-9} + \frac{3x}{x^2-9}$

12. $\frac{3x}{x^2-1} - \frac{3}{x^2-1}$

13. $\frac{2h-i}{h-i} - \frac{3h-2i}{h-i}$

14. $\frac{5t+4w}{2t-w} - \frac{9t+2w}{2t-w}$

15. $\frac{3a^2}{a+1} - \frac{a^2+2}{a+1}$

16. $\frac{x^2+3}{2x-3} - \frac{5x^2-6}{2x-3}$

17. $\frac{3d+4}{2d^2-8} - \frac{7d+12}{2d^2-8}$

18. $\frac{8+5h}{3h^2-48} - \frac{6h+4}{3h^2-48}$

B

19. $\frac{5c-4}{2c^2-c-3} + \frac{1-3c}{2c^2-c-3}$

20. $\frac{4m+3}{9m^2-3m-2} - \frac{m+5}{9m^2-3m-2}$

21. $\frac{3d+7}{9d^2+9d-4} - \frac{11-9d}{9d^2+9d-4}$

22. $\frac{z^2+3z}{10z^2+11z-6} + \frac{5z^2+6z}{10z^2+11z-6}$

23. $\frac{7n-3}{8n^2+2n-3} + \frac{4-9n}{8n^2+2n-3}$

24. $\frac{3w^2-7}{4w^2-15w-4} - \frac{2w^2+9}{4w^2-15w-4}$

25. $\frac{2e}{e-5} - \left(\frac{e-2}{e-5} + \frac{2}{e-5}\right)$

26. $\frac{3p}{(p+1)^2} - \left(\frac{5p+1}{(p+1)^2} - \frac{3p+2}{(p+1)^2}\right)$

27. $\frac{6v-1}{v^2-9} - \frac{2v+7}{v^2-9} + \frac{5-3v}{v^2-9}$

28. $\frac{7g+3f}{f^2-g^2} - \frac{5g+3f}{f^2-g^2} + \frac{f-g}{f^2-g^2}$

C

29. Find a rational expression which when added to $\frac{5r-1}{5-3r}$ gives a sum of $\frac{2r-3}{5-3r}$.

30. Find a rational expression which when added to $\frac{u^2-3v^2}{u^2-9v^2}$ gives a sum of $\frac{u+v}{u+3v}$.

31. Show that for all real numbers x and y such that $x \neq y$,

$$\frac{1}{x-y} + \frac{1}{y-x} = 0.$$

32. Simplify $\frac{5x}{2x-1} - \frac{1-x-6x^2}{1-4x^2}$.

8–8 Sums and Differences of Rational Expressions with Unequal Denominators

The **least common denominator (LCD)** of two or more fractions is the *least common multiple* of their denominators. To add or subtract rational numbers having unequal denominators, first find the LCD of their denominators. Then express each fraction as an equivalent fraction with the LCD as denominator.

EXAMPLE 1 Simplify $\frac{3}{8} + \frac{5}{14}$.

SOLUTION 1. Find the LCM by factoring 8 and 14 over the set of primes.

$$8 = 2^3 \text{ and } 14 = 2 \cdot 7$$
$$\text{LCM} = 2^3 \cdot 7 = 56$$

Thus, the LCD is 56.

2. Express each fraction as an equivalent fraction with the LCD as denominator.

$$\frac{3}{8} = \frac{3 \cdot 7}{8 \cdot 7} = \frac{21}{56}; \qquad \frac{5}{14} = \frac{5 \cdot 4}{14 \cdot 4} = \frac{20}{56}$$

3. Simplify.

$$\frac{3}{8} + \frac{5}{14} = \frac{21}{56} + \frac{20}{56} = \frac{21 + 20}{56} = \frac{41}{56}$$

The method used in Example 1 to find the sum of two rational numbers can also be used to find the sum or difference of rational expressions that have unequal denominators. The **least common multiple (LCM)** of two or more polynomials is the polynomial of least degree and least positive constant factor that has each of the given polynomials as a factor. The least common denominator of two or more rational expressions is the LCM of their denominators.

To find the LCM of two or more polynomials you first factor each of the polynomials completely. The LCM will be the product of all the different factors, each factor occurring in the product the greatest number of times that it occurs in the complete factorization of any one of the given polynomials.

EXAMPLE 2 Find the LCM of $4x^2 - 36$ and $6x^2 - 36x + 54$.

SOLUTION $4x^2 - 36 = 2^2(x + 3)(x - 3)$

$6x^2 - 36x + 54 = 2 \cdot 3(x - 3)^2$

$\text{LCM} = 2^2 \cdot 3(x + 3)(x - 3)^2 = 12(x + 3)(x - 3)^2$

EXAMPLE 3 Simplify $2 + \frac{5r}{3r + 12} - \frac{r - 1}{r^2 - 16}$.

SOLUTION

1. Find the LCM of the denominators.

$$3r + 12 = 3(r + 4); \qquad r^2 - 16 = (r + 4)(r - 4)$$
$$\text{LCM} = 3(r + 4)(r - 4)$$

Thus, the LCD $= 3(r + 4)(r - 4)$.

2. Express each fraction as an equivalent fraction with the LCD as denominator.

$$2 = \frac{2 \cdot 3(r + 4)(r - 4)}{1 \cdot 3(r + 4)(r - 4)} = \frac{6(r + 4)(r - 4)}{3(r + 4)(r - 4)}$$

$$\frac{5r}{3r + 12} = \frac{5r \cdot (r - 4)}{3(r + 4) \cdot (r - 4)} = \frac{5r(r - 4)}{3(r + 4)(r - 4)}$$

$$\frac{r - 1}{r^2 - 16} = \frac{(r - 1) \cdot 3}{(r + 4)(r - 4) \cdot 3} = \frac{3(r - 1)}{3(r + 4)(r - 4)}$$

3. Simplify.

$$2 + \frac{5r}{3r + 12} - \frac{r - 1}{r^2 - 16}$$
$$= \frac{6(r + 4)(r - 4)}{3(r + 4)(r - 4)} + \frac{5r(r - 4)}{3(r + 4)(r - 4)} - \frac{3(r - 1)}{3(r + 4)(r - 4)}$$
$$= \frac{6(r + 4)(r - 4) + 5r(r - 4) - 3(r - 1)}{3(r + 4)(r - 4)}$$
$$= \frac{6r^2 - 96 + 5r^2 - 20r - 3r + 3}{3(r + 4)(r - 4)}$$
$$= \frac{11r^2 - 23r - 93}{3(r + 4)(r - 4)}, \text{ or } \frac{11r^2 - 23r - 93}{3r^2 - 48}$$

Unless instructed otherwise, you need not number and describe steps as in Example 3. The way that you usually give the solution is shown in Example 4.

EXAMPLE 4 Simplify $\frac{3x}{x^2 - 2xy + y^2} - \frac{2y}{x^2 - y^2}$.

SOLUTION

$$\frac{3x}{x^2 - 2xy + y^2} - \frac{2y}{x^2 - y^2} = \frac{3x}{(x - y)^2} - \frac{2y}{(x + y)(x - y)}$$
$$= \frac{3x\,(x + y)}{(x - y)^2\,(x + y)} - \frac{2y\,(x - y)}{(x + y)(x - y)\,(x - y)}$$
$$= \frac{3x^2 + 3xy - 2xy + 2y^2}{(x - y)^2(x + y)}$$
$$= \frac{3x^2 + xy + 2y^2}{(x - y)^2(x + y)}$$

Note that, in adding and subtracting rational expressions, *any* common multiple of the denominators may be used as a common denominator. However, it is often most convenient to use the *least* common denominator.

Oral Exercises

Name the LCM of each of the following.

1. 16, 24, 30
2. 10, 44, 56
3. $18x^2y$, $15x^3y^3$
4. $10c$, $25c^2d$
5. $4(x-3)^2$, $(x+3)^2(x-3)$
6. $5a^2(a-2)^3$, $6(a-2)^2$

Name the LCD of the fractions in each expression.

7. $\frac{1}{4}+\frac{1}{6}$
8. $\frac{3}{4a}-\frac{5}{2ab}$
9. $\frac{a}{b^2c}+\frac{b}{a^2c^2}$
10. $3+\frac{1}{a}$
11. $x+\frac{1}{x}$
12. $\frac{2}{x+1}+\frac{3}{x-1}$
13. $\frac{t}{2(t-1)}-\frac{3t}{1-t}$
14. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}$
15. $\frac{3}{(x+1)(x-4)}+\frac{5}{3(x-4)}$
16. $\frac{5s}{6(s-1)^2(s+2)^3}+\frac{5s}{8(s+2)^5}$
17. $\frac{3}{x^2y}-\frac{1}{y^2z}+\frac{5}{xyz}$
18. $4-\frac{1}{x(x+1)}+\frac{1}{x^2-1}$

Replace each __?__ with the factor(s) that will make the fractions equivalent.

19. $\frac{2}{3a}=\frac{?}{15a^2b^2}$
20. $\frac{4}{c-3}=\frac{?}{c(c-3)}$
21. $\frac{3d}{d+1}=\frac{?}{(d+1)(d-2)}$
22. $\frac{4a+1}{(a-1)^2}=\frac{?}{(a-1)^3(a+1)}$
23. $\frac{5(x-1)}{6(x+1)^2(x-2)}=\frac{?}{12(x+1)^3(x-2)^3}$
24. $\frac{3t}{t^2-1}=\frac{?}{4(t+1)^2(t-1)}$

Written Exercises

Simplify.

A

1. $\frac{5x}{2}+\frac{5x}{6}$
2. $\frac{5p}{7}-\frac{3}{11}$
3. $\frac{5}{c}+\frac{1}{d}$
4. $\frac{3e}{4}-\frac{5}{e}$
5. $3+\frac{1}{t}$
6. $5s-\frac{1}{5s}$
7. $\frac{1}{2h}+\frac{1}{h^2}$
8. $\frac{3}{fg}-\frac{2}{f^2g^2}$
9. $\frac{u-v}{u}+\frac{u+v}{v}$
10. $\frac{r+s}{2r}-\frac{r+s}{2s}$
11. $\frac{4}{6-3a}-\frac{1}{a-2}$
12. $\frac{3}{2t-6}+\frac{4}{3t-9}$

13. $\frac{t}{t-3} + \frac{t-1}{t+3}$

14. $\frac{3b}{2b-4} - \frac{b+1}{b-2}$

15. $b - \frac{3}{b-2}$

16. $3x - \frac{1}{1-3x}$

17. $\frac{3}{a(a-3)} - \frac{5}{a-3}$

18. $\frac{5t}{t-1} + \frac{3}{t(t-1)}$

19. $\frac{h}{g-3} - \frac{2h}{3-g}$

20. $\frac{2}{a-b} + \frac{3}{2(b-a)}$

21. $\frac{3}{w+2} - \frac{3w}{w^2-4}$

22. $\frac{2}{9q-6} + \frac{1+q}{6q^2-4q}$

23. $\frac{4}{3x-6} - \frac{x+1}{2x^2-4x}$

24. $\frac{x-1}{x^2+5x} - \frac{x+1}{x^2-25}$

B

25. $\frac{5m}{m^2-2mn+n^2} - \frac{3}{m-n}$

26. $\frac{1}{u^2+u-6} + \frac{1}{u^2-9}$

27. $\frac{b}{b^2-b-20} - \frac{b}{b^2+9b+20}$

28. $\frac{p+1}{p^2+4p+3} + \frac{1}{p^2-2p-15}$

29. $c - \frac{25c-5c^2}{c^2-10c+25}$

30. $\frac{2d-12}{d^2-5d-6} + \frac{1}{d-1}$

31. $\frac{a+5}{a^2-a-6} + \frac{a+3}{a^2+7a+10}$

32. $\frac{a+3b}{a^2-7ab+12b^2} - \frac{a-3b}{a^2-ab-12b^2}$

33. $\frac{p+1}{p} - \frac{p+q}{q} + \frac{q-1}{p}$

34. $\frac{2}{3+c} - \frac{5}{3-c} - \frac{2c+1}{9-c^2}$

35. $s - \frac{s-1}{s-6} - \frac{s^2-6}{s^2-36}$

36. $\frac{3}{e+2} - e + \frac{3e}{e^2-4}$

C

37. $\left(\frac{1}{s} + \frac{1}{t}\right) \div \left(\frac{1}{s} - \frac{1}{t}\right)$

38. $\left(1 + \frac{3}{x-2}\right)\left(2 - \frac{6}{x+1}\right)$

39. $\left(1 + \frac{ab}{a^2-ab+b^2}\right)\left(1 + \frac{ab}{a^2+b^2}\right)$

40. $\left(x - 4 - \frac{21}{x}\right) \div \left(x - 8 + \frac{7}{x}\right)$

41. $\left(\frac{1}{a+b} - \frac{1}{a-b}\right)^{-1}\left(1 - \frac{a^2-3b^2}{2a^2-ab-3b^2}\right)$

42. $\left(\frac{1}{y^2-4} + \frac{1}{y^2+2y}\right)^{-1}\left(\frac{1}{y} - \frac{y-2}{y^2+y-2}\right)$

43. a. Is the sum of two rational expressions always a rational expression?
    b. Is the difference of two rational expressions always a rational expression?

44. Explain why the set of rational numbers is closed with respect to addition and subtraction.

Computer Exercises

For students with computer experience

1. Write a program that will compute the sum of two fractions with equal denominators. The output should be in the form of a fraction, not a decimal, and should be in simplest form.
2. Modify the program that you wrote for Exercise 1 so that it will compute the sum of *any* two fractions that you input, whether or not their denominators are equal.

PROGRAMMING IN BASIC

Given the program that follows, it is possible to use a computer to find the coefficients of the product of two polynomials in a single variable. The program uses an array of *subscripted variables* to store the coefficients of the polynomials and to execute the necessary multiplications and additions. Specifically, A(I) represents the coefficients of the first polynomial being multiplied, B(I) the coefficients of the second, and P(I) the coefficients of the product.

For example, to multiply the polynomials $3x + 1$ and $5x + 4$, you input the degree of the polynomials as 1 and their coefficients as:

A(1) = 3   B(1) = 5
A(2) = 1   B(2) = 4

$$
\begin{array}{rrr}
 & (3)x & +\ (1) \\
 & \underline{(5)x} & \underline{+\ (4)} \\
(5)(3)x^2 & +\ (5)(1)x & \\
 & (4)(3)x & +\ (4)(1) \\
\hline
15x^2\ + & 17x\ + & 4
\end{array}
$$

The coefficients of the product are then output as:

P(3) = 15   P(2) = 17   P(1) = 4

To understand how the program computes these coefficients, it may be helpful to study the multiplication as shown at the right above.

In using the program, notice that each of the polynomials being multiplied is considered as having the same degree, n, and the same number of terms, $n + 1$. Thus, coefficients of "missing terms" must be input as zero. Also note that, if the degree of the polynomials being multiplied is greater than 4, a DIMension statement may be needed.

```
10  PRINT "TO MULTIPLY TWO"
20  PRINT "POLYNOMIALS IN X:"
30  PRINT
40  PRINT "INPUT DEGREE N (N <= 4):";
50  INPUT N
60  REM *INPUT COEFFICIENTS
70  FOR I = N + 1 TO 1 STEP -1
80  PRINT "INPUT A(";I;"), B(";I;"):";
90  INPUT A(I), B(I)
100  NEXT I
110  REM *SET P(I) = 0
120  FOR I = 1 TO 2*N + 1
130  LET P(I) = 0
140  NEXT I
150  REM *COMPUTE COEFFICIENTS
160  REM *OF THE PRODUCT
170  FOR J = N + 1 TO 1 STEP -1
180  FOR I = N + 1 TO 1 STEP -1
190  LET M = I + J - 1
200  LET P(M) = P(M) + A(I) * B(J)
```

```
210  NEXT I
220  NEXT J
230  PRINT "COEFFICIENTS OF PRODUCT:"
240  FOR I = 2*N + 1 TO 1 STEP -1
250  PRINT P(I);" ";
260  NEXT I
270  END
```

Exercises

Type in the program as given. Then RUN it to find the product of each of the following pairs of polynomials.

1. $3x^2 + 2x + 1$
$\underline{3x^2 + 2x + 1}$

2. $3x^3 - 4x^2 + 5x - 2$
$\underline{2x^3 + x^2 - x + 3}$

3. $6x^4 + 5x^3 - 2x^2 + x - 1$
$\underline{x^4 + 3x^3 + x^2 - 3x + 1}$

4. $x^3 - 2x^2 + 0x + 3$
$\underline{0x^3 + 0x^2 + x + 3}$

5. $2x^2 + 3x - 1$
$\underline{x - 2}$

6. $x^3 + x^2 + x + 1$
$\underline{x - 1}$

8–9 Complex Fractions

If a fraction has a numerator or denominator that contains a fraction or a term with a negative exponent, the fraction is called a **complex fraction.** Complex fractions may be completely numerical expressions or may involve variables. The following expressions are examples of complex fractions.

$$\frac{\frac{1}{2} - \frac{1}{3}}{\frac{3}{5}}, \quad \frac{\frac{a}{b^2}}{\frac{b}{c^2}}, \quad \frac{5}{2 + \frac{1}{a}}, \quad \frac{d^{-2} + 1}{d - 1}$$

Any complex fraction can be transformed into an expression that is free of negative exponents and fractions in the numerator and denominator. Such an expression is called a **simple fraction.** The complex fractions in the examples just given can be transformed, respectively, into the following simple fractions. (See Oral Exercises 6, 8, 9, and 11 on page 417.)

$$\frac{5}{18}, \quad \frac{ac^2}{b^3}, \quad \frac{5a}{2a + 1}, \quad \frac{d^2 + 1}{d^3 - d^2}$$

Throughout this section the variables will be restricted to values for which each expression is defined. You can therefore assume that the complex fraction and its simplified form are equivalent expressions. Two common methods used to simplify complex fractions are illustrated in Example 1 on the following page.

EXAMPLE 1 Simplify $\dfrac{\dfrac{r}{2s} - \dfrac{s}{2r}}{\dfrac{1}{r} + \dfrac{1}{s}}$.

SOLUTION (Method 1)

$$\frac{\dfrac{r}{2s} - \dfrac{s}{2r}}{\dfrac{1}{r} + \dfrac{1}{s}} = \frac{\dfrac{r^2 - s^2}{2rs}}{\dfrac{s + r}{rs}} \longleftarrow$$

Simplify the numerator and denominator separately.

$$= \frac{r^2 - s^2}{2rs} \div \frac{s + r}{rs} \longleftarrow$$

Rewrite the fraction as a quotient using the ÷ sign.

$$= \frac{r^2 - s^2}{2rs} \cdot \frac{rs}{s + r}$$

$$= \frac{(r + s)(r - s)}{2rs} \cdot \frac{rs}{s + r}$$

$$= \frac{r - s}{2}$$

(Method 2)

$$\frac{\dfrac{r}{2s} - \dfrac{s}{2r}}{\dfrac{1}{r} + \dfrac{1}{s}} = \frac{\left(\dfrac{r}{2s} - \dfrac{s}{2r}\right) \cdot 2rs}{\left(\dfrac{1}{r} + \dfrac{1}{s}\right) \cdot 2rs} \longleftarrow$$

Multiply the numerator and denominator by the LCD of all the simple fractions contained in the complex fraction.

$$= \frac{r^2 - s^2}{2s + 2r}$$

$$= \frac{(r + s)(r - s)}{2(s + r)}$$

$$= \frac{r - s}{2}$$

If the complex fraction contains terms with negative exponents, first rewrite the expression using positive exponents only.

EXAMPLE 2 Simplify $\dfrac{d^{-2} + 1}{d - 1}$.

SOLUTION

$$\frac{d^{-2} + 1}{d - 1} = \frac{\left(\dfrac{1}{d^2} + 1\right)}{(d - 1)}$$

$$= \frac{\left(\dfrac{1}{d^2} + 1\right) \cdot d^2}{(d - 1) \cdot d^2}$$

$$= \frac{1 + d^2}{d^3 - d^2}, \text{ or } \frac{d^2 + 1}{d^3 - d^2}$$

If the numerator or denominator of a complex fraction is itself a complex fraction, simplify the fraction in steps.

EXAMPLE 3 Simplify $\dfrac{\dfrac{5}{1-x}}{1+\dfrac{1}{1+\dfrac{1}{x}}}$.

SOLUTION First, simplify $\dfrac{1}{1+\dfrac{1}{x}}$: $\quad \dfrac{1}{1+\dfrac{1}{x}} = \dfrac{1 \cdot x}{\left(1+\dfrac{1}{x}\right) \cdot x} = \dfrac{x}{x+1}$

Then:
$$\frac{\dfrac{5}{1-x}}{1+\dfrac{1}{1+\dfrac{1}{x}}} = \frac{\left(\dfrac{5}{1-x}\right) \cdot (1-x)(1+x)}{\left(1+\dfrac{x}{x+1}\right) \cdot (1-x)(1+x)}$$
$$= \frac{5(1+x)}{(1-x)(1+x)+x(1-x)}$$
$$= \frac{5+5x}{1-x^2+x-x^2}$$
$$= \frac{5+5x}{1+x-2x^2}$$

Oral Exercises

a. Use Method 1 to simplify each complex fraction.
b. Use Method 2 to simplify each complex fraction.

1. $\dfrac{\frac{5}{9}}{\frac{7}{9}}$

2. $\dfrac{\frac{4}{7}}{\frac{2}{7}}$

3. $\dfrac{\frac{5}{6}}{\frac{3}{2}}$

4. $\dfrac{\frac{15}{8}}{\frac{25}{12}}$

5. $\dfrac{1+\frac{5}{16}}{1-\frac{3}{4}}$

6. $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{3}{5}}$

7. $\dfrac{\frac{a}{b^2}}{\frac{a}{b^3}}$

8. $\dfrac{\frac{a}{b^2}}{\frac{b}{c^2}}$

9. $\dfrac{5}{2+\frac{1}{a}}$

10. $\dfrac{1+\frac{1}{a}}{3+\frac{1}{a}}$

11. $\dfrac{d^{-2}+1}{d-1}$

12. $\dfrac{x^{-3}+\frac{1}{x}}{x-\frac{1}{x}}$

Written Exercises

Simplify.

A

1. $\dfrac{\frac{4}{5}}{\frac{8}{15}}$
2. $\dfrac{\frac{7}{9}}{\frac{15}{14}}$
3. $\dfrac{\frac{k}{3}}{\frac{2k}{9}}$
4. $\dfrac{\frac{2d}{5}}{\frac{3d}{10}}$
5. $\dfrac{\frac{c^2}{d}}{\frac{c}{d^3}}$
6. $\dfrac{\frac{4r}{3}}{\frac{5r^2}{6}}$
7. $\dfrac{\frac{e}{3}+\frac{f}{4}}{\frac{e}{2}-\frac{f}{3}}$
8. $\dfrac{\frac{2}{a}-\frac{1}{b}}{\frac{1}{a}-\frac{3}{b}}$
9. $\dfrac{\frac{5w}{6}-\frac{3z}{2}}{\frac{w}{3}+\frac{z}{6}}$
10. $\dfrac{\frac{2c}{5}+\frac{3d}{10}}{\frac{3c}{2}-\frac{7d}{10}}$
11. $\dfrac{\frac{u}{3v}-\frac{v}{3u}}{\frac{1}{3v}+\frac{1}{3u}}$
12. $\dfrac{1-\frac{25}{k^2}}{1-\frac{5}{k}}$
13. $\dfrac{d^{-1}+3}{1-d^{-1}}$
14. $\dfrac{t^{-1}}{t^{-1}-(3t)^{-1}}$
15. $\dfrac{xy^{-1}+2}{yx^{-1}-2}$
16. $\dfrac{5-r(2s)^{-1}}{4+r(3s)^{-1}}$
17. $\dfrac{9-c^{-2}}{3c^{-1}-c^{-2}}$
18. $\dfrac{2j^{-1}+5j^{-2}}{4-25j^{-2}}$

B

19. $\dfrac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{1}{x}-\frac{6}{x^2}}$
20. $\dfrac{1-\frac{4}{a^2}}{1-\frac{5}{a}+\frac{6}{a^2}}$
21. $(r^{-1}-s^{-1})^{-1}$
22. $(x^{-1}+y^{-1})^{-2}$
23. $\dfrac{1}{(u^{-2}+v^{-2})^{-1}}$
24. $\dfrac{1}{(c^{-1}-d^{-1})^{-2}}$
25. $\dfrac{\frac{c}{c-d}-\frac{d}{c+d}}{\frac{cd}{c^2-d^2}}$
26. $\dfrac{\frac{r}{r-2}-\frac{2}{r+2}-\frac{8}{r^2-4}}{\frac{1}{r-2}}$
27. $[(e^2-1)^{-1}+1][(e-1)^{-1}+1]^{-1}$
28. $[a(a-1)^{-1}+1][3(a^2-1)^{-1}+4]^{-1}$

C

29. $1-\dfrac{1}{1-\frac{1}{z-2}}$
30. $2+\dfrac{1}{1+\frac{2}{m+\frac{1}{m}}}$
31. $\dfrac{1}{1-\frac{1}{2-\frac{1}{3-t}}}$
32. $1-\dfrac{1}{1-\frac{1}{1-\frac{1}{1-n}}}$
33. If $r=\dfrac{1-s}{1+s}$ and $s=\dfrac{1+t}{1-t}$, show that $r+t=0$.
34. Sketch the graphs of $f(x)=\dfrac{1}{2+\frac{1}{x}}$ and $g(x)=\dfrac{x}{2x+1}$. State how the two graphs are alike and how they are different.

Self-Test 3

VOCABULARY least common denominator (LCD) (p. 410)
least common multiple (LCM) of polynomials (p. 410)
complex fraction (p. 415)
simple fraction (p. 415)

Simplify. Assume that no variable has a value that results in division by zero.

1. $\frac{3x + 3}{2x} \cdot \frac{x - 1}{x + 1}$ 2. $\frac{c^2 - 16}{3c} \div \frac{c + 4}{2c}$ *Obj. 1, p. 403*

3. $\frac{5d - 2}{4d^2 + 8d - 5} - \frac{3d - 7}{4d^2 + 8d - 5}$ 4. $\frac{3}{2e} + \frac{2}{3e^3}$ *Obj. 2, p. 403*

5. $\frac{3b + \frac{2}{b}}{b + \frac{1}{2}}$ 6. $\frac{a^{-2} + 1}{1 - a^{-1}}$ *Obj. 3, p. 403*

Check your answers with those at the back of the book.

Chapter Summary

1. The *basic property of quotients* asserts that, for all real numbers r and s and all nonzero real numbers t and u, $\frac{rs}{tu} = \frac{r}{t} \cdot \frac{s}{u}$.
2. If a is a nonzero real number and m and n are positive integers, the following *laws of exponents* for division are true:
   1. If $m = n$, then $\frac{a^m}{a^n} = 1$.
   2. If $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$.
   3. If $m < n$, then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.
3. One polynomial can be divided by another using a division algorithm similar to the long division algorithm used for integers.
4. A *rational expression* is any expression that can be written as the quotient of two polynomials, provided that the denominator is not zero. Rational expressions can be added, subtracted, multiplied, or divided using methods similar to the methods used in numerical computations.

5. Any fraction whose numerator and denominator are integers or polynomials can be simplified by dividing the numerator and denominator by their greatest common factor (GCF).

6. For every nonzero real number a, $a^0 = 1$.

7. For every nonzero real number a and every integer n, $a^{-n} = \frac{1}{a^n}$.

8. A number is said to be written in *scientific notation* if it is expressed in the form $k \times 10^n$, where $1 \leq k < 10$ and n is an integer.

9. Let a and b denote real numbers and let n denote an integer.

   1. If $b \neq 0$ and a and n are not both 0, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

   2. If $a \neq 0$ and $b \neq 0$, then $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n}$.

   3. If $a \neq 0$ and $b \neq 0$, then $\left(\frac{a}{b}\right)^0 = \frac{a^0}{b^0} = 1$.

10. A *complex fraction* has a numerator or denominator that contains a fraction or a term with a negative exponent. Any complex fraction can be transformed into a *simple fraction* that is free of negative exponents and fractions in its numerator and denominator.

Chapter Review

Write the letter of the correct answer.

1. Simplify $\frac{26a^3b^7}{39ab^9}$ using only positive exponents. *8-1*

   a. $\frac{2a^3}{3b^2}$ b. $\frac{2a^2}{3b^2}$ c. $\frac{2a^2b^2}{3}$ d. $\frac{13a^2}{b^2}$

2. Simplify $\frac{(-2ax)^3(-3a^2x^3)^2}{(-9a^3x^4)(4ax^2)}$ using only positive exponents.

   a. $2a^3x^3$ b. a^3x^3 c. $\frac{a^3x^2}{2}$ d. $-\frac{a^3x^2}{2}$

3. Express $\frac{28u^7 - 16u^5 + 20u^2}{-4u^5}$ as a sum. *8-2*

   a. $-7u^2 - 4 + 5u^3$ b. $-7u^2 + 4 - 5u^3$

   c. $-7u^2 + 4 - \frac{5}{u^3}$ d. $-7u^2 - 4 + \frac{5}{u^3}$

4. Divide $x^2 + 6x + 5$ by $x + 1$, assuming $x + 1 \neq 0$. *8-3*

   a. $x + 5$ b. $x + 5 + \frac{10}{x + 1}$

   c. $x - 5$ d. $x + 5 - \frac{10}{x + 1}$

5. Simplify $\frac{5x + 10}{15(x + 2)^2}$. 8-4

a. $\frac{x}{(x + 2)^2}$ b. $\frac{3}{x + 2}$ c. $\frac{1}{3(x + 2)}$ d. $\frac{x + 10}{3(x + 2)^2}$

6. Simplify $\frac{3f^2 + 6f - 45}{27 - 3f^2}$.

a. $\frac{f + 5}{f - 3}$ b. $-\frac{f + 5}{f + 3}$ c. $\frac{3f + 5}{f + 9}$ d. $-\frac{f - 5}{9 - 3f}$

7. Simplify $\frac{16a^{-2}}{8a^{-3}}$ using only positive exponents. 8-5

a. $2a$ b. $\frac{2}{a^5}$ c. $\frac{a}{2}$ d. $\frac{2}{a}$

8. Simplify $(x^0y^{-2})^{-1}$ using only positive exponents.

a. y^2 b. $\frac{y^2}{x}$ c. $\frac{1}{y^2}$ d. $\frac{1}{xy^3}$

9. Express 0.000437 in scientific notation.

a. 437×10^6 b. 437×10^{-6} c. 4.37×10^{-4} d. 0.437×10^3

10. Simplify $\frac{v^2 - 6v + 8}{v^2 + 4v} \cdot \frac{9v}{6 - 3v}$. 8-6

a. -3 b. $3v^2 - 48$ c. $\frac{3(v - 4)}{v + 4}$ d. $\frac{3(4 - v)}{v + 4}$

11. Simplify $\frac{4m^2n^3}{3xw^2} \div \frac{20m^4n^2}{9x^3w}$, assuming m, n, x, and $w \neq 0$.

a. $\frac{3x^2}{5w}$ b. $\frac{3nx^2}{5m^2w}$ c. $\frac{3m^2w}{5nx^2}$ d. $\frac{100m^6n^5}{27x^4w^3}$

12. Simplify $\frac{7m}{2} - \frac{5m - 1}{2}$. 8-7

a. $\frac{2m - 1}{2}$ b. $\frac{2m + 1}{4}$ c. $\frac{2m + 1}{2}$ d. $\frac{m + 1}{2}$

13. Simplify $\frac{c - 6}{c^2 - 3c - 10} + \frac{1}{c^2 - 3c - 10}$.

a. $c + 2$ b. $c - 2$ c. $\frac{1}{c - 2}$ d. $\frac{1}{c + 2}$

14. Simplify $\frac{2}{x - 1} - \frac{3}{x^2 - 1}$. 8-8

a. $\frac{2x - 3}{x^2 - 1}$ b. $\frac{2x - 1}{x^2 - 1}$ c. $\frac{2x + 5}{x^2 - 1}$ d. $\frac{2}{x + 1}$

15. Simplify $\frac{2 - x^2}{x^2 - 4x} + \frac{2x - 1}{2x - 8}$.

a. $-\frac{1}{2x}$ b. $\frac{-2}{x - 4}$ c. $\frac{1 + 2x - x^2}{2x^2 - 8x}$ d. $\frac{1 + 2x - x^2}{x^2 - 2x - 8}$

16. Simplify $\dfrac{\frac{r-s}{r}}{\frac{1}{2}+\frac{s}{2r}}$. 8-9

a. $\dfrac{r+s}{r-s}$ b. $\dfrac{r-s}{r+s}$ c. $\dfrac{r+s}{2r-2s}$ d. $\dfrac{2r-2s}{r+s}$

Chapter Test

Simplify using only positive exponents.

1. $\dfrac{5x}{-60x^2}$ 2. $\dfrac{-(v^2)^3}{v^4}$ 3. $\dfrac{21a^5bc^3}{0.3ab^2c^2}$ 8-1

Express each quotient as a sum.

4. $\dfrac{6a^3+24a^5}{6a^4}$ 5. $\dfrac{11c^3d-33cd-55c^2d^2}{-11cd}$ 8-2

Divide the first polynomial by the second. Assume that no divisor equals zero.

6. $6y^2-7y+5;\ 2y-3$ 7. $t^3-5t^2+10t-12;\ t-3$ 8-3

Simplify using only positive exponents.

8. $\dfrac{4b^2-12b}{24-8b}$ 9. $\dfrac{2a^2-7a-4}{6a^2+a-1}$ 8-4

10. $(-4g^3h^{-2})^0$ 11. $\dfrac{(-3)^0a^4b^{-1}}{a^{-1}bc^{-2}}$ 8-5

12. Express the product $500 \times 50{,}000 \times 0.0005$ in scientific notation.

Simplify using only positive exponents. Assume that no variable has a value that results in division by zero.

13. $\dfrac{-24m^3n^2}{14n^4}\cdot\dfrac{21n^3}{4m^5}$ 14. $\dfrac{e^2-2e+1}{e^2}\div(e-1)$ 8-6

15. $\dfrac{2x+3}{x-5}-\dfrac{3x-2}{x-5}$ 16. $\dfrac{3-x}{x^2-16}+\dfrac{2x-7}{x^2-16}$ 8-7

17. $\dfrac{3}{g+2}+g$ 18. $\dfrac{w}{w^2-25}-\dfrac{1}{2w+10}$ 8-8

19. $\dfrac{\frac{m}{4n}-\frac{n}{4n}}{\frac{1}{n}-\frac{1}{m}}$ 20. $\dfrac{(4x)^{-1}-xy^{-2}}{2x-y}$ 8-9

Mixed Review

Simplify.

1. $(3c - 5)(2c + 3)$
2. $-d^3 + 4d - 5 - (6d^3 + 6d - 5)$
3. $(3xy)^2(2x^3y)(-2x)$
4. $(4ab)(-3a^2b)^2 - (2ab)^3(5a^2)$
5. $(2x - 3)^2$
6. $(x + 3)^3$
7. $(7x - 2y)(7x + 2y)$
8. $(r + 2s)(r^2 - 3rs + s^2)$

Factor completely.

9. $32ax^3 - 18ax$
10. $3x^2 - 5x - 8$
11. $x^4 - 2x^3 - 24x^2$
12. $3m - 2mn - 4n^2 + 6n$
13. $18 - 21x - 4x^2$
14. $9x^2 - 15x - 24$
15. $y^4 - 16$
16. $84z^2 + 7z - 42$
17. $6a^3 + 9a^2 - 3a$
18. $4b^3 - 48b^2 + 80b$
19. $5r^3 - 40s^3$
20. $t^2(t - 1) + 4(1 - t)$

Solve.

21. $4w - 3(1 - w) = -17$
22. $3 - 5y \geq (2 - y)4 + 7$
23. $\frac{3}{4}(4z - 8) - 4 < 5z + 8$
24. $\frac{2}{3}(3y - 6) = \frac{1}{2}(10y + 4)$
25. $|3u - 7| < 2$
26. $3x > 4x - 3$ and $2x \leq 5x + 6$
27. $m^2 - 15 = 2m$
28. $(4n - 1)(3n + 2) = (6n + 5)(2n - 3)$

Solve each system of equations.

29. $3x + 4y = -7$
$-2x + 3y = 16$
30. $3a - 4b = 5$
$a + 7b = 10$

Solve for the variable in color.

31. $m = n(t + v)$
32. $x = 5(y + z)$
33. $T = 3a - 4b$
34. $P = q + qnt$
35. $d = \frac{ef}{2bc}$
36. $h = \frac{s + w}{u}$

Solve.

37. One train traveling at an average speed of 60 km/h left a station 2 h after another train left from the same station. Traveling in the same direction along a parallel track, the second train overtook the first in 1 h. What was the average speed of the first train?
38. The sum of the digits of a two-digit number is 9. When the digits are reversed, the new number is 27 more than the original number. Find the original number.

PREPARING FOR COLLEGE ENTRANCE EXAMS

Strategy for Success: The method that is used in scoring a particular multiple-choice exam determines whether or not it is worthwhile to guess an answer. If you find that it *is* worthwhile, you may be able to guess by using your knowledge of algebra to eliminate one or more of the answer choices. For example, if you know that a certain answer must be a positive integer, you can eliminate any choice that is a negative number, zero, or a fraction or decimal.

Decide which is the best of the choices given and write the corresponding letter on your answer sheet.

1. How many integral values of k are there for which the polynomial $x^2 + kx + 36$ is factorable?
(A) 4 (B) 10 (C) 3 (D) 2 (E) 5

2. Which of the following polynomials is (are) irreducible?
I. $6x^2 + 55x + 51$ II. $8x^2 - 43x + 57$ III. $143a^3 + 297$
(A) I only (B) II only (C) III only (D) I and II only (E) I and III only

3. For what value of n is the sentence $4^{3n-2} = 16^n$ true?
(A) 4 (B) 2 (C) 1 (D) 3 (E) 0

4. The product of two consecutive positive odd integers is 74 more than the square of the lesser integer. Find the greater integer.
(A) 51 (B) 37 (C) 43 (D) 39 (E) 45

5. Which of the following are factors of $x^3 - 9x - 7x^2 + 63$?
I. $x - 7$ II. $x + 3$ III. $x - 3$
(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

6. Find an integral value of c such that $3x^2 - 23x + c$ and $9x^2 + 6x + 1$ will have a common binomial factor.
(A) -8 (B) 8 (C) 5 (D) -5 (E) -7

7. The perimeter of a rectangle is 86 m and its area is 432 m^2. What is the length of the longer side of the rectangle?
(A) 16 m (B) 32 m (C) 27 m (D) 21 m (E) 14 m

8. Jill is x years old. Her brother Jim's age is the square of her age. Five years from now, Jim's age will be two years less than twice Jill's age at that time. How old is Jim now?
(A) 3 years (B) 8 years (C) 14 years (D) 9 years (E) 7 years

9. Express $\frac{2x - 3}{15 + 7x - 2x^2} \div \frac{(x - 5)^{-1}}{(2x - 3)^{-2}}$ in lowest terms. Assume that no variable has a value that results in division by zero.
(A) -1 (B) $\frac{3 - 2x}{3 + 2x}$ (C) $\frac{1}{9 - 4x^2}$ (D) $\frac{2x - 3}{5 - x}$ (E) $\frac{x - 5}{(5 - x)(3 + 2x)}$