Chapter 7

Polynomials and Factoring

Addition, Subtraction, and Multiplication of Polynomials

OBJECTIVES for Sections 7-1 through 7-4:
1. *To add and subtract polynomials.*
2. *To multiply monomials.*
3. *To find powers of a monomial.*
4. *To multiply a polynomial by a monomial.*
5. *To multiply polynomials.*

7–1 Adding and Subtracting Polynomials

A numeral, a variable, or an indicated product of a numeral and one or more variables is called a **monomial.** For example,

$$-3, \quad x, \quad 5y^2, \quad 12a^2b, \quad \text{and} \quad -\frac{1}{4}m^5n^2$$

are monomials. A monomial that contains no variable, such as -3, is called a **constant monomial,** or a **constant**.

The number of times that a variable occurs as a factor in a monomial is called the **degree of the variable** in that monomial. For example, in the monomial

$$-4x^2yz^3 \quad \begin{cases} x \text{ has degree 2,} \\ y \text{ has degree 1, and} \\ z \text{ has degree 3.} \end{cases}$$

The **degree of a monomial** is the total number of times that its variables occur as factors. Thus, the degree of $-4x^2yz^3$ is $2 + 1 + 3$, or 6. The degree of any nonzero constant monomial is 0. The monomial 0 has *no* degree.

The numerical factor of a monomial is called the **coefficient,** or **numerical coefficient,** of the monomial. Monomials that are identical or that differ only in their coefficients are called **similar,** or **like.** Thus,

$$-2xy^3, \qquad xy^3, \qquad -xy^3, \qquad \text{and} \qquad 7y^3x$$

are similar, but

$$5xy^3 \qquad \text{and} \qquad 5x^3y$$

are not similar.

A monomial or a sum of monomials is called a **polynomial.** Each of the monomials is called a **term** of the polynomial, and the coefficients of the terms are called the **coefficients** of the polynomial. The coefficient of a nonzero constant term is defined to be the constant itself. Thus, the terms of the polynomial

$$8x^3 + (-3x^2) + 0x + (-4)$$

are $8x^3$, $-3x^2$, $0x$, and -4; the coefficients are 8, -3, 0, and -4. Note that this polynomial is more commonly written

$$8x^3 - 3x^2 - 4,$$

where the term with coefficient 0 is omitted and the connecting + signs are taken to be understood.

It is sometimes convenient to say that a monomial is a polynomial of *one* term. A polynomial of *two* terms is called a **binomial,** and a polynomial of *three* terms is called a **trinomial.** For example, $x^2 + y^2$ is a binomial, and $x^2 + 2xy + y^2$ is a trinomial.

A polynomial is said to be simplified, or in **simplest form,** when no two of its terms are similar. For example,

$$8y^2 - 4y + 1$$

is in simplest form, but

$$9z^2 + 5z - 16z - 3$$

is not in simplest form. You simplify a polynomial by using the distributive axiom to add similar terms.

EXAMPLE 1 Simplify $9z^2 + 5z - 16z - 3$.

SOLUTION
$$\begin{aligned} 9z^2 + 5z - 16z - 3 &= 9z^2 + [5 + (-16)]z - 3 \\ &= 9z^2 - 11z - 3 \end{aligned}$$

The **degree of a polynomial** is the greatest of the degrees of its terms after it has been simplified. Thus, $9z^2 - 11z - 3$ is of degree two.

To *add* two or more polynomials, you write the sum and simplify by adding similar terms.

EXAMPLE 2 Add $9n^2 + 5n$ and $n^2 - 6n + 12$.

SOLUTION 1 $(9n^2 + 5n) + (n^2 - 6n + 12) = (9n^2 + n^2) + [5n + (-6n)] + 12$
$= 10n^2 - n + 12$

SOLUTION 2 You can also line up the similar terms vertically and add.

$$\begin{array}{r} 9n^2 + 5n \\ n^2 - 6n + 12 \\ \hline 10n^2 - n + 12 \end{array}$$

To *subtract* one polynomial from another, you add the opposite of each term of the polynomial that you are subtracting. Then simplify the sum.

EXAMPLE 3 Subtract $t^2 - 8t + 2$ from $4t^2 + 7t$.

SOLUTION 1 $(4t^2 + 7t) - (t^2 - 8t + 2) = 4t^2 + 7t + (-t^2) + 8t + (-2)$
$= (4t^2 - t^2) + (7t + 8t) - 2$
$= 3t^2 + 15t - 2$

SOLUTION 2 $\begin{array}{r} 4t^2 + 7t \\ t^2 - 8t + 2 \\ \hline \end{array} \longrightarrow \begin{bmatrix} \text{change to the} \\ \text{opposite and add} \end{bmatrix} \longrightarrow \begin{array}{r} 4t^2 + 7t \\ -t^2 + 8t - 2 \\ \hline 3t^2 + 15t - 2 \end{array}$

Note that the terms of a simplified polynomial are usually arranged in order of *decreasing* or *increasing* degree of one of the variables. For example:

$7x^4 - 2x^3 - x + 5$ is in order of decreasing degree in x.

$5 - x - 2x^3 + 7x^4$ is in order of increasing degree in x.

$-4a^3 + a^2b + 6ab^2 + 5b^3$ is in order of $\begin{cases} \text{decreasing degree in } a, \\ \text{increasing degree in } b. \end{cases}$

Oral Exercises

Tell whether or not the given expression is a monomial. If the expression is a monomial, state its degree and coefficient.

1. $3a$
2. $-3bc$
3. x^4
4. $-y$
5. 12
6. 0
7. $\frac{1}{2}m^2$
8. $\frac{n^3}{7}$
9. $-\frac{2}{z}$
10. $\frac{pq^2}{r}$
11. $4rs^2$
12. $5u + v^2$

Name the similar monomials.

13. $3b,\ 3b^2,\ -2b,\ ab,\ -2b^2,\ 3ab$
14. $4xy,\ -2yz,\ 9yx,\ 4xz,\ -zy,\ 2zx$
15. $-m^2n^2,\ 3mn^2,\ -4m^2n,\ 2m^2n^2,\ 8mn,\ -2m^2n,\ 3mn,\ -2mn^2$
16. $rs^3,\ -5r^3s^2,\ 7r^3s,\ -2r^2s^3,\ -2r^3s,\ 3r^2s^3,\ -8rs^3,\ 2r^3s^2$

Tell whether the given polynomial is a monomial, binomial, trinomial, or none of these. Then state the degree and the coefficients of the polynomial.

17. $5c^2d$
18. $5c^2 + d$
19. $8p^2 - 2p + 4$
20. $3q + 7q^3 - 5q^2$
21. $6abc - 5a^2b^2c^2$
22. $-6a^3 + 2a^2b - ab^2 + 9b^3$

Compute the following for each pair of polynomials.
a. the sum when the second is added to the first
b. the difference when the second is subtracted from the first

23. $\begin{array}{r} 4a - 3 \\ \underline{a + 1} \end{array}$
24. $\begin{array}{r} 2b + 7 \\ \underline{3b - 2} \end{array}$
25. $\begin{array}{r} 5r + 3s - t \\ \underline{r - 3s - 2t} \end{array}$
26. $\begin{array}{r} u - 4v + 6w \\ \underline{2u - 4v - w} \end{array}$
27. $\begin{array}{r} 2x^2 - 3x - 5 \\ \underline{-3x^2 + 2x \phantom{{}-5}} \end{array}$
28. $\begin{array}{r} 5y^4 + 2y^2 \phantom{{}-3} \\ \underline{5y^4 - y^2 - 3} \end{array}$

Arrange the given polynomial in order of *decreasing* degree in x.

29. $3x + x^2 - 7$
30. $9x^2 - 2x^4 + 8 - 5x^3$
31. $x^2y - 3y^3 + 4x^3 - 2xy^2$
32. $3x^2y^2 - 8xy^3 + x^4 - y^4 + 2x^3y$

Written Exercises

Replace each $\underline{\ ?\ }$ with the number or variable that makes a pair of similar monomials.

A

1. $8c^3d^5;\ -2c^?d^?$
2. $-3u^2v^4;\ 7\underline{\ ?\ }^2\underline{\ ?\ }^4$
3. $3j^5k^?;\ -2\underline{\ ?\ }^5k^3$
4. $10m^6\underline{\ ?\ }^2;\ -3\underline{\ ?\ }^6n^2$
5. $8x^2;\ 8x^?$
6. $-5y^3;\ \underline{\ ?\ }y^3$

Copy the given polynomial and underline similar terms in the same way. Then simplify the polynomial.

EXAMPLE $\quad 5ac^2 + 3ac - 7ac^2 - 4ac + 8$

SOLUTION $\quad \underline{5ac^2} + \underline{\underline{3ac}} - \underline{7ac^2} - \underline{\underline{4ac}} + 8 = (5ac^2 - 7ac^2) + (3ac - 4ac) + 8$
$= -2ac^2 - ac + 8$

7. $3rs + 2 - 2rs - 3$
8. $-5bc + d - bc - 4d$
9. $b^2 - 7b - 6b^2 + b$
10. $4m^2 + 2m^3 - 2m^2 - m^3$

11. $-3t + t^3 - 5t - t^3 - 1$

12. $2w^2 - w^4 - 3w^4 + 4w^2 + w$

13. $9ab + bc + 2ab + bc + 7$

14. $7xy - 3yz - 6xy + 4yz - xz$

15. $4p^3q^2 - 5p^2q^3 - 9pq + p^2q^3 - 2p^3q^2$

16. $8m^2 + 6m^2n - 3mn^2 - 2m^2n + 5mn^2$

17. $9rs - 3st + 6rt - 5st - 6rt - rs$

18. $12yz - yz^2 + 7y^2z - 7yz - 2y^2z - 5yz$

Add.

19. $\begin{array}{r} 2a + 5b - c \\ \underline{5a + 3b + 2c} \end{array}$

20. $\begin{array}{r} 4r - 3s + 5t \\ \underline{-3r - 2s + 2t} \end{array}$

21. $\begin{array}{r} 9s^2 + 4s - 5 \\ \underline{3s^2 \qquad\quad + 3} \end{array}$

22. $\begin{array}{r} 8t^2 - 2t + 1 \\ \underline{2t - 3} \end{array}$

23. $\begin{array}{r} 6m^4 - 5m^2 \qquad \\ \underline{-2m^4 + m^2 - 1} \end{array}$

24. $\begin{array}{r} 4n^5 \qquad\qquad + 5n \\ \underline{-5n^5 - 3n^3 + 2n} \end{array}$

25–30. In Exercises 19–24, subtract the second polynomial from the first.

Add or subtract the polynomials as indicated.

B

31. $(5p - 2q + 7r) + (3q - r - s)$

32. $(3a^2 - 2a + 7) + (a^3 - 2a^2 + a)$

33. $(2x^2 - 5x - 4) - (3x^2 + x + 1)$

34. $(3y^2 + 2y + 1) - (4y^2 - 3y - 1)$

35. $(5ab - 7b + 9a) + (3a + 4b - 4ab)$

36. $(-2v^2w^2 + 3v - w) + (-4v + 2w + 3v^2w^2)$

37. $(2m + 9) + (-3m - 2) - (4m - 1)$

38. $(4n - 7) - (n + 3) + (3n + 8)$

39. $(2bc - 4bc^2) - (2b^2c - bc) + (5b^2c + bc^2)$

40. $(2x^2y^2 - 5x^2y) - (2xy^2 - x^2y^2) - (x^2y - xy^2)$

41. $(6x^3 - 5x + 2x^2 - 1) - (3 - 7x - x^2 - x^3)$

42. $(5y^4 - y^6 + 6 - y^2) + (5y^6 - 1 + y^2 - y^4)$

Solve.

43. $(3x - 1) - (5x + 1) = -6$

44. $(2y + 5) - (3 - y) = -7$

45. $(4 - 2a) - (3a + 1) = 6 - 6a$

46. $(3b - 5) - (8 - 3b) = 8b - 9$

47. $(j^2 + j + 3) - (j^2 + 2j - 1) = 8$

48. $(k^3 + 8k - 3) - (k^3 - 6k + 5) = -1$

C

49. Subtract the sum of $2c - 3cd - d^2$ and $c + 5cd + 3d^2$ from $8cd - c^2d^2$.

50. Subtract the sum of $3v - 2w$ and $2v^2 + w$ from the sum of $5w - w^2$ and $4v - w$.

51. What polynomial must be added to $3r^2 - 5rs - s^2$ to obtain the polynomial $r^2 + 2rs + 3r^2s^2 - 4s^2$?

52. What polynomial must be subtracted from $t^2 - 3t + 2$ to obtain the polynomial $6t + 2t^2 - 5t^3$?

53. If two polynomials of degree three are added, must their sum be a polynomial of degree three? Explain.

54. If the difference between two polynomials is a polynomial of degree two, must one or both of the polynomials be of degree two?

7–2 Multiplying Monomials

In Section 1-6 you learned that, if n is a positive integer, the nth *power* of a real number a is defined as follows.

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

If you now apply this definition of power to the product

$$a^2 \cdot a^3,$$

you obtain the following.

$$a^2 \cdot a^3 = \underbrace{\overbrace{(a \cdot a)}^{2 \text{ factors}} \cdot \overbrace{(a \cdot a \cdot a)}^{3 \text{ factors}}}_{2 + 3 = 5 \text{ factors}} = a^5$$

In general, for all real numbers a, if m and n are positive integers:

$$a^m \cdot a^n = \underbrace{\overbrace{(a \cdot a \cdot \ldots \cdot a)}^{m \text{ factors}}\overbrace{(a \cdot a \cdot \ldots \cdot a)}^{n \text{ factors}}}_{m + n \text{ factors}} = a^{m+n}$$

Thus, to find the *product of powers* of the same number, you add the exponents. You can use this fact together with the commutative and associative axioms for multiplication to simplify many products.

EXAMPLE 1 Simplify $(6x^2y)(-4x^5y^3)$.

SOLUTION

$$\begin{aligned}(6x^2y)(-4x^5y^3) &= [6 \cdot (-4)](x^2 \cdot x^5)(y \cdot y^3) \\ &= -24(x^{2+5})(y^{1+3}) \\ &= -24x^7y^4\end{aligned}$$

Now consider the expression

$$(a^5)^3.$$

If you apply the definition of power and the preceding result to this expression, you obtain the following.

$$(a^5)^3 = a^5 \cdot a^5 \cdot a^5 = a^{5+5+5} = a^{15} = a^{5 \cdot 3}$$

In general, for all real numbers a, if m and n are positive integers:

$$(a^m)^n = \overbrace{(a^m)(a^m) \ldots (a^m)}^{a^m \text{ is a factor } n \text{ times}} = a^{\overbrace{m+m+\ldots+m}^{n \text{ terms}}} = a^{mn}$$

Thus, to find a *power of a power* of a real number, you multiply the exponents.

EXAMPLE 2 Simplify. **a.** $(x^4)^3$ **b.** $(-y^2)^5$

SOLUTION **a.** $(x^4)^3 = x^{4 \cdot 3} = x^{12}$
b. $(-y^2)^5 = -y^{2 \cdot 5} = -y^{10}$

To simplify the expression

$$(ab)^3,$$

you can apply the definition of power and the commutative and associative axioms for multiplication.

$$(ab)^3 = (ab)(ab)(ab) = (a \cdot a \cdot a)(b \cdot b \cdot b) = a^3b^3$$

In general, for all real numbers a and b, if m is a positive integer:

$$(ab)^m = \overbrace{(ab)(ab) \cdot \ldots \cdot (ab)}^{ab \text{ is a factor } m \text{ times}} = \overbrace{(a \cdot \ldots \cdot a)}^{m \text{ factors}}\overbrace{(b \cdot \ldots \cdot b)}^{m \text{ factors}} = a^mb^m$$

That is, to find a *power of a product* of real numbers, you find the power of each factor and then multiply.

EXAMPLE 3 Simplify. **a.** $(5x)^3$ **b.** $(-2rs)^4$

SOLUTION **a.** $(5x)^3 = 5^3 \cdot x^3 = 125x^3$
b. $(-2rs)^4 = (-2)^4 \cdot r^4 \cdot s^4 = 16r^4s^4$

The rules that have been used to simplify expressions in the three preceding examples can be summarized as the following *laws of exponents* for multiplication.

Laws of Exponents for Multiplication

For all real numbers a and b, if m and n are positive integers:

1. $a^m \cdot a^n = a^{m+n}$ **2.** $(a^m)^n = a^{mn}$ **3.** $(ab)^m = a^mb^m$

Sometimes you need to use more than one of these laws of exponents in simplifying a single expression.

EXAMPLE 4 Simplify. **a.** $(c^2d)(2cd)^3$ **b.** $(-2x^2y^3)(3xy^2)^3$

SOLUTION **a.**
$$\begin{aligned}(c^2d)(2cd)^3 &= (c^2d)(2^3c^3d^3)\\ &= 2^3(c^2 \cdot c^3)(d \cdot d^3)\\ &= 8c^5d^4\end{aligned}$$

b.
$$\begin{aligned}(-2x^2y^3)(3xy^2)^3 &= (-2x^2y^3)[3^3x^3(y^2)^3]\\ &= (-2x^2y^3)(27x^3y^6)\\ &= (-2 \cdot 27)(x^2 \cdot x^3)(y^3 \cdot y^6)\\ &= -54x^5y^9\end{aligned}$$

Oral Exercises

Simplify. Assume that any variable used as an exponent represents a positive integer.

1. $z^5 \cdot z^2$
2. $c \cdot c^4$
3. $2r^2 \cdot r$
4. $b^3 \cdot 5b^2$
5. $3w^4 \cdot 2w$
6. $4d^3 \cdot 3d^5$
7. $a^2 \cdot a^3 \cdot a^5$
8. $t \cdot t^4 \cdot t^2$
9. $b^2 \cdot b^x$
10. $j^r \cdot j^s$
11. $m \cdot m^a \cdot m^b$
12. $u^y \cdot u \cdot u^y$
13. $(cd)^3$
14. $(rs)^5$
15. $(5pq)^2$
16. $(-2uv)^3$
17. $(a^3)^2$
18. $(z^2)^5$
19. $(2s^3)^2$
20. $(5b^2)^3$
21. $(a^3)^x$
22. $(x^j)^k$
23. $(3c)^d$
24. $(4c^2)^r$

State the simplest form of the square of each monomial.

25. c^3
26. $-5m^4$
27. $3ab$
28. $4u^3v^5$

State the simplest form of the cube of each monomial.

29. t^6
30. $-2n^2$
31. $-4st$
32. $5r^3s^4$

Written Exercises

Simplify.

A

1. $(2b^2)(-5b)$
2. $(3z^3)(2z^2)$
3. $(6mn)(-2m)^5$
4. $(5cd)(-3d^2)$
5. $(3a^2z^3)(3az)$
6. $(-2pq^6)(3p^2q)$
7. $(4y)(-y^3)(2y^3)$
8. $(5c^3)(-2c^2)(-c^5)$
9. $(-3uv^2)(5u^2v^2)$
10. $(4a^3b)(-3a^2b)$
11. $(2c^2)^2(c^5)$
12. $(-3r^3)^2(r^4)$
13. $(3g^2)^2(2g)^3$
14. $(5f^3)^2(4f^2)^2$
15. $(a^2b)^3(ab^2)^4$
16. $(u^3v^2)^2(uv^4)^3$
17. $(-2a^2b)^2(-a^4b^3)^3$
18. $(3r^4s)^3(-5r^3s)^2$
19. $(x^2y)(xy^2)(x^2y^2)$
20. $(a^3b^3)(a^2b)(a^2b^3)$
21. $(-2mn)(3m^3n^2)(2m^4n)$
22. $(-3jk^2)(7j^3k)(-j^4k^2)$
23. $(r^2s^2t)(r^3s)(s^4t^3)$
24. $(-p^2q^3)(-qr^4)(-p^5r^2)$

Find a monomial that is equivalent to the given expression.

25. $(2z^3)(3z) + 5z^4$
26. $8m^5 - 2m^2(3m^3)$
27. $(5b^3c)(2bc^3) - (2bc)^4$
28. $(-3r^2s^2)^2 + (3rs^2)(2r^3s^2)$
29. $(2xy^2)^2(5x^5y) - (6xy^2)(x^2y)^3$
30. $(3cd)(5c^3d^2)^3 - (4c^2d)^3(2c^2d^2)^2$

Find a simplified polynomial that is equivalent to the given expression.

B 31. $(3y^4)(-y)^5 - (4y)(2y^2)^3$

32. $(5m^3)(3m)^4 + (-2m^2)(3m^2)^3$

33. $b^3(b^2)^2 + (2b)^4 - b(3b)^3$

34. $q(2q^2)^3 - (3q^3)^2 - q(q^3)^2$

35. $(3c)^2(cd)^2 - (2c^2d)^2 + (5cd^2)^3$

36. $(5u)^2(2v^2)^3 + (3u^2v^2)^3 - (4uv^3)^2$

37. $-3(-2r^2st)^2(-rs^3)^3(rt^2)^4 + (-3s^4t^2)^2(-rs)^3(r^4t^3)^2$

38. $(-x^4yz^2)^3(xy^2)^5(5yz^3)^2 + (-x^3z)^2(3xy^5)^3(-y^2z)^4$

Simplify. Assume that each expression that is an exponent represents a positive integer.

C 39. $x^n \cdot x^n \cdot x^n$

40. $y^{n+1} \cdot y^{n-1}$

41. $(z^n)^2$

42. $(a^{n+1})^2$

43. $(r^{n+1})^2 \cdot r^{n-2}$

44. $(s^2)^{n+1} \cdot s^{1-n}$

Solve for x.

45. $2^x \cdot 2^2 = 2^8$

46. $3^x \cdot 3^x = 3^{16}$

47. $(3^x)^5 = 3^{20}$

48. $(5^x)^x = 5^{16}$

49. $2 \cdot 2^{x-1} = 2^7$

50. $5^x \cdot 5 = 5^{11}$

Computer Exercises

For students with computer experience

1. Without using the computer's exponentiation operation, write a program that will compute the value of a^n when you input a value for a and a positive integral value for n.
2. Modify the program that you wrote for Exercise 1 so that it will allow you to input *two* numbers, a and b, and it will compute the values of a^n, b^n, and $(ab)^n$.
3. Write a program that will evaluate a polynomial of the form $ax^3 + bx^2 + cx + d$ when you input values for a, b, c, and d as well as a value for x. (*Note*: Use the computer's exponentiation operation, but be aware that it will often produce only *approximations* of the correct value because of the method that the computer uses to calculate the result.)
4. Write a program that will perform the same task as the program that you wrote for Exercise 3, but this time do *not* use the computer's exponentiation operation. The method that you use should be based on the fact that

$$ax^3 + bx^2 + cx + d = [(ax + b)x + c]x + d.$$

5. Modify the program that you wrote for Exercise 4 so that it will evaluate any polynomial of degree n in the variable x when you input a value for n. (*Hint*: Write the program so that the values of x and n are input first. Then have the program evaluate the polynomial as the coefficients are input, adding each coefficient to the running total and multiplying this total by x.)

7–3 Multiplying Polynomials

To multiply a polynomial *by a monomial*, you use the distributive axiom together with the laws of exponents for multiplication. For example:

$$\begin{aligned} 2x(3x + 7) &= 2x(3x) + 2x(7) \\ &= 6x^2 + 14x \end{aligned}$$

The expression $6x^2 + 14x$ is called the *product* of the monomial $2x$ and the polynomial $3x + 7$.

EXAMPLE 1 Simplify $-3a^3(2a^2 - 5a + 2)$.

SOLUTION

$$\begin{aligned} -3a^3(2a^2 - 5a + 2) &= -3a^3(2a^2) + (-3a^3)(-5a) + (-3a^3)(2) \\ &= -6a^5 + 15a^4 - 6a^3 \end{aligned}$$

EXAMPLE 2 Simplify $2m^2(3m + 1) - 5m(m + 4)$.

SOLUTION Follow the order of operations and first perform the indicated multiplications. Then simplify the resulting polynomial by adding the similar terms.

$$\begin{aligned} &2m^2(3m + 1) - 5m(m + 4) \\ &\qquad = [2m^2(3m) + 2m^2(1)] - [5m(m) + 5m(4)] \\ &\qquad = (6m^3 + 2m^2) - (5m^2 + 20m) \\ &\qquad = 6m^3 + 2m^2 + (-5m^2) + (-20m) \\ &\qquad = 6m^3 - 3m^2 - 20m \end{aligned}$$

To multiply a polynomial *by a polynomial*, you multiply each term of one of the polynomials by each term of the other, then add similar terms as shown in Example 2. For example:

$$\begin{aligned} (x + 4)(2x + 5) &= x(2x + 5) + 4(2x + 5) \\ &= x(2x) + x(5) + 4(2x) + 4(5) \\ &= 2x^2 + 5x + 8x + 20 \\ &= 2x^2 + 13x + 20 \end{aligned}$$

If you wish, you can use a vertical arrangement when you multiply two polynomials, as shown in the following example.

EXAMPLE 3 Simplify $(y + 2)(y^3 - 5y^2 + 2)$.

SOLUTION

$$\begin{array}{rrrrrl} y^3 & -5y^2 & & +2 & & \\ y & +2 & & & & \\ \hline y^4 & -5y^3 & & +2y & & \longleftarrow \{\, y(y^3 - 5y^2 + 2) \\ & 2y^3 & -10y^2 & & +4 & \longleftarrow \{\, 2(y^3 - 5y^2 + 2) \\ \hline y^4 & -3y^3 & -10y^2 & +2y & +4 & \longleftarrow \{(y + 2)(y^3 - 5y^2 + 2) \end{array}$$

Notice in the first line of the solution to Example 3 that a space was left for the "missing term" $0y$.

Sometimes you will need to multiply polynomials in the process of solving an equation.

EXAMPLE 4 Solve $4n(n + 2) = 4(n^2 - 4)$.

SOLUTION

$$\begin{aligned} 4n(n + 2) &= 4(n^2 - 4) \\ 4n^2 + 8n &= 4n^2 - 16 \\ 4n^2 + 8n - 4n^2 &= 4n^2 - 16 - 4n^2 \\ 8n &= -16 \\ n &= -2 \end{aligned}$$

Check:

$$\begin{aligned} 4n(n + 2) &= 4(n^2 - 4) \\ 4(-2)(-2 + 2) &\stackrel{?}{=} 4[(-2)^2 - 4] \\ 4(-2)(0) &\stackrel{?}{=} 4(4 - 4) \\ 0 &= 0 \quad \checkmark \end{aligned}$$

$\therefore$ the solution set is $\{-2\}$.

Oral Exercises

Simplify.

1. $3(w - 2)$
2. $-2(a + 5)$
3. $-4(3 - c)$
4. $(5 - z)5$
5. $(3a + 2)(-2)$
6. $-3(4d - 1)$
7. $f(2f - 3)$
8. $g(5 - 4g)$
9. $u^2(2u + 7)$
10. $-p^2(3p - 4)$
11. $3c(4 - 5c^2)$
12. $5v(3v^2 + 8)$

Replace each $\underline{\ ?\ }$ with the number or expression that makes a true statement.

13. $(3b + 2)(2b + 3) = \underline{\ ?\ }(2b + 3) + \underline{\ ?\ }(2b + 3)$
14. $(7a + 1)(2a - 3) = (7a + 1)\underline{\ ?\ } - (7a + 1)\underline{\ ?\ }$
15. $(5w - 3)(3w - 2) = 5w(\underline{\ ?\ }) - 3(\underline{\ ?\ })$
16. $(8v - 9)(v + 5) = (8v - 9)\underline{\ ?\ } + (8v - 9)\underline{\ ?\ }$
17. $(2f - 5)(f^2 + 3f - 1) = \underline{\ ?\ }(f^2 + 3f - 1) - \underline{\ ?\ }(f^2 + 3f - 1)$
18. $(4u + 3)(2u^2 - u + 2) = 4u(\underline{\ ?\ }) + 3(\underline{\ ?\ })$
19. $(2m + 3)(m + 4) = 2m(m + 4) + 3(m + 4)$
    $= \underline{\ ?\ }m^2 + \underline{\ ?\ }m + \underline{\ ?\ }$
20. $(3d - 2)(2d + 5) = 3d(2d + 5) - 2(2d + 5)$
    $= \underline{\ ?\ }d^2 + \underline{\ ?\ }d + \underline{\ ?\ }$

Written Exercises

Simplify.

A

1. $2m(m^2 + 5m - 9)$
2. $(4n^2 - n + 7)(-3n)$
3. $z^2(z^2 - 5z + 2)$
4. $y^4(5 - 2y - y^2)$
5. $-a^3(3 - 7a - 4a^2)$
6. $(c^4 + 3c^2 - 5)(-c^5)$
7. $4s^2(3s^3 - 2s^2 + s - 5)$
8. $-3t^3(2t^3 + 5t^2 - 3t - 1)$
9. $2pq(p^3 - p^2q + pq^2 - q^3)$
10. $3c^2d(2d^3 - 5cd^2 + c^2d - 4c^3)$
11. $-2u^2v^3(5v^4 - 2u^2v^2 + 3u^4)$
12. $5j^3k^4(3j^4 - j^3k + jk^3 - 7k^4)$
13. $x^2(2x + 1) + 2x(x - 5)$
14. $y^3(4y - 3) - y^2(y^2 + 2)$
15. $2a^2(4a - 3) - a(6a^2 - a)$
16. $b^3(3b - 1) - 3b^2(b^2 - 7b)$
17. $2j(4j^2 + 3j) - (j + 1)j^2$
18. $3k^2(k - 2) - (4k - 1)k^2$
19. $(a + 3)(a + 1)$
20. $(z - 1)(z - 4)$
21. $(m + 2)(m - 4)$
22. $(c + 5)(c - 2)$
23. $(2b + 1)(b + 3)$
24. $(3w + 2)(w - 4)$
25. $(4t - 3)(2t + 3)$
26. $(5c + 3)(5c - 3)$
27. $(3z - 7)(3z + 7)$
28. $(2x - 5)(5x + 6)$
29. $(2c + 5)^2$
30. $(3d - 2)^2$

B

31. $(r + 3)(r^2 - 2r - 2)$
32. $(s - 2)(3s^2 + 2s - 3)$
33. $(2c^2 - 3c + 1)(c - 3)$
34. $(3n^2 + 5n + 2)(n + 4)$
35. $(u - v)(u^2 + uv - v^2)$
36. $(x + y)(x^2 - xy + y^2)$
37. $(2c + d)(3c^2 - 2cd - 5d^2)$
38. $(3a - b)(5a^2 + 3ab - 2b^2)$

Solve.

39. $3a(a + 1) = 3(a^2 + 5)$
40. $2(b^2 - 8) = 2b(b + 4)$
41. $2x(x + 3) = 2x(x + 1) - 20$
42. $4y(y + 1) = y(4y + 3) - 7$
43. $r(3r + 8) - 15 = 3r(r + 1)$
44. $2t(2t + 3) + 14 = t(4t - 1)$

Simplify.

C

45. $(x + 1)(x - 2)(x + 3)$
46. $(y - 2)(y + 3)(y - 5)$
47. $(m + 2)(2m - 1)(3m + 1)$
48. $(4n - 1)(n + 2)(3n - 2)$
49. $(r + 2)^3$
50. $(2s + 1)^3$
51. $(x^2 - 2x + 3)(x^2 + 2x + 1)$
52. $(3a^2 + 2ab + b^2)(a^2 - ab - b^2)$

Simplify. Assume that *n* is a positive integer.

53. $x^n(x^n + 1)$
54. $x^n(x^n + x)$
55. $x^{n+1}(x^n + 1)$
56. $x^{n+1}(x^n + x)$
57. Explain why the degree of the product of two nonzero polynomials is equal to the sum of the degrees of the polynomials.

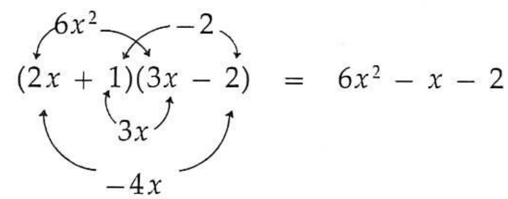
7–4 Multiplying Binomials Mentally

When you multiply two binomials, you usually can save time if you learn to perform the multiplication mentally. For example, the multiplication at the right shows the steps used to find the product of the binomials $2x + 1$ and $3x - 2$. The procedure for obtaining each term of the product is outlined below.

$$\begin{array}{rrrr} & 2x & +\ 1 & \\ & 3x & -\ 2 & \\ \hline 6x^2 & +\ 3x & & \\ & -\ 4x & -\ 2 & \\ \hline 6x^2 & -\ x & -\ 2 & \end{array}$$

1. Multiply the *first terms* of the binomials.
2. Multiply the *first term* of each binomial by the *second term* of the other; add the products if possible.
3. Multiply the *second terms* of the binomials.

When the binomials are arranged horizontally, this procedure can be shown as follows.



Special patterns emerge when you multiply certain pairs of binomial factors. For example, the multiplication at the right shows the steps used to *square the binomial* $a + b$. Again, notice how each term of the product is obtained.

$$\begin{array}{rrrr} a & +\ b & & \\ a & +\ b & & \\ \hline a^2 & +\ ab & & \\ & ab & +\ b^2 & \\ \hline a^2 & +\ 2ab & +\ b^2 & \end{array}$$

1. Square the first term of the binomial.
2. Double the product of the two terms.
3. Square the second term of the binomial.

Thus,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Similarly, when you square the binomial $a - b$, you obtain the result

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Because each of the expressions $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ is a trinomial that can be obtained by squaring a binomial, a polynomial that can be written in one of these forms is called a **trinomial square.** For example, since

$$m^2 + 6m + 9 = m^2 + 2(3)m + 3^2 = (m + 3)^2,$$

the trinomial $m^2 + 6m + 9$ is a trinomial square.

EXAMPLE 1 Simplify.

a. $(a + 5)^2$ b. $(3b - 2)^2$ c. $(2x - 5y)^2$ d. $(4z^3 + 1)^2$

SOLUTION

a. $(a + 5)^2 = a^2 + 2(a)(5) + 5^2$
$= a^2 + 10a + 25$

b. $(3b - 2)^2 = (3b)^2 - 2(3b)(2) + 2^2$
$= 9b^2 - 12b + 4$

c. $(2x - 5y)^2 = (2x)^2 - 2(2x)(5y) + (5y)^2$
$= 4x^2 - 20xy + 25y^2$

d. $(4z^3 + 1)^2 = (4z^3)^2 + 2(4z^3)(1) + 1^2$
$= 16z^6 + 8z^3 + 1$

You can extend the procedure for finding the square of a binomial to finding higher powers. For example, to find the *cube* of the binomial $a + b$, observe the following.

$$(a + b)^3 = (a + b)(a + b)^2$$
$$= (a + b)(a^2 + 2ab + b^2)$$

$$\begin{array}{rrrr} a^2 + & 2ab + & b^2 & \\ a + & b & & \\ \hline a^3 + 2a^2b + & ab^2 & & \\ & a^2b + & 2ab^2 + & b^3 \\ \hline a^3 + 3a^2b + & 3ab^2 + & b^3 & \end{array}$$

To complete the multiplication, you can use a vertical arrangement as shown at the right. Thus,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Similarly, the cube of the binomial $a - b$ is

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Another special pattern occurs when you multiply two binomials in which *the first terms are the same*, but *the second terms are opposites of each other*. For example, the multiplication at the right shows the steps used to find the product of the binomials $a + b$ and $a - b$. Notice that two terms of the product, ab and $-ab$, are opposites, and so their sum is zero. Thus, you obtain the result

$$\begin{array}{rrr} a + & b & \\ a - & b & \\ \hline a^2 + & ab & \\ & - ab & - b^2 \\ \hline a^2 & & - b^2 \end{array}$$

$$(a + b)(a - b) = a^2 - b^2.$$

A polynomial that can be written in the form $a^2 - b^2$ is called a **difference of squares.**

EXAMPLE 2 Simplify.

a. $(s + 4)(s - 4)$
b. $(3t - 1)(3t + 1)$
c. $(m^2 + 7n)(m^2 - 7n)$

SOLUTION

a. $(s + 4)(s - 4) = s^2 - 4^2 = s^2 - 16$
b. $(3t - 1)(3t + 1) = (3t)^2 - 1^2 = 9t^2 - 1$
c. $(m^2 + 7n)(m^2 - 7n) = (m^2)^2 - (7n)^2 = m^4 - 49n^2$

Sometimes you can mentally find the product of two real numbers if you recognize that one factor is the *sum* of two particular numbers and the other factor is the *difference* of these same numbers. You can compute such a product by finding the difference of the squares of these numbers.

EXAMPLE 3 Simplify $51 \cdot 49$.

SOLUTION

$51 = 50 + 1$

$49 = 50 - 1$

$51 \cdot 49 = (50 + 1)(50 - 1) = 50^2 - 1^2 = 2500 - 1 = 2499$

$\therefore 51 \cdot 49 = 2499$

In solving a word problem, you sometimes need to multiply polynomials.

EXAMPLE 4 A certain rectangle is three times as long as it is wide. A second rectangle is 4 cm longer and 1 cm narrower than the first rectangle, but the areas of the two rectangles are equal. Find the dimensions of the first rectangle.

SOLUTION

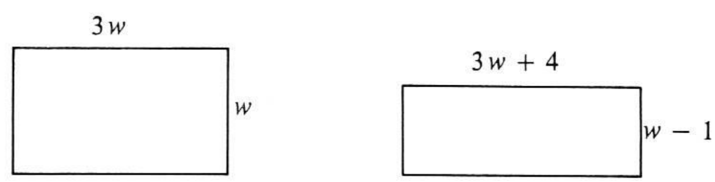
Step 1 The problem asks for the dimensions of the first rectangle.

Step 2 Let w = the width of the first rectangle. Then:

$3w$ = the length of the first rectangle

$w - 1$ = the width of the second rectangle

$3w + 4$ = the length of the second rectangle



Step 3 $\underbrace{\text{Area of first rectangle}}_{3w(w)}$ equals $\underbrace{\text{area of second rectangle}}_{(3w + 4)(w - 1)}$

$$3w(w) \;=\; (3w + 4)(w - 1)$$

Step 4

$$\begin{aligned} 3w(w) &= (3w + 4)(w - 1) \\ 3w^2 &= 3w^2 + w - 4 \\ 3w^2 - 3w^2 &= 3w^2 + w - 4 - 3w^2 \\ 0 &= w - 4 \\ 4 &= w \end{aligned}$$

width = 4

length = 3(4) = 12

Step 5 Checking the results is left to you.

$\therefore$ the first rectangle is 4 cm wide and 12 cm long.

Oral Exercises

Name the coefficient of x in each simplified product.

1. $(x + 3)(x + 1)$
2. $(x - 3)(x - 1)$
3. $(x + 3)(x - 1)$
4. $(x - 3)(x + 1)$
5. $(x + 3)(x - 3)$
6. $(x - 3)(x + 3)$
7. $(x + 3)(x + 3)$
8. $(x - 3)(x - 3)$

Complete.

9. $(a + 1)(a + 4) = a^2 + \underline{?}\,a + 4$
10. $(b - 2)(b - 3) = b^2 + \underline{?}\,b + 6$
11. $(m - 5)(m + 2) = m^2 + \underline{?}\,m - 10$
12. $(n + 4)(n - 3) = n^2 + n + \underline{?}$
13. $(3x + 2)(x + 1) = 3x^2 + \underline{?}\,x + 2$
14. $(y - 4)(2y + 5) = 2y^2 + \underline{?}\,y - 20$
15. $(r - 5)(r - 5) = r^2 - 10r + \underline{?}$
16. $(s - 3)(s + 3) = s^2 + \underline{?}\,s - 9$
17. $(v + 7)^2 = v^2 + \underline{?}\,v + 49$
18. $(3w - 1)^2 = 9w^2 + \underline{?}\,w + 1$
19. $(2c + 3)^2 = 4c^2 + \underline{?}\,c + 9$
20. $(4d - 1)(3d + 2) = 12d^2 + \underline{?}\,d - 2$

Written Exercises

Simplify.

A

1. $(j + 5)(j - 3)$
2. $(k - 2)(k + 5)$
3. $(3 + u)(7 + u)$
4. $(7 - v)(1 - v)$
5. $(s + 11)(s - 11)$
6. $(t - 9)(t + 9)$
7. $(m - 7)^2$
8. $(n + 10)^2$
9. $(3y - 1)(y + 2)$
10. $(2z - 5)(z + 3)$
11. $(2c + 3)^2$
12. $(3s - 7)^2$
13. $(2p + 1)(3p - 2)$
14. $(5q - 3)(5q + 3)$
15. $(4a - 7)(4a + 7)$
16. $(4b - 3)(2b + 3)$
17. $(4r - 3s)(3r + 2s)$
18. $(3j + 4k)(2j - 3k)$
19. $(6c - 5d)(2c - 3d)$
20. $(3m + 5n)(4m + 3n)$
21. $(5b + 2c)^2$
22. $(4x - 3y)^2$
23. $(3a^2 + 1)(3a^2 - 1)$
24. $(2z^3 - 5)(2z^3 + 5)$
25. $(4m^2 + 3)^2$
26. $(5n^3 - 1)^2$
27. $(j^2 - k^2)^2$
28. $(s^3 + t^3)^2$
29. $(5c^3 - d^3)^2$
30. $(p^2 + 3q^2)^2$

B

31. $(x + 2)^3$
32. $(y - 1)^3$
33. $(3m - 1)^3$
34. $(2n + 3)^3$
35. $(a^2 + 1)^3$
36. $(b^2 - 2)^3$

Compute each product by first rewriting it as a difference of squares.

37. $99 \cdot 101$
38. $38 \cdot 42$
39. $28 \cdot 32$
40. $81 \cdot 79$
41. $87 \cdot 93$
42. $16 \cdot 24$
43. $8.9 \cdot 9.1$
44. $48\frac{1}{2} \cdot 51\frac{1}{2}$

Solve.

45. $(4x + 1)(x - 2) = (2x - 3)(2x + 3)$
46. $(4y - 1)(4y + 1) = (8y - 3)(2y + 1)$
47. $(8a - 3)(a + 1) = (4a + 5)(2a - 1)$
48. $(6b - 5)(2b + 1) = (4b - 3)(3b + 1)$
49. $(2m - 1)^2 = (4m + 1)(m - 2)$
50. $(n - 2)(4n + 7) = (2n + 1)^2$
51. $(x + 1)^2 - x^2 = 3(x - 2)$
52. $y^2 - (y - 1)^2 = 3(y + 1)$
53. $(r - 1)^2 - (r + 1)^2 = 2(r + 4)$
54. $(s + 1)^2 - (s - 1)^2 = 2(s - 3)$

Simplify. Assume that n represents a positive integer.

C 55. $(x^n + y^n)^2$

56. $(x^{n+1} - y)^2$

57. $(x^n - y^n)^3$

58. $(x^{2n} + y)(x^{2n} - y)$

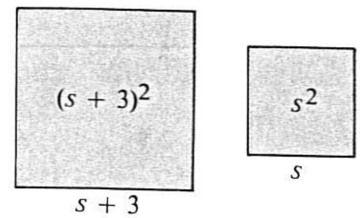
59. Show that the absolute value of the difference of the squares of two consecutive integers is equal to the absolute value of the sum of the integers.

60. Show that the absolute value of the difference of the squares of two consecutive even integers is equal to twice the absolute value of the sum of the integers.

Problems

Solve.

A 1. The side of one square is 3 cm longer than the side of a second square, and the area of the second square is 51 cm² less than the area of the first square. Find the length of a side of the second square.



Ex. 1

2. A rectangle is 2 cm wider and 5 cm longer than a certain square. The area of the rectangle is 38 cm² greater than that of the square. What are the dimensions of the rectangle?

3. The difference of the squares of two consecutive positive even integers is 164. Find the integers.

4. The difference of the squares of two consecutive negative integers is 9. Find the integers.

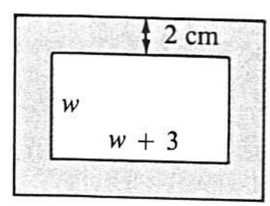
B 5. The product of two consecutive positive odd integers is 38 less than the square of the greater integer. Find the integers.

6. The product of three consecutive integers is 22 less than the cube of the middle integer. Find the integers.

7. The height of a rectangular box is 2 cm less than the width. The length of the box is 5 cm greater than the width. The sum of the areas of the top and bottom of the box is 98 cm² greater than the sum of the areas of the ends. Find the dimensions of the box.

8. A rectangular box is 4 cm longer and 3 cm narrower than a certain cube. The rectangular box and the cube have equal heights and equal surface areas. Find the length and width of the rectangular box.

C 9. A rectangular picture is 3 cm longer than it is wide. The picture is surrounded by a frame that is 2 cm wide. The area of the picture and the frame together is 100 cm² greater than the area of the picture. Find the width of the picture.



Ex. 9

10. A certain castle, twice as long as it is wide, is surrounded by a moat 6 m wide. The area of the castle floor (the area of the rectangle inside the moat) is 864 m^2 less than the area of the region bounded by the outer edge of the moat. Find the dimensions of the castle.

11. Find three consecutive integers such that the product of the first and third is one less than the square of the second.

12. Kevin has three boxes, each the shape of a cube. An edge of the largest box is 2 cm longer than an edge of the middle box, which in turn is 2 cm longer than an edge of the smallest box. The surface area of the largest box is 240 cm^2 greater than the surface area of the smallest box. Find the measure of an edge of each box.

Self-Test 1

VOCABULARY

monomial (p. 323)
constant monomial or constant (p. 323)
degree of a variable in a monomial (p. 323)
degree of a monomial (p. 324)
coefficient or numerical coefficient of a monomial (p. 324)
similar or like monomials (p. 324)
polynomial (p. 324)
term of a polynomial (p. 324)
coefficients of a polynomial (p. 324)
binomial (p. 324)
trinomial (p. 324)
simplest form of a polynomial (p. 324)
degree of a polynomial (p. 324)
laws of exponents for multiplication (p. 329)
trinomial square (p. 335)
difference of squares (p. 336)

Add or subtract the polynomials as indicated.

1. $(3r^2 - 2r + 6) + (r^2 + r - 2)$ *Obj. 1, p. 323*
2. $(5c^2 + 2c) - (4c^3 + 2c^2 + c - 1)$
3. $(8u^2v - 3uv^2) + (2u^2 - 3u^2v + v^2)$

Simplify.

4. $p^2 \cdot p^5$ 5. $(4a^3)(-2a^2)$ 6. $(3x^2y)(xy^3)$ *Obj. 2, p. 323*
7. $(m^2)^5$ 8. $(-3st)^2$ 9. $(2u^2v^3)^3$ *Obj. 3, p. 323*
10. $5y(y^2 - 2y)$ 11. $a^2b(3a^2 - 2ab + b^2)$ *Obj. 4, p. 323*
12. $(c - 2)(c + 3)$ 13. $(2d - 5)(d + 4)$ *Obj. 5, p. 323*
14. $(3x - 2)^2$ 15. $(2m - 5n)(2m + 5n)$

Check your answers with those at the back of the book.

Factoring

OBJECTIVES for Sections 7-5 through 7-9:

1. *To factor integers over the set of prime numbers.*
2. *To find the greatest common factor and the least common multiple of two or more monomials.*
3. *To find the greatest monomial factor of a polynomial.*
4. *To factor quadratic polynomials completely.*

7–5 Factoring Monomials

As you have learned, two or more numbers that are multiplied to form a product are called the *factors* of the product. Thus, when a number is expressed as the product of two or more members of a given set, the number is said to be **factored** over that set. The set of numbers from which the factors are selected is called the **factor set.** For example, the number 6 can be factored over the set of *integers* as follows.

$$6 = 1 \cdot 6 \qquad 6 = (-1)(-6) \qquad 6 = 2 \cdot 3 \qquad 6 = (-2)(-3)$$

Although it is also true that

$$6 = \tfrac{1}{3} \cdot 18 \qquad \text{and} \qquad 6 = (-0.5)(-12),$$

in these cases 6 is *not* factored over the set of integers because $\frac{1}{3}$ and -0.5 are not integers. *In this book, integers will be factored over the set of integers unless some other factor set is specified.* When a number is factored over the set of integers, the factors are called *integral factors* of the number.

A set of numbers that is often used as a factor set is the set of *prime numbers.* A **prime number,** or **prime,** is an integer *greater than 1* that has no positive integral factors other than itself and 1. For example, the first ten prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \text{ and } 29.$$

Over the set of primes, then, the only factors of 6 are 2 and 3. Thus, 2 and 3 are called the *prime factors* of 6. The expression of a positive integer as the product of prime factors is called the **prime factorization** of the integer.

To find the prime factorization of a positive integer, you can sometimes proceed in more than one way. Consider the following example.

$$\begin{aligned} 108 &= 2 \cdot 54 \\ &= 2 \cdot 2 \cdot 27 \\ &= 2 \cdot 2 \cdot 3 \cdot 9 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \end{aligned} \qquad \begin{aligned} 108 &= 9 \cdot 12 \\ &= 3 \cdot 3 \cdot 12 \\ &= 3 \cdot 3 \cdot 4 \cdot 3 \\ &= 3 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \end{aligned}$$

Either way, you find that $108 = 2^2 \cdot 3^3$.

In fact, no matter how you proceed, you will obtain the prime factorization $2^2 \cdot 3^3$; it is essentially the *one and only one*, or *unique*, factorization of 108 over the set of prime numbers. Other prime factorizations merely vary the order in which the prime factors 2 and 3 appear.

Once you have found the prime factorization of a number, you can use it to find all the positive integral factors of the number.

EXAMPLE 1 List all the positive integral factors of 108.

SOLUTION The prime factorization of 108 is $2^2 \cdot 3^3$.

The positive integral factors of 108 can be listed as follows.

1	$2^2 = 4$	$2 \cdot 3 = 6$	$2^2 \cdot 3 = 12$
2	$3^2 = 9$	$2 \cdot 3^2 = 18$	$2^2 \cdot 3^2 = 36$
3	$3^3 = 27$	$2 \cdot 3^3 = 54$	$2^2 \cdot 3^3 = 108$

$\therefore$ the positive integral factors of 108 are
1, 2, 3, 4, 9, 27, 6, 18, 54, 12, 36, and 108.

You can also use prime factorization to find the *greatest common factor* and the *least common multiple* of two or more integers. The greatest integer that is a factor of each of the given integers is called their **greatest common factor (GCF)**. The least positive integer that is a multiple of each of the given integers is called their **least common multiple (LCM)**.

EXAMPLE 2 Find the GCF and the LCM of 48 and 60.

SOLUTION First find the prime factorization of each integer.

$$48 = 2^4 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$

The GCF is the product of the lesser powers of each *common* prime factor.

$$\text{GCF} = 2^2 \cdot 3 = 12$$

The LCM is the product of the greater powers of *each* prime factor.

$$\text{LCM} = 2^4 \cdot 3 \cdot 5 = 240$$

$\therefore$ the GCF is 12 and the LCM is 240.

The factoring of monomials is similar to the factoring of integers. For example, since

$$-2x^2 = -2x \cdot x,$$

$-2x^2$ is a *multiple of* $-2x$ and of x, and $-2x$ and x are *factors* of $-2x^2$. In fact, if the factor set is specified as the set of all monomials with integral coefficients, you can list all the factors of $-2x^2$ as follows:

$$1,\ -1,\ 2,\ -2,\ x,\ -x,\ 2x,\ -2x,\ x^2,\ -x^2,\ 2x^2,\ -2x^2$$

In this book, monomials with integral coefficients will be factored over the set of all monomials with integral coefficients unless some other factor set is specified.

Using methods similar to those used with integers, you can also find the *greatest common factor* and the *least common multiple* of two or more monomials. The monomial with the greatest degree and the greatest numerical coefficient that is a factor of each of the given monomials is called their **greatest common factor (GCF).** The monomial with the least degree and the least positive numerical coefficient that is a multiple of each of the given monomials is called their **least common multiple (LCM).**

The GCF of $9ab^2$ and $-6a^2b$ is $3ab$.

The LCM of $9ab^2$ and $-6a^2b$ is $18a^2b^2$.

Note that, although numerical coefficients of monomials may be negative, the numerical coefficients of the GCF and LCM of monomials are always positive.

EXAMPLE 3 **a.** Find the GCF of $27x^2y^2$ and $18x^3yz$.

b. Find the LCM of $27x^2y^2$ and $18x^3yz$.

SOLUTION First find the prime factorizations of the numerical coefficients.

$$27 = 3^3 \qquad 18 = 2 \cdot 3^2$$

a. Find the GCF of the numerical coefficients.

$$\text{GCF} = 3^2 = 9$$

Compare the powers of each variable that occurs in *both* monomials, and choose the power with the *lesser* exponent.

Compare x^2 and x^3; choose x^2.

Compare y^2 and y; choose y.

The GCF of the two monomials is the product of the GCF of the numerical coefficients and the lesser powers of the common variables.

$$\text{GCF} = 9x^2y$$

b. Find the LCM of the numerical coefficients.

$$\text{LCM} = 2 \cdot 3^3 = 54$$

Compare the powers of each variable that occurs in *either* monomial, and choose the power with the *greater* exponent.

Compare x^2 and x^3; choose x^3.

Compare y^2 and y; choose y^2.

Choose z.

The LCM of the two monomials is the product of the LCM of the numerical coefficients and the greater powers of the variables.

$$\text{LCM} = 54x^3y^2z$$

Oral Exercises

Tell whether the given statement is true or false.

1. 2 is a factor of 10 over the set of integers.
2. 2 is a factor of 21 over the set of integers.
3. 0 is a factor of 5 over the set of integers.
4. 5 is a factor of 0 over the set of integers.
5. 2 is a factor of every even integer.
6. 1 is a factor of every odd integer.
7. 1 is a prime number.
8. 7 has no prime factors.
9. 3 and 7 are the prime factors of 21.
10. 1 and 3 are the prime factors of 3.
11. Every integer is a factor of itself.
12. 1 is a factor of every integer.

Give the prime factorization of each integer.

13. 15
14. 24
15. 32
16. 60

Name the GCF and the LCM of each pair of integers.

17. 3 and 7
18. 6 and 14
19. 7 and 49
20. 14 and 49

Name the GCF and the LCM of each pair of monomials.

21. x and y
22. xy and xz
23. x and x^2
24. xy and y^2

Written Exercises

Factor each integer over the set of prime numbers.

A

1. 75
2. 98
3. 154
4. 195
5. 144
6. 200
7. 1296
8. 1024

List all the positive integral factors of each number.

9. 18
10. 36
11. 54
12. 59
13. 101
14. 105
15. 96
16. 216

For each monomial, list all the factors that are monomials with integral coefficients.

17. $-8a$
18. $5b^2$
19. $3rs^2$
20. $2u^2v^2$

Find the GCF and the LCM of each pair of integers.

21. 22, 33
22. 14, 35
23. 45, 75
24. 28, 98
25. 6, 35
26. 15, 22
27. 90, 135
28. 68, 119

Find the GCF and the LCM of each pair of monomials.

29. $6ab^2$, $45a^3b$
30. $48m^2n^2$, $80mn^4$
31. $-48r^5s^2$, $144qr^3s^2$
32. $34u^2v$, $-85u^3v^3w$
33. $-110ab^2c^5$, $-154a^2bc^3$
34. $-105r^2s^2t^3$, $-175rst^4$

Name the monomial factor, if any, by which the first monomial can be multiplied so that the product is the second monomial. In Exercises 47–52, assume that *n* is a positive integer.

B 35. $5j$; $15jk$
36. $-6p$; $-54pq$
37. $9rs^2$; $-45r^3s^3$
38. $-8b^2c$; $96b^2c^2$
39. $-10s^3t^5$; $-130s^6t^6$
40. $-3x^4y$; $-57x^6y^8$
41. $-12uv^2w$; $60u^2v^3w^2$
42. $-8p^2qr$; $56p^3q^3r^3$
43. $9a^2c$; $54a^2b^3c^4$
44. $-14f^3g^2$; $-42f^4g^4h^4$
45. $-16r^2s^2t^3$; 0
46. 0; $-4x^2y^4z^3$

C 47. $5x^n$; $15x^{n+1}$
48. $-6y^n$; $-48y^{n+2}$
49. $2a^n$; $-18a^{2n}$
50. $-7z^n$; $42z^{3n}$
51. $-3j^n$; $-18j^{2n+1}$
52. $-4k^n$; $-32k^{5n+3}$

53. The GCF of two monomials is $3a^3b^4$ and their LCM is $15a^7b^6$. If one of the monomials is $15a^3b^6$, find the other monomial.

Computer Exercises

For students with computer experience

1. Write a program that will determine whether or not one positive integer is a factor of another by determining whether any integral multiple of the first integer is equal to the second.
2. Write a program that will determine *all* the positive integral factors of a given positive integer.
3. Modify the program that you wrote for Exercise 2 so that it will determine all the *common* factors of two given positive integers.
4. Write a program that will determine the LCM of two given positive integers. (*Hint*: Start with either of the two integers, say a, and determine whether or not it is a multiple of the other. If it is, print it. If it is not, try $2a$, $3a$, and so on until you find a multiple of a that is also a multiple of the other integer.)
5. Write a program that will determine whether or not a given positive integer is a prime number.
6. Modify the program that you wrote for Exercise 2 so that it will determine all the *prime* factors of a given positive integer. Be sure that repeated factors are listed the number of times that they occur in the unique prime factorization of the integer. (*Hint*: Try each prime number starting with 2 and determine whether or not it is a factor of the given integer. If it is, print it, then try it as a factor again. If it is not, go on to the next greater prime number.)

7–6 Factoring Monomials from Polynomials

To factor a polynomial, you express it as a product of polynomials that are members of a specified factor set. *In this book, polynomials with integral coefficients will be factored over the set of all polynomials with integral coefficients unless some other factor set is specified.*

The first step in factoring a polynomial that is in simplest form is to determine its *greatest monomial factor*. The **greatest monomial factor** of a polynomial is the greatest common factor of its terms. If there is a greatest monomial factor other than 1, you use the distributive axiom to rewrite the given polynomial as the product of this greatest monomial factor and a polynomial whose greatest monomial factor is 1.

EXAMPLE 1 Factor. **a.** $6x^4 - 15x^3 + 3x^2$ **b.** $4m^3n - 7m^2n^2$

SOLUTION **a.** The GCF of all the terms is $3x^2$.

$\therefore 6x^4 - 15x^3 + 3x^2 = 3x^2(2x^2 - 5x + 1)$

b. The GCF of both terms is m^2n.

$\therefore 4m^3n - 7m^2n^2 = m^2n(4m - 7n)$

Sometimes you can use the distributive axiom to factor a polynomial that is *not* in simplest form by recognizing a common *binomial* factor.

EXAMPLE 2 Factor. **a.** $y(y - 3) + 7(y - 3)$ **b.** $a(z^2 + 5) - (z^2 + 5)$

SOLUTION **a.** The common binomial factor is $(y - 3)$.

$\therefore y(y - 3) + 7(y - 3) = (y - 3)(y + 7)$

b. The common binomial factor is $(z^2 + 5)$.

$$\begin{aligned} a(z^2 + 5) - (z^2 + 5) &= a(z^2 + 5) - 1(z^2 + 5) \\ &= (z^2 + 5)(a - 1) \end{aligned}$$

$\therefore a(z^2 + 5) - (z^2 + 5) = (z^2 + 5)(a - 1)$

In working with common binomial factors, you should learn to recognize factors that are opposites of each other. For example:

$$x - y = x + (-y) = -y + x = -(y - x)$$

EXAMPLE 3 Factor $n(n - 3) - 7(3 - n)$.

SOLUTION Notice that $n - 3$ and $3 - n$ are opposites.

$$\begin{aligned} n(n - 3) - 7(3 - n) &= n(n - 3) - 7[-(n - 3)] \\ &= n(n - 3) + 7(n - 3) \\ &= (n - 3)(n + 7) \end{aligned}$$

Oral Exercises

Name the greatest monomial factor of each polynomial.

1. $3x - 12$
2. $90 + 15y$
3. $a^2 + 5a$
4. $3b - 4b^2$
5. $7c^2 - 21c$
6. $25d + 30d^3$
7. $7t^2 + 15$
8. $9m^2 + n^2$
9. $18rs - 24r$
10. $30g + 54gh$
11. $24uv + 40u^2v^2$
12. $27jk^2 - 18j^2k$
13. $28c^2d^2 - 21c^2d$
14. $36y^3z + 48y^2z^2$
15. $6p + 8p^2 - 4p^3$
16. $12a^3 - 15a^2 - 7a$
17. $4v^4 - 13v^3 + 8v^2$
18. $6w + 10w^3 + 14w^5$

Name a binomial factor of each polynomial.

19. $a(a - 6) + 2(a - 6)$
20. $5(2 + b) - b(2 + b)$
21. $x(x + 8) + (x + 8)$
22. $y(5 - y) - (5 - y)$
23. $m(m - 2) + 7(2 - m)$
24. $n(7 - n) - 10(n - 7)$

Written Exercises

Write each polynomial as the product of its greatest monomial factor and another polynomial.

A

1. $7r + 14s$
2. $3w - 3y^2$
3. $5m^2 - 6m$
4. $9c + 4c^2$
5. $15b^2 + 6b$
6. $12u - 20uv$
7. $10pq - 12p^2q$
8. $6f^2g + 9fg$
9. $36r^3s^2 - 60r^2s^3$
10. $27u^4v^5 - 63u^3v^6$
11. $16x^4y^5 + 16x^2y^2$
12. $10a^2b - 10a^5b^2$
13. $5b^3 - 35b^2 + 10b$
14. $12r - 30r^2 + 6r^3$
15. $21z^5 - 77z^3 - 49z$
16. $20p^4 - 28p^5 + 44p^6$
17. $6b^2c - 4bc^2 + 4bc$
18. $8r^3s^2 - 12rs^3 + 4r^2s$
19. $15y^4z + 10y^3z^2 - 20y^3z^3$
20. $14a^5b^2 + 28a^2b^4 - 21a^2b^3$
21. $45a^2b^4 + 18a^4b^3 - 81a^3b^4$
22. $40u^4v^6 - 48u^3v^5 - 16u^5v^3$
23. $42r^2s^5t^3 + 54r^3s^2t$
24. $84xy^4z^3 - 60x^4yz^2$

Write each polynomial as the product of two binomials.

25. $z(z - 1) + 2(z - 1)$
26. $6(3 + r) + r(3 + r)$
27. $2m(m + 5) - 3(m + 5)$
28. $5t(t - 4) - 6(t - 4)$
29. $d(d - 5) + 7(5 - d)$
30. $p(p - 2) - 4(2 - p)$
31. $9(1 - q) - q(q - 1)$
32. $4(m + 9) - m(9 + m)$

Write each polynomial as the product of two binomials.

EXAMPLE $(z-2)^2 + 9(z-2)$

SOLUTION $(z-2)^2 + 9(z-2) = (z-2)[(z-2)+9]$
$= (z-2)(z+7)$

B

33. $(x+7)^2 + 5(x+7)$
34. $(y-2)^2 - 3(y-2)$
35. $(a-6)^2 - 4(6-a)$
36. $9(b-5) + (5-b)^2$
37. $(m-8)^2 + (m-8)$
38. $(n-10)^2 + (10-n)$
39. $4(z+1)^2 + 3(z+1)$
40. $7(w-3)^2 + 5(w-3)$
41. $2(p-3) + 5(p-3)^2$
42. $8(q+1) - 2(q+1)^2$
43. $3(a-1)^2 + 2(1-a)$
44. $5(z-4)^2 - (4-z)$

Replace each $\underline{\ ?\ }$ with the binomial that will make the sentence a true statement. Assume that n is a positive integer.

C

45. $x^n + x^{n+1} = x^n(\underline{\ ?\ })$
46. $y^{n+3} - y^{n+2} = y^n(\underline{\ ?\ })$
47. $j^n - j^{2n} = j^n(\underline{\ ?\ })$
48. $k^{4n} - k^{5n} = k^{4n}(\underline{\ ?\ })$
49. $r^{9n}s^2 - r^{5n}s^4 = r^{5n}s(\underline{\ ?\ })$
50. $a^3b^{3n} - ab^{2n} = ab^{2n}(\underline{\ ?\ })$
51. $v^nw^{n+3} + v^{n+1}w^n = v^nw^n(\underline{\ ?\ })$
52. $c^{n+5}d^{n+2} - c^{n+1}d^{n+4} = c^{n+1}d^{n+2}(\underline{\ ?\ })$
53. $a^{n+2}b^{3n+1} - a^{n+1}b^{3n+4} = a^nb^{3n}(\underline{\ ?\ })$
54. $x^{4n}y^{2n+1} + x^{2n}y^{2n+5} = x^{2n}y^{2n}(\underline{\ ?\ })$

François Viète

1540–1603

François Viète (or Vieta) was born in Poitou, France. A lawyer by profession, he served as councilor to the parliaments at Rennes and Tours and as royal privy councilor. His hobby was algebra. During the war with Spain, Viète aided Henry IV by decoding Spanish messages. To Viète, code breaking was simply a matter of solving algebraic equations.

Banished from court by his political enemies from 1584 to 1589, Viète turned his full attention to mathematics. He introduced the use of letters to express known numbers, such as constants and coefficients, as well as unknown numbers. He used vowels for the unknown numbers and consonants for the known numbers. Viète also used signs of operation to indicate addition, subtraction, multiplication, and division. Viète simplified the subject of algebra by this use of signs and symbols to replace words.

READING ALGEBRA Prefixes

Many of the words that are used in algebra have *prefixes*. A prefix is a letter or a group of letters that is placed before a *base word* or a *root* to make another word. Usually the prefix changes or modifies the meaning of the base word or root. For example, the prefix *in*, meaning "not," is placed before the base word *consistent* to form the word *inconsistent*, meaning "not consistent."

In your study of algebra, knowing the meaning of the prefix of a word may help you to better understand and remember the word. The following chart lists some of the common prefixes and their meanings, as well as examples of how these prefixes are used in algebra.

Prefix	*Meaning*	*Example*
bi	two	binomial
co	together	coefficient
equi	the same	equivalent
in	not	inequality
mono	one	monomial
poly	many	polynomial
quad	four	quadrant
tri	three	trinomial

Exercises

Identify each term whose definition is given.

1. a mathematical sentence which states that two expressions name the same number
2. an operation that pairs any two real numbers with a third real number
3. two lines that have all their points in common
4. a set that is not finite
5. the number that corresponds to a point on a number line

6. Why is a trinomial of the form $ax^2 + bx + c$ called a *quadratic* trinomial?

7–7 Factoring Special Polynomials

When the greatest monomial factor of a polynomial in simplest form is 1, you may still be able to factor the polynomial if you recognize a special *factor pattern*. In particular, the special products of binomials that you learned in Section 7-4 are used frequently in the factoring of polynomials. That is:

$$\left.\begin{aligned} a^2 + 2ab + b^2 &= (a + b)^2 \\ a^2 - 2ab + b^2 &= (a - b)^2 \end{aligned}\right\} \longrightarrow \textit{trinomial squares}$$

$$a^2 - b^2 = (a + b)(a - b) \;\} \longrightarrow \textit{difference of squares}$$

EXAMPLE 1 Factor.

a. $x^2 + 6x + 9$ **b.** $9y^2 - 12y + 4$ **c.** $z^4 - 25$

SOLUTION

a. $x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$ ⟵ trinomial square
$= (x + 3)^2$

b. $9y^2 - 12y + 4 = (3y)^2 - 2(3y)(2) + 2^2$ ⟵ trinomial square
$= (3y - 2)^2$

c. $z^4 - 25 = (z^2)^2 - 5^2$ ⟵ difference of squares
$= (z^2 + 5)(z^2 - 5)$

Two other special factor patterns occur when a polynomial is a *sum of cubes* or a *difference of cubes*. A **sum of cubes** is a polynomial that can be written in the form $a^3 + b^3$. Similarly, a **difference of cubes** is a polynomial that can be written in the form $a^3 - b^3$. To factor sums and differences of cubes, you use the following patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

EXAMPLE 2 Factor.

a. $r^3 + 27$ **b.** $8s^3 - t^3$ **c.** $q^6 + 1$

SOLUTION

a.
$$\begin{aligned} r^3 + 27 &= r^3 + 3^3 \\ &= (r + 3)[r^2 - (r)(3) + 3^2] \\ &= (r + 3)(r^2 - 3r + 9) \end{aligned}$$

b.
$$\begin{aligned} 8s^3 - t^3 &= (2s)^3 - t^3 \\ &= (2s - t)[(2s)^2 + (2s)(t) + t^2] \\ &= (2s - t)(4s^2 + 2st + t^2) \end{aligned}$$

c.
$$\begin{aligned} q^6 + 1 &= (q^2)^3 + 1^3 \\ &= (q^2 + 1)[(q^2)^2 - (q^2)(1) + 1^2] \\ &= (q^2 + 1)(q^4 - q^2 + 1) \end{aligned}$$

Sometimes you will be able to use more than one method in factoring a given polynomial. For example, the terms of the polynomial

$$3x^5 - 75x^3$$

have a greatest monomial factor, $3x^3$, and so you can write

$$3x^5 - 75x^3 = 3x^3(x^2 - 25).$$

In this factorization, however, notice that the binomial factor, $x^2 - 25$, is itself a difference of squares. Therefore, you can further factor the polynomial as follows.

$$3x^5 - 75x^3 = 3x^3(x + 5)(x - 5)$$

EXAMPLE 3 Factor.

a. $6a^2 + 12a + 6$ **b.** $15b^4 + 15b$ **c.** $c^4 - 81$

SOLUTION

a. $6a^2 + 12a + 6 = 6(a^2 + 2a + 1)$ ← trinomial square

$= 6(a + 1)^2$

b. $15b^4 + 15b = 15b(b^3 + 1)$ ← sum of cubes

$= 15b(b + 1)(b^2 - b + 1)$

c. $c^4 - 81 = (c^2 + 9)(c^2 - 9)$ ← difference of squares

$= (c^2 + 9)(c + 3)(c - 3)$

When you are unable to factor a polynomial either by using a greatest monomial factor or a special factor pattern, you may still be able to *factor by grouping* the terms of the polynomial. In the next example, for instance, notice that the first three terms of the polynomial, when grouped, form a trinomial square. When you factor this trinomial, the original polynomial can then be factored as a difference of squares.

EXAMPLE 4 Factor $m^2 - 6m + 9 - n^2$.

SOLUTION

$$\begin{aligned} m^2 - 6m + 9 - n^2 &= (m^2 - 6m + 9) - n^2 \\ &= (m - 3)^2 - n^2 \\ &= [(m - 3) + n][(m - 3) - n] \\ &= (m + n - 3)(m - n - 3) \end{aligned}$$

You may also find it helpful to rearrange terms in a polynomial before trying to factor it. In the following example, the arrows indicate an appropriate grouping of the terms.

EXAMPLE 5 Factor $3xy - 20zw - 15xz + 4yw$.

SOLUTION

$$\begin{aligned} 3xy - 20zw - 15xz + 4yw &= (3xy - 15xz) + (4yw - 20zw) \\ &= 3x(y - 5z) + 4w(y - 5z) \\ &= (y - 5z)(3x + 4w) \end{aligned}$$

Oral Exercises

Tell whether or not each polynomial is a difference of squares. If it is, give the factored form of the polynomial.

1. $x^2 - 36$
2. $y^2 + 25$
3. $9 - a^2$
4. $8 - b^2$
5. $16m^2 - 49$
6. $1 - 64n^2$
7. $25g^2 - 4h^2$
8. $c^6 - 100$

Tell whether or not the given polynomial is a trinomial square. If it is, give the factored form of the polynomial.

9. $x^2 + 10x + 25$
10. $y^2 - 8y + 16$
11. $z^2 - 7z + 49$
12. $a^2 + 2a + 1$
13. $b^2 + 4b + 4$
14. $x^2 - 2xy + y^2$

State the value or values of k, if any, for which the given polynomial is a trinomial square.

15. $m^2 + 6m + k$
16. $n^2 - 16n + k$
17. $p^2 + kp + 100$
18. $q^2 + kq - 81$
19. $s^2 + kst + t^2$
20. $u^2 + kuv - v^2$

Written Exercises

Factor each polynomial.

A

1. $64z^2 - 25$
2. $100 - 9m^2$
3. $121c^4 - 1$
4. $144 - 49d^6$
5. $64u^2 - 121v^2$
6. $16j^6 - 81k^8$
7. $a^2 + 14a + 49$
8. $b^2 - 16b + 64$
9. $36 - 12r + r^2$
10. $81 + 18s + s^2$
11. $m^2 + 2mn + n^2$
12. $p^2 - 2pq + q^2$
13. $4c^2 + 4c + 1$
14. $9d^2 - 6d + 1$
15. $4g^2 - 20g + 25$
16. $16 + 24h + 9h^2$
17. $25y^2 - 40yz + 16z^2$
18. $9m^2 - 42mn + 49n^2$
19. $r^3 + 8$
20. $z^3 - 27$
21. $64u^3 - 1$
22. $1 + 125v^3$
23. $27j^3 - k^3$
24. $a^9 + 8b^6$

Factor each polynomial in two steps. The first step should be to write the polynomial as the product of its greatest monomial factor and another polynomial.

25. $3c^2 - 75$
26. $18 - 2s^2$
27. $80 + 40x + 5x^2$
28. $7y^2 - 14y + 7$
29. $45v^2 - 120v + 80$
30. $100w^2 - 80w + 16$

B

31. $16m^3 - 36m$
32. $20ab - 45ab^3$
33. $3x^3 + 6x^2 + 3x$
34. $20y^2 - 20y^3 + 5y^4$
35. $50j^3 + 40j^2 + 8j$
36. $27m^2n - 36mn + 12n$

Factor each polynomial by grouping.

37. $a^2 + 12a + 36 - b^2$
38. $c^2 - 4cd + 4d^2 - 25$
39. $25x^2 - 10x + 1 - y^2$
40. $4 + 28s + 49s^2 - t^2$

41. $3a^2 - 3ab + 2a - 2b$

42. $c^2 - c + 3cd - 3d$

43. $5r + r^2 - 5s - rs$

44. $p^2 - 2p + pq - 2q$

45. $10ac + 6bd + 15bc + 4ad$

46. $2xz - 21wy - 14yz + 3wx$

47. $2p^2 + 15qr + 10pq + 3pr$

48. $6a^2 - 5bc + 2ab - 15ac$

Factor each polynomial. Assume that *n* represents a positive integer.

C 49. $x^{2n} - 1$

50. $x^{2n} - y^2$

51. $x^{2n} - 2x^n + 1$

52. $x^{4n} - 6x^{2n} + 9$

53. $x^{3n} - y^3$

54. $x^{4n} - y^2$

Computer Exercises For students with computer experience

1. Write a program that will compute the product of two binomials of the form $px + q$ and $rx + s$ when you input values for p, q, r, and s. A sample output would be

   5X ↑ 2 + 7X + 2.

   (*Note*: Some computers will display a different exponent symbol.)

2. Write a program that will allow you to input values for a, b, and c and will determine whether or not a trinomial of the form $ax^2 + bx + c$ is a trinomial square. If it is, the program should display the binomial factors of the trinomial. A sample output would be

   (4X + 1) ↑ 2.

   If it is *not* a trinomial square, the output should so state.

3. Modify the program that you wrote for Exercise 1 so that it will compute the product of a binomial of the form $px + q$ and a trinomial of the form $rx^2 + sx + t$ when you input values for p, q, r, s, and t. A sample output would be

   6X ↑ 3 + 7X ↑ 2 + 4X + 1.

7–8 Factoring Quadratic Trinomials of the Form $x^2 + bx + c$

A polynomial that can be expressed in the form

$$ax^2 + bx + c,\ a \neq 0,$$

is called a **quadratic polynomial.** For example,

$$x^2 - 5x + 4, \qquad 3y^2 + 9y, \qquad \text{and} \qquad -5z^2$$

are quadratic polynomials. In a quadratic polynomial that is expressed in simplest form, the term of *degree two* is called the **quadratic term;** the term of *degree one* is called the **linear term;** and the numerical term is called the **constant term.**

Thus, in the quadratic polynomial $ax^2 + bx + c$:

ax^2 is the quadratic term,
bx is the linear term, and
c is the constant term.

A quadratic polynomial in which $b \neq 0$ and $c \neq 0$ is called a **quadratic trinomial.** In Section 7-7 you learned to factor a special type of quadratic trinomial, the *trinomial square*. In this section you will learn to factor other quadratic trinomials in which the coefficient of the quadratic term is 1. That is, you will learn a method of factoring quadratic trinomials of the general form $x^2 + bx + c$.

If a quadratic trinomial of the form $x^2 + bx + c$ can be factored, its factors will be binomials of the form $x + p$ and $x + q$. That is,

$$x^2 + bx + c = (x + p)(x + q).$$

Applying the distributive axiom to the right side of this equation, you obtain:

$$\begin{aligned} x^2 + bx + c &= (x + p)x + (x + p)q \\ &= x^2 + px + qx + pq \\ &= x^2 + (p + q)x + pq \end{aligned}$$

Therefore,

$$x^2 + bx + c = x^2 + (p + q)x + pq,$$

and the following relationships must hold true:

$$\begin{aligned} b &= p + q \\ c &= pq \end{aligned}$$

These relationships suggest the following technique for factoring a quadratic trinomial of the form $x^2 + bx + c$.

1. List the pairs of factors of the constant term, c, that have a product equal to c.
2. Find the pair of factors in the list that have a *sum* equal to the coefficient of the linear term, b.
3. Write the factorization as $(x + p)(x + q)$, using the factors in the chosen pair as the values of p and q.

Note that, once you have written a factorization, you should check that it is correct by multiplying the binomial factors to determine if their product is the original trinomial.

EXAMPLE 1 Factor.

a. $x^2 - 5x + 6$ **b.** $y^2 + y - 12$

SOLUTION **a.** 1. The constant term is 6.
Thus the possible factor pairs are:
$1, 6 \quad -1, -6 \quad 2, 3 \quad -2, -3$

2. The coefficient of the linear term is -5.
Choose the factor pair whose sum is -5:
$-2 + (-3) = -5$

3. Write the factorization as $(x - 2)(x - 3)$.

Check: $(x - 2)(x - 3) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$

$\therefore x^2 - 5x + 6 = (x - 2)(x - 3)$

b. 1. The constant term is -12.
Thus the possible factor pairs are:
$1, -12 \quad -1, 12 \quad 2, -6 \quad -2, 6 \quad 3, -4 \quad -3, 4$

2. The coefficient of the linear term is 1.
Choose the factor pair whose sum is 1:
$-3 + 4 = 1$

3. Write the factorization as $(y - 3)(y + 4)$.

Check: $(y - 3)(y + 4) = y^2 - 3y + 4y - 12 = y^2 + y - 12$

$\therefore y^2 + y - 12 = (y - 3)(y + 4)$

As you gain experience in factoring, you will probably learn to review the factors of the constant term mentally instead of writing them down.

When the coefficient of the quadratic term is -1, it is usually helpful to begin by factoring -1 from each term of the trinomial.

EXAMPLE 2 Factor $8 + 2w - w^2$.

SOLUTION First, rearrange the terms of the trinomial in order of decreasing degree.

$$8 + 2w - w^2 = -w^2 + 2w + 8$$

Then factor -1 from each term.

$$-w^2 + 2w + 8 = -(w^2 - 2w - 8)$$

Now factor the trinomial within parentheses.

$$w^2 - 2w - 8 = (w + 2)(w - 4)$$

Finally, return the factor -1 to this product.

$$-(w^2 - 2w - 8) = -(w + 2)(w - 4)$$

$\therefore 8 + 2w - w^2 = -(w + 2)(w - 4)$

Each of the quadratic trinomials in Examples 1 and 2 is said to be *reducible* over the set of polynomials with integral coefficients. A polynomial is **reducible** over a given factor set if it can be expressed as the product of two or more polynomials *of lower positive degree* taken from that set. A polynomial that is *not* reducible over a given factor set is said to be **irreducible** over that set.

EXAMPLE 3 Factor $t^2 - 6t - 5$.

SOLUTION The constant term is -5, and so the only possible factor pairs to consider are 1, -5 and -1, 5.

The coefficient of the linear term is -6, but neither pair of factors has a sum of -6.

$\therefore t^2 - 6t - 5$ is irreducible.

Sometimes a common monomial factor "conceals" a quadratic trinomial in which the coefficient of the quadratic term is 1. Therefore, it is generally a good practice to begin any factorization of a polynomial by determining the greatest monomial factor of its terms.

EXAMPLE 4 Factor.

a. $2a^3 + 10a^2 - 28a$ b. $12b + 15 - 3b^2$ c. $3cd^2 + 6cd + 6c$

SOLUTION

a. $$2a^3 + 10a^2 - 28a = 2a(a^2 + 5a - 14)$$
$$= 2a(a + 7)(a - 2)$$

b. $$12b + 15 - 3b^2 = -3b^2 + 12b + 15$$
$$= -3(b^2 - 4b - 5)$$
$$= -3(b - 5)(b + 1)$$

c. $$3cd^2 + 6cd + 6c = 3c(d^2 + 2d + 2)$$

In part (c) of Example 4, notice that there is no further factorization of $3c(d^2 + 2d + 2)$ because the trinomial factor, $d^2 + 2d + 2$, is irreducible.

Oral Exercises

For each of the following quadratic trinomials, name the quadratic term, the linear term, and the constant term.

1. $x^2 + 5x + 4$
2. $y^2 - 5y - 14$
3. $10 + 3z - z^2$
4. $8 + 2w^2 - 7w$
5. $9u^2 + 7u$
6. $4 + v^2$

For each of the following quadratic trinomials, find a pair of integers whose product is the constant term and whose sum is the coefficient of the linear term.

7. $a^2 + 9a + 18$
8. $b^2 - 7b + 12$
9. $c^2 - 5c - 6$
10. $d^2 + 3d - 10$
11. $f^2 + f - 2$
12. $g^2 - g - 20$

Tell whether each of the following quadratic trinomials is reducible or irreducible.

13. $m^2 + 6m + 8$
14. $n^2 - 6n + 8$
15. $p^2 - 7p + 8$
16. $q^2 + 7q - 8$
17. $x^2 - 9x - 8$
18. $y^2 - 6y - 8$

Written Exercises

Factor. If the trinomial is irreducible, so state.

A

1. $a^2 + 7a + 12$
2. $b^2 - 7b + 10$
3. $r^2 + 4r - 5$
4. $s^2 - s - 12$
5. $j^2 - j + 2$
6. $k^2 - 5k - 6$
7. $g^2 - 5g - 24$
8. $h^2 - 7h + 18$
9. $m^2 - 9m + 20$
10. $n^2 - 2n - 24$
11. $-x^2 - 11x - 24$
12. $-y^2 + 12y - 32$
13. $6 + 7x + x^2$
14. $21 - 10y + y^2$
15. $36 + 9j - j^2$
16. $48 - 2k - k^2$
17. $m^2 + 3mn + 2n^2$
18. $a^2 - 2ab - 3b^2$

Factor each trinomial in two steps. The first step should be to write the trinomial as the product of its greatest monomial factor and another trinomial.

19. $4c^2 - 44c + 120$
20. $3d^2 - 18d - 48$
21. $k^3 - 8k^2 + 15k$
22. $a^3 + 13a^2 + 42a$
23. $m^4 - 7m^3 - 18m^2$
24. $r^5 + 10r^4 + 24r^3$
25. $3v^3 - 3v^2 - 60v$
26. $4w^3 - 28w^2 - 120w$
27. $5b^4 - 15b^3 + 10b^2$
28. $3c^5 - 18c^4 - 48c^3$
29. $-6x^5 + 24x^4 - 18x^3$
30. $-2y^3 + 22y^2 + 24y$

B

31. $2x^2y^2 - 8x^2y - 90x^2$
32. $az^3 + 9az^2 - 22az$
33. $15st + 8s^2t + s^3t$
34. $9q^2 - 10pq^2 + p^2q^2$
35. $40u^3 + 16u^3v - 2u^3v^2$
36. $30xy - 5x^2y - 5x^3y$
37. $s^3 - s^2t - 12st^2$
38. $a^3 + 3a^2b - 28ab^2$
39. $m^4 - 5m^3n + 6m^2n^2$
40. $b^2c^3 + 8bc^4 + 12c^5$
41. $2j^2 - 10j^2k + 8j^2k^2$
42. $3c^4 + 12c^3d - 36c^2d^2$

Determine all positive and negative integral values of *k* for which the given trinomial is reducible.

43. a. $x^2 + kx + 2$
    b. $x^2 + kx - 2$
44. a. $x^2 + kx + 6$
    b. $x^2 + kx - 6$
45. a. $x^2 + kx + 12$
    b. $x^2 + kx - 12$
46. a. $x^2 + kx + 30$
    b. $x^2 + kx - 30$

Factor. In Exercises 51 and 52, assume that *n* represents a positive integer.

C

47. $x^4 - x^2 - 2$
48. $x^6 + 2x^3 - 15$
49. $x^8 + 5x^4 + 6$
50. $x^{10} - x^5 - 30$
51. $x^{2n} + 3x^n - 10$
52. $x^{4n} + 3x^{2n} + 2$

7–9 Factoring Quadratic Trinomials of the Form $ax^2 + bx + c$

In Section 7-8 you learned to factor quadratic trinomials in which the coefficient of the quadratic term is 1. As you will see, the factoring technique developed in that section can be extended to the factoring of quadratic trinomials in which the coefficient of the quadratic term is an integer other than 1.

If a quadratic trinomial of the form $ax^2 + bx + c$ can be factored, its factors will be binomials of the form $px + r$ and $qx + s$. That is,

$$ax^2 + bx + c = (px + r)(qx + s).$$

Applying the distributive axiom to the right side of this equation, you obtain:

$$\begin{aligned} ax^2 + bx + c &= (px + r)qx + (px + r)s \\ &= pqx^2 + rqx + psx + rs \\ &= pqx^2 + (rq + ps)x + rs \end{aligned}$$

Therefore,

$$ax^2 + bx + c = pqx^2 + (rq + ps)x + rs,$$

and the following relationships must hold true:

$$\begin{aligned} a &= pq \\ b &= rq + ps \\ c &= rs \end{aligned}$$

These equations suggest the method for finding the values of p, q, r, and s that is illustrated in the following example.

EXAMPLE 1 Factor $5x^2 + x - 4$.

SOLUTION The coefficient of the quadratic term is 5. The only *positive* factor pair that has a product of 5 is 5, 1. Thus the factorization of the trinomial must *begin* as:

$$(5x \quad)(x \quad)$$

The constant term is -4. The factor pairs that have a product of -4 are 1, -4; -1, 4; and 2, -2. Thus the factorization of the trinomial must *end* as:

$$\begin{aligned} &(\quad 1)(\quad -4), \\ &(\quad -1)(\quad 4), \\ \text{or } &(\quad 2)(\quad -2) \end{aligned}$$

There are six *trial* factorizations to check to determine which gives a linear term of x in its product.

Trial Factorization	*Linear Term*
$(5x + 1)(x - 4)$	$-20x + 1x = -19x$
$(5x - 1)(x + 4)$	$20x + (-1x) = 19x$
$(5x + 4)(x - 1)$	$-5x + 4x = -1x$
$(5x - 4)(x + 1)$	$5x + (-4x) = 1x$
$(5x + 2)(x - 2)$	$-10x + 2x = -8x$
$(5x - 2)(x + 2)$	$10x + (-2x) = 8x$

The fourth factorization gives a linear term of $1x$, or x.

$\therefore 5x^2 + x - 4 = (5x - 4)(x + 1)$

The factorization of a polynomial is said to be a **complete factorization** when each factor is either a monomial or an irreducible polynomial whose greatest monomial factor is 1.

EXAMPLE 2 Factor $12y^3 - 14y^2 - 10y$ completely.

SOLUTION The greatest monomial factor is $2y$. Thus:

$$12y^3 - 14y^2 - 10y = 2y(6y^2 - 7y - 5)$$

Find the binomial factors, if any, of $6y^2 - 7y - 5$.

Trial Factorization	*Linear Term*
$(6y + 1)(y - 5)$	$-30y + 1y = -29y$
$(6y - 1)(y + 5)$	$30y + (-1y) = 29y$
$(6y + 5)(y - 1)$	$-6y + 5y = -1y$
$(6y - 5)(y + 1)$	$6y + (-5y) = 1y$
$(3y + 1)(2y - 5)$	$-15y + 2y = -13y$
$(3y - 1)(2y + 5)$	$15y + (-2y) = 13y$
$(3y + 5)(2y - 1)$	$-3y + 10y = 7y$
$(3y - 5)(2y + 1)$	$3y + (-10y) = -7y$

The last factorization gives a linear term of $-7y$, and so $6y^2 - 7y - 5 = (3y - 5)(2y + 1)$. The greatest monomial factor of each of these binomials is 1.

$\therefore$ factored completely, $12y^3 - 14y^2 - 10y = 2y(3y - 5)(2y + 1)$.

Except for trivial changes in the factors, such as changing their order or introducing -1 as a factor, the complete factorization of a polynomial is *unique*. Thus, each of the following is an equivalent form of the factorization given as an answer in Example 2.

$$2y(2y + 1)(3y - 5)$$
$$2y(-5 + 3y)(1 + 2y)$$
$$-2y(5 - 3y)(1 + 2y)$$

EXAMPLE 3 Factor $15x^4 + 3x^2 - 18$ completely.

SOLUTION
$$\begin{aligned}15x^4 + 3x^2 - 18 &= 3(5x^4 + x^2 - 6)\\ &= 3(5x^2 + 6)(x^2 - 1)\\ &= 3(5x^2 + 6)(x + 1)(x - 1)\end{aligned}$$

Oral Exercises

Name the pairs of expressions that you would try as linear terms and as constant terms of the binomials when looking for binomial factors of the given polynomial.

EXAMPLE $6x^2 + 7x - 10$

SOLUTION Linear terms: $6x$, x and $3x$, $2x$
Constant terms: $1, -10; -1, 10; 2, -5;$ and $-2, 5$

1. $2c^2 + 3c + 1$
2. $3d^2 - 5d + 2$
3. $2m^2 - m - 10$
4. $5n^2 + 3n - 14$
5. $5p^2 - 16p + 12$
6. $2q^2 + 13q + 18$
7. $6a^2 + a - 7$
8. $14b^2 - 3b - 2$
9. $9x^2 - 9x - 10$
10. $4y^2 + 4y - 15$
11. $10r^2 - r - 21$
12. $15s^2 - 2s - 8$

Written Exercises

Factor completely.

A

1. $5a^2 + 7a + 2$
2. $3b^2 - 7b + 2$
3. $2c^2 - 3c - 5$
4. $5d^2 + 2d - 3$
5. $3x^2 - 5x - 8$
6. $2y^2 + 7y + 6$
7. $6r^2 - r - 2$
8. $15s^2 + s - 2$
9. $4z^2 - 16z + 15$
10. $4w^2 + 5w - 21$
11. $8m^2 + 2m - 15$
12. $18n^2 - n - 4$
13. $10j^2 - 37j - 12$
14. $8k^2 - 26k + 21$
15. $12p^2 + 23p + 10$
16. $12q^2 + 19q - 10$
17. $20u^2 + 17u - 10$
18. $20v^2 - 37v - 6$
19. $4x^2 + 24x + 20$
20. $6y^2 - 12y - 18$
21. $4m^2 + 14m + 6$
22. $6n^2 - 22n + 12$
23. $12a^2 + 2a - 4$
24. $12b^2 - 24b - 15$
25. $12m^3 + 6m^2 - 6m$
26. $9n^3 - 3n^2 - 6n$
27. $18z^4 + 12z^3 + 2z^2$
28. $27d^5 - 18d^4 + 3d^3$
29. $16a^5 - 4a^3$
30. $20b^4 - 45c^2$

B

31. $6x^5 - 22x^3 - 8x$
32. $6y^5 + 3y^3 - 9y$
33. $6z^6 - 21z^4 - 12z^2$
34. $3w^6 - 9w^4 + 3w^2$
35. $-12b^5 - b^3 + b$
36. $-18c^7 - c^5 + 4c^3$
37. $6u^3 + 23u^2v - 18uv^2$
38. $10r^3 + 3r^2s - 18rs^2$
39. $12w^4 - 35w^3z + 8w^2z^2$
40. $20x^4 - 7x^3y - 6x^2y^2$
41. $6y^3z + 10y^2z^2 - 4yz^3$
42. $8a^3b^2 + 2a^2b^3 - 3ab^4$

C 43. $x^4 - y^4$ 44. $x^8 - y^8$ 45. $x^6 - y^6$ 46. $x^{10} - y^{10}$

47. $y^4 + y^2 + 1$ (*Hint:* $y^4 + y^2 + 1 = y^4 + 2y^2 + 1 - y^2$)

48. $x^4 - 7x^2 + 9$ (*Hint:* $x^4 - 7x^2 + 9 = x^4 - \underline{\ ?\ } + 9 - \underline{\ ?\ }$)

49. Find integral values of a, b, and c such that $ax + b$ is a factor of both $2x^2 - 5x + c$ and $4x^2 + 4x + 1$.

50. Show that there is an infinite set of integral values of c for which the trinomial $x^2 + x + c$ can be factored over the set of polynomials with integral coefficients.

Self-Test 2

VOCABULARY

factor a number (p. 341)
factor set (p. 341)
prime number, or prime (p. 341)
prime factorization (p. 341)
greatest common factor (GCF) of integers (p. 342)
least common multiple (LCM) of integers (p. 342)
greatest common factor (GCF) of monomials (p. 343)
least common multiple (LCM) of monomials (p. 343)
greatest monomial factor (p. 346)
sum of cubes (p. 350)
difference of cubes (p. 350)
quadratic polynomial (p. 353)
quadratic term (p. 353)
linear term (p. 353)
constant term (p. 353)
quadratic trinomial (p. 354)
reducible (p. 355)
irreducible (p. 355)
complete factorization (p. 359)

1. Factor 240 over the set of prime numbers. *Obj. 1, p. 341*

2. Find the greatest common factor and the least common multiple of $48c^3d^5$ and $32bc^2d^6$. *Obj. 2, p. 341*

Write each polynomial as the product of its greatest monomial factor and another polynomial.

3. $25m^2 - 40m$ 4. $6x^4y^2 - 10x^5y^3 + 14x^6y$ *Obj. 3, p. 341*

Factor completely.

Obj. 4, p. 341

5. $s^2 + 6s + 9$ 6. $81m^2 - 49$
7. $x^2 - 2xy + y^2$ 8. $n^3 - 1$
9. $t^2 - 7t + 12$ 10. $2u^2 - 3u - 20$
11. $3g^3 - 27g$ 12. $6h^3 - 10h^2 + 8h$

Check your answers with those at the back of the book.

PROGRAMMING IN BASIC

The following program can be used to factor a quadratic trinomial of the form $ax^2 + bx + c$ when you input values for a, b, and c. The program uses a *subroutine* in lines 300–350. That is, the line

200 GOSUB 300

takes the execution of the program to line 300, and the line

350 RETURN

takes the execution back to line 210.

```
10  PRINT "TO FIND FACTORS OF"
20  PRINT "A TRINOMIAL OF THE FORM"
30  PRINT "AX ↑ 2 + BX + C"
40  PRINT "INPUT A(>0), B, C(<>0):";
50  INPUT A, B, C
60  IF A <= 0 THEN 40
70  IF C = 0 THEN 40
80  PRINT
90  REM *FIND FACTORS OF A
100  FOR A1 = 1 TO A
110  LET A2 = A/A1
120  IF A2 <> INT(A2) THEN 250
130  IF A2 < A1 THEN 380
140  PRINT "A1 = ";A1,"A2 = ";A2
150  REM *FIND FACTORS OF C
160  FOR C1 = 1 TO ABS(C)
170  LET C2 = C/C1
180  IF C2 <> INT(C2) THEN 240
190  LET C3 = C1
200  GOSUB 300
210  LET C3 = -C3
220  LET C2 = -C2
230  GOSUB 300
240  NEXT C1
250  NEXT A1
260  GOTO 380
270  REM *SUBROUTINE
280  REM *COMPUTE AND PRINT
290  REM *POSSIBLE MIDDLE TERMS
300  PRINT A1;" X ";C2;" + ";
310  PRINT A2;" X ";C3;" = ";
320  LET M = A1 * C2 + A2 * C3
330  PRINT M;" B =";B
340  IF M = B THEN 410
```

```
350  RETURN
360  REM *END OF SUBROUTINE
370  REM *OUTPUT
380  PRINT
390  PRINT "IRREDUCIBLE"
400  GOTO 440
410  PRINT
420  PRINT "(";A1;"X + (";C3;"))";
430  PRINT "(";A2;"X + (";C2;"))"
440  END
```

Exercises

Type in the program as given. Then RUN the program to factor the following quadratic trinomials.

1. $x^2 + 3x + 4$

2. $x^2 + 4x + 4$

3. $x^2 - 12x + 36$

4. $2x^2 + 14x - 36$

5. $8x^2 + 5x - 6$

6. $6x^2 + 29x + 35$

Applications of Factoring

OBJECTIVES for Sections 7-10 and 7-11:
1. To solve polynomial equations by factoring.
2. To use quadratic equations to solve word problems.

7–10 Solving Polynomial Equations by Factoring

An equation that can be written equivalently in the form

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

is called a **quadratic equation.** In this section you will learn to solve certain quadratic equations by using the factoring techniques that have been presented in this chapter. The basis of this method of solution is the *zero-product property* of real numbers.

To understand the zero-product property, first note that the multiplicative property of zero guarantees the following theorem to be true.

> *Theorem.* For all real numbers a and b,
>
> if $a = 0$ or $b = 0$, then $ab = 0$.

Compare the theorem on the preceding page with the following one, which will be proved in Exercise 53 on page 367.

Theorem. For all real numbers a and b,

$$\text{if } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

As you can see, the hypothesis of the first theorem, $a = 0$ or $b = 0$, is the conclusion of the second theorem. Moreover, the hypothesis of the second theorem, $ab = 0$, is the conclusion of the first theorem. Thus, each of these theorems is said to be the *converse* of the other. Two statements are **converses** when the hypothesis of each statement is the conclusion of the other statement.

A statement and its converse can be combined by using the phrase *if and only if*. Thus the following *zero-product property* is a combination of the two preceding theorems.

Zero-Product Property

For all real numbers a and b,

$$ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

The zero-product property can be shown to be true for any number of factors. In general, then, the zero-product property states that *a product is equal to zero if and only if at least one of its factors is equal to zero.* This property is often used in solving quadratic equations.

EXAMPLE 1 Solve $(2x - 3)(x + 5) = 0$.

SOLUTION Using the zero-product property, write the equivalent disjunction.

$$2x - 3 = 0 \quad \text{or} \quad x + 5 = 0$$

$$x = \frac{3}{2} \quad \Big| \quad x = -5$$

Check *both* solutions in the original equation.

$$(2x - 3)(x + 5) = 0$$
$$\left[2\left(\frac{3}{2}\right) - 3\right]\left(\frac{3}{2} + 5\right) \stackrel{?}{=} 0$$
$$(0)\left(\frac{13}{2}\right) \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

$$(2x - 3)(x + 5) = 0$$
$$[2(-5) - 3](-5 + 5) \stackrel{?}{=} 0$$
$$(-13)(0) \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

$\therefore$ the solution set is $\left\{\frac{3}{2}, -5\right\}$.

EXAMPLE 2 Solve $y^2 = 5y + 36$.

SOLUTION

1. Transform the given equation into an equivalent equation that has 0 as one side. $\quad y^2 - 5y - 36 = 0$
2. Factor the trinomial completely. $\quad (y + 4)(y - 9) = 0$
3. Solve the equivalent disjunction.

$$y + 4 = 0 \quad \text{or} \quad y - 9 = 0$$
$$y = -4 \quad \Big| \quad y = 9$$

4. Check both solutions in the original equation.

$$y^2 = 5y + 36 \qquad y^2 = 5y + 36$$
$$(-4)^2 \stackrel{?}{=} 5(-4) + 36 \qquad 9^2 \stackrel{?}{=} 5(9) + 36$$
$$16 = 16 \ \checkmark \qquad 81 = 81 \ \checkmark$$

$\therefore$ the solution set is $\{-4, 9\}$.

In general, a **polynomial equation** is any equation that can be written equivalently with 0 as one side and a polynomial as the other side. Polynomial equations are often named by the term of highest degree. Thus, as you have learned, an equation of the form

$ax + b = 0, \quad a \neq 0,$ is a *linear equation;*

$ax^2 + bx + c = 0, \quad a \neq 0,$ is a *quadratic equation.*

Similarly, an equation of the form

$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0,$ is a **cubic equation.**

Since the zero-product property is true for any number of factors, it can also be used in solving cubic equations and other polynomial equations of higher degree.

EXAMPLE 3 Solve $10a^3 + a^2 = 3a$.

SOLUTION

1. Transform the given equation into an equivalent equation that has 0 as one side. $\quad 10a^3 + a^2 - 3a = 0$
2. Factor the polynomial completely.

$$a(10a^2 + a - 3) = 0$$
$$a(5a + 3)(2a - 1) = 0$$

3. Solve the equivalent disjunction.

$$a = 0 \quad \text{or} \quad 5a + 3 = 0 \quad \text{or} \quad 2a - 1 = 0$$
$$a = 0 \quad \Big| \quad a = -\frac{3}{5} \quad \Big| \quad a = \frac{1}{2}$$

4. Checking the results is left to you.

$\therefore$ the solution set is $\left\{0, -\frac{3}{5}, \frac{1}{2}\right\}$.

Oral Exercises

State a disjunction that is equivalent to the given open sentence.

1. $c(c - 5) = 0$
2. $4x(x + 6) = 0$
3. $(m - 2)(m + 3) = 0$
4. $(w + 1)(w - 1) = 0$
5. $(2a + 1)(a + 3) = 0$
6. $n(3n - 2)(n + 1) = 0$

For each statement in Exercises 7–14:

a. Tell whether the statement is true or false for all real values of the variables.

b. Give the converse of the statement and tell whether it is true or false.

c. If a statement and its converse are *both* true, reword them in combined form using the words "if and only if."

7. If $a + c = b + c$, then $a = b$.
8. If $a = b$, then $ac = bc$.
9. If $x = 3$ and $y = 4$, then $x + y = 7$.
10. If $x^2 = 2x$, then $x^2 - 2x = 0$.
11. If $rs = 1$, then $r = 1$ and $s = 1$.
12. If $a < b$, then $a - c < b - c$.
13. If $-a > 0$, then $a < 0$.
14. If $a > b$, then $ac > bc$.

Reword each theorem in the form of two theorems that are converses.

15. For all real numbers a and b, $a > b$ if and only if $a - b > 0$.
16. For all real numbers a and b, $ab > 0$ if and only if either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.
17. For all positive real numbers a and b, $a < b$ if and only if $\frac{1}{a} > \frac{1}{b}$.
18. For all real numbers a and b and all positive real numbers c and d, $\frac{a}{c} < \frac{b}{d}$ if and only if $ad < bc$.

Written Exercises

Solve.

A

1. $(x - 4)(x + 1) = 0$
2. $(y + 3)(y - 2) = 0$
3. $(5m - 1)(2m - 3) = 0$
4. $(3n + 4)(5n + 4) = 0$
5. $a(a - 2)(a + 5) = 0$
6. $b(2b - 1)(b + 3) = 0$
7. $3p(p + 7)(5p - 3) = 0$
8. $5q(3q + 5)(4q + 3) = 0$
9. $w^2 + 3w - 10 = 0$
10. $z^2 + 2z - 24 = 0$
11. $s^2 - 9 = 0$
12. $t^2 - 1 = 0$
13. $3c^2 + 5c - 12 = 0$
14. $8d^2 - 2d - 3 = 0$
15. $6u^2 + 7u = 3$
16. $3v^2 - v = 10$
17. $20f^2 = -21f - 4$
18. $6g^2 = 11g - 5$

19. $29v = 10 + 10v^2$

20. $14w = -16 - 3w^2$

21. $4j^2 = 25$

22. $9k^2 = 49$

23. $3g^2 = 12g$

24. $2h^2 = 5h$

B

25. $a^3 - 5a^2 - 36a = 0$

26. $b^3 - 7b^2 + 12b = 0$

27. $r^3 = 4r^2 - 3r$

28. $s^3 - 4s^2 = 32s$

29. $4x^3 + 2x^2 = 6x$

30. $9y^3 - 3y^2 = 12y$

31. $9z^3 = 25z$

32. $18w^3 = 8w$

33. $x^4 - 8x^2 + 16 = 0$

34. $y^5 - 2y^3 + y = 0$

35. $p^4 - 5p^2 = -4$

36. $q^4 + 36 = 13q^2$

37. $(2m - 1)(m - 3) = 18$

38. $(4n - 3)(2n + 1) = 63$

39. $(3w + 2)^2 = 9$

40. $(5w - 2)^2 = 16$

41. $(p + 3)^2 - p = 15$

42. $(4q - 1)^2 + 2 = 8q$

43. $(4a - 3)^2 + 11a = (a + 2)(a - 2) + 33$

44. $(3b - 5)^2 + 9b = (4 + b)(4 - b)$

Write a quadratic equation that has the given solution set. The equation should be written in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

C

45. $\{3, 7\}$

46. $\{-5, -6\}$

47. $\{-2, 3\}$

48. $\{7, -2\}$

49. $\{\frac{1}{2}, \frac{3}{4}\}$

50. $\{-\frac{2}{5}, -\frac{2}{3}\}$

51. $\{-\frac{3}{2}, \frac{3}{5}\}$

52. $\{\frac{5}{3}, -\frac{3}{4}\}$

Give the reason that justifies each statement in the following proof.

53. Prove: For all real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$.

PROOF

Case 1: If $a = 0$, the conclusion is true whether or not $b = 0$.
Case 2: If $a \neq 0$, reason as follows.

Statements	*Reasons*
1. $\frac{1}{a}$ is a real number.	?
2. $ab = 0$	?
3. $\frac{1}{a}(ab) = \frac{1}{a}(0)$	?
4. $\frac{1}{a}(ab) = b$	?
5. $\frac{1}{a}(0) = 0$	?
6. $\therefore b = 0$	Substitution principle

Therefore, given $ab = 0$, it follows that $a = 0$ or $b = 0$.

54. Prove: For all real numbers a and b, $ab > 0$ if and only if either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

7–11 Solving Problems by Factoring

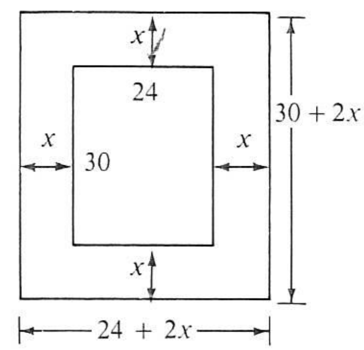
The zero-product property of the real numbers is also used in solving problems that can be represented by quadratic equations. When you use a quadratic equation to solve a problem, you may find that the equation has more than one solution. Remember that each solution of the equation is only a *possible* solution of the problem. Each solution of the equation must be checked against the conditions of the problem to determine the *actual* solutions of the problem.

EXAMPLE 1 The flower garden in the Bay Area Park measures 24 m by 30 m. The garden is surrounded by a paved walk of uniform width. If the combined area of the paved walk and the garden is 1080 m², what is the width of the walk?

SOLUTION

Step 1 The problem asks for the width of the walk.

Step 2 Let $x =$ width of the walk in meters. Then:
$30 + 2x =$ total length in meters;
$24 + 2x =$ total width in meters.



Step 3 The total area of the paved walk and the garden is 1080 m.

$$(30 + 2x)(24 + 2x) = 1080$$

Step 4

$$720 + 48x + 60x + 4x^2 = 1080$$
$$4x^2 + 108x + 720 = 1080$$
$$4x^2 + 108x - 360 = 0$$
$$4(x^2 + 27x - 90) = 0$$
$$4(x + 30)(x - 3) = 0$$

Since $4 \neq 0$, $x + 30 = 0$ or $x - 3 = 0$

$x = -30$ | $x = 3$

Step 5 Since it is not meaningful to have a width of −30, that value is discarded as a possible solution. Check the value 3 in the original problem.
Is the total area 1080 m² when the walk is 3 m wide?

$$\text{total width in meters} = 24 + 2(3) = 30$$
$$\text{total length in meters} = 30 + 2(3) = 36$$
$$\text{total area in square meters} = 30 \cdot 36 = 1080$$

$\therefore$ the width of the walk is 3 m.

In the following example, both solutions of the equation satisfy the conditions of the original problem.

EXAMPLE 2 The height h in meters that an object will reach in t seconds when it is thrown upward from the ground with an initial speed of r meters per second is given by the formula

$$h = rt - 4.9t^2.$$

In how many seconds after it is thrown will an object that is thrown upward from the ground with an initial speed of 34.3 m/s be 49 m above the ground?

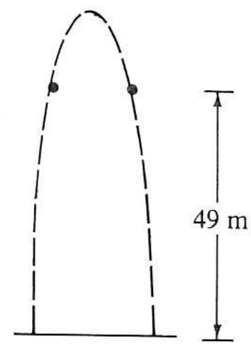
SOLUTION

Step 1 The problem asks for the number of seconds after being thrown upward when the object will be 49 m above the ground.

Step 2 Let t = required number of seconds. Then:

h = height in meters = 49

r = initial speed in m/s = 34.3



Step 3 The given formula is $h = rt - 4.9t^2$

Step 4

$$49 = 34.3t - 4.9t^2$$

$$490 = 343t - 49t^2 \longleftarrow \text{Multiply both sides by 10}$$

$$49t^2 - 343t + 490 = 0$$

$$49(t^2 - 7t + 10) = 0$$

$$49(t - 2)(t - 5) = 0$$

$$t - 2 = 0 \quad \text{or} \quad t - 5 = 0$$

$$t = 2 \quad \Big| \quad t = 5$$

Step 5 Checking the results is left to you.

$\therefore$ the object will be 49 m above the ground both 2 s and 5 s after being thrown upward.

Problems

Solve.

A

1. Find two consecutive positive integers whose product is 132.
2. Find two negative integers that differ by 5 and whose product is 126.
3. The square of a positive integer exceeds four times the integer by 32. Find the integer.
4. The square of a negative integer is 28 greater than three times the integer. Find the integer.
5. Find two consecutive negative integers such that the sum of their squares is 113.
6. Find two integers that differ by 6 while their squares differ by 132.

7. A rectangular garden is 12 m long and 10 m wide. Surrounding the garden is a paved walk of uniform width. The combined area of the garden and the walk is 168 m^2. Find the width of the walk.
8. A rectangular picture is 4 cm longer than it is wide. It is surrounded by a mat that is 2 cm wide. The combined area of the picture and the mat is 140 cm^2. Find the dimensions of the picture.
9. The width of a rectangle is 7 m less than twice the length. The area of the rectangle is 30 m^2. Find the length of the rectangle.
10. A certain rectangle is 3 cm longer than it is wide. The area of the rectangle is 550 cm^2. Find the dimensions of the rectangle.
11. The sum of two integers is 20 and their product is 36. Find the integers.
12. Find two positive integers whose product is 240 and whose difference is 8.

For problems 13–16, use the formula $h = rt - 4.9t^2$.

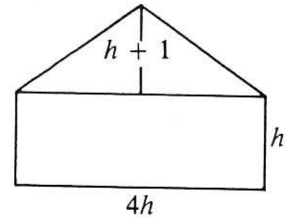
13. A ball is thrown upward with an initial speed of 24.5 m/s. When is the ball 29.4 m high?
14. A projectile is fired upward with an initial speed of 2940 m/s. After how many minutes does it hit the ground?

B

15. A signal flare is fired upward with an initial speed of 245 m/s. A helicopter pilot at a height of 1960 m sees the flare pass on its way upwards. Assuming that the helicopter remains at the same height, how long will it be before the flare passes the helicopter on its way down?

16. A ball is thrown upward from the top of a tower that is 98 m high with an initial speed of 39.2 m/s. When does it hit the ground? (*Hint*: If h is the height of the ball above the tower, then $h = -98$ when the ball hits the ground.)
17. The length of one leg of a right triangle is 2 cm less than three times the length of the other leg. The area of the triangle is 48 cm^2. Find the length of each leg.
18. The length of one side of a triangle is 2 cm less than twice the length of the altitude to that side. The area of the triangle is 30 cm^2. Find the length of the altitude.
19. Marie made a rectangular pen for her dog using a side of the barn for one side and 26 m of fencing for the remaining three sides. If the area enclosed was 72 m^2, find the dimensions of the pen.

20. A side of a house is in the shape of a triangle on top of a rectangle. The rectangle is four times as long as it is high, and the altitude of the triangular part is 1 m greater than the height of the rectangle. The total area of the side of the house is 60 m². Find the height of this side of the house.



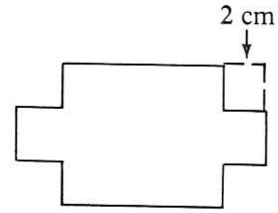
Ex. 20

21. Jackie is k years old. Her brother Joe is k^2 years old. In 7 years, Joe will be one year older than twice Jackie's age at that time. How old is each now?

22. Ken is z years old. His aunt is $(z - 1)^2$ years old. In two years, his aunt's age will be six times Ken's age then. How old is each now?

23. Dan owns a vacant lot that is 32 m wide and 40 m long. He makes a rectangular basketball court in the middle by subtracting equal amounts from the length and width. The area of the court is 560 m². How far from the edge of the lot is an edge of the court?

24. An open rectangular box with length four times its width is made from a rectangular piece of metal by cutting a 2 cm square from each corner and turning up the sides. If the volume of the box is 128 cm³, find the dimensions of the original piece of metal.



Ex. 24

C 25. Show that the sum of the squares of any two consecutive integers is one greater than a multiple of four.

26. Show that the square of an odd integer is one greater than a multiple of eight.

Self-Test 3

VOCABULARY quadratic equation (p. 363)
converse (p. 364)
zero-product property (p. 364)
polynomial equation (p. 365)
cubic equation (p. 365)

Solve.

1. $(m + 2)(m - 7) = 0$
2. $n(3n + 5) = 0$ *Obj. 1, p. 363*
3. $x^2 - 3x - 18 = 0$
4. $2y^3 - 5y^2 - 25y = 0$
5. $6g^2 + 5g = 6$
6. $2h^3 = h^2 + 6h$
7. The width of a certain rectangle is 13 m less than its length. The area of the rectangle is 48 m². Find the dimensions of the rectangle. *Obj. 2, p. 363*
8. Find two consecutive positive integers such that the sum of their squares is 145.

Check your answers with those at the back of the book.

EXTRA

Transformations of the Plane: Reflections

Recall from page 256 that a *translation* is a type of transformation, or mapping, of a plane. In this section you will learn about another type of mapping that is called a **reflection.** In a reflection, each point on a plane is transformed into its mirror image across an *axis of reflection.* For example, if the axis of reflection is the x-axis, as in Figure 1, then the square with vertices (1, 1), (2, 1), (2, 2), and (1, 2) is mapped onto the square with vertices (1, −1), (2, −1), (2, −2), and (1, −2).

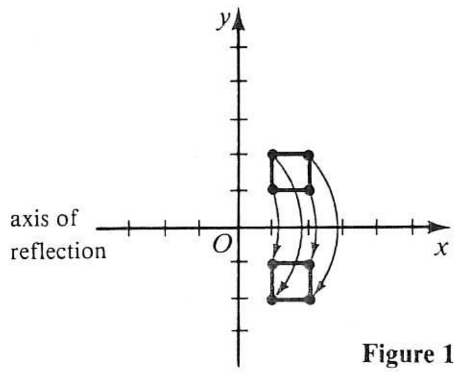


Figure 1

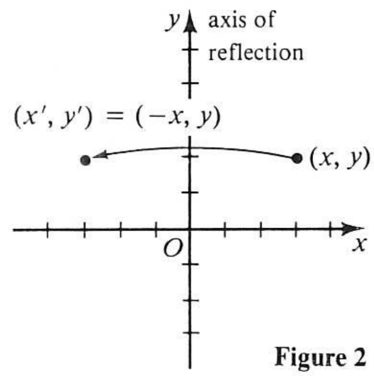


Figure 2

A reflection across the x-axis maps each point (x, y) on the plane onto a new point (x', y') whose coordinates are related to those of the point (x, y) by the *equations of reflection*

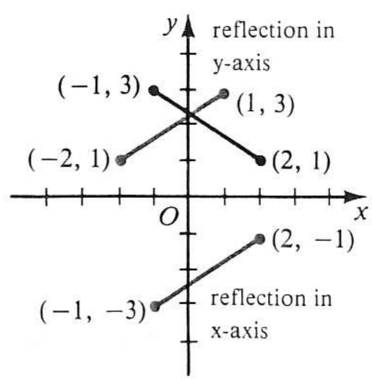
$$x' = x \quad \text{and} \quad y' = -y.$$

Similarly, as suggested by Figure 2, the equations for a reflection across the y-axis are

$$x' = -x \quad \text{and} \quad y' = y.$$

EXAMPLE Sketch the line segment with endpoints (−1, 3) and (2, 1) and then sketch the reflections of this segment across the x-axis and the y-axis.

SOLUTION A reflection across the x-axis maps the endpoints (−1, 3) and (2, 1) onto (−1, −3) and (2, −1), respectively. These points are the endpoints of the reflection of the segment across the x-axis. A reflection across the y-axis maps the endpoints (−1, 3) and (2, 1) onto (1, 3) and (−2, 1), respectively. These points are the endpoints of the reflection of the segment across the y-axis. The reflections of the segment are sketched on the coordinate plane at the right.



The following result is proved in more advanced courses.

Under a reflection:

1. Every line in the plane is mapped (reflected) onto a line in the plane.
2. Every line segment is mapped (reflected) onto a line segment of equal length.
3. Every angle is mapped (reflected) onto an angle of equal measure.

Translations and reflections of the plane are called *rigid transformations* because they preserve the size and shape of geometric figures on the plane. Another rigid transformation of the plane that you will study in later courses is a *rotation* of the plane. In Figure 3, you see the effect of a 45° rotation of the plane on the square with vertices (1, 1), (2, 1), (2, 2), and (1, 2).

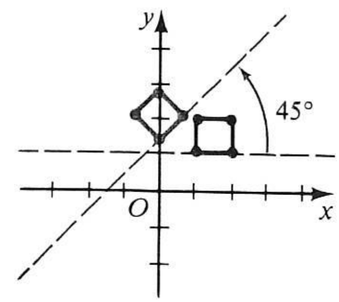


Figure 3

Exercises

In Exercises 1–8, the coordinates of the endpoints of a line segment are given.

a. Sketch the segment on a coordinate plane and label its endpoints with their coordinates.

b. On the same coordinate plane, sketch the reflections of the segment across the x-axis and the y-axis. Label the endpoints of each reflected segment with their coordinates.

1. (3, 1), (5, 0)
2. (0, 2), (3, 1)
3. (−2, 3), (1, 4)
4. (4, −2), (1, 1)
5. (−1, −1), (1, 1)
6. (−1, 1), (1, −1)
7. (1, −4), (3, 2)
8. (−3, 2), (−1, −2)
9. What must be true of a line other than an axis if it is transformed into itself by a reflection across the y-axis? the x-axis?
10. What must be true of a line other than an axis if it is transformed into a parallel line by a reflection across the y-axis? the x-axis?
11. What is the reflection of the line segment with endpoints (2, 1) and (5, 2) across the line $y = x$?
12. What is the reflection of the line segment with endpoints (2, 1) and (5, 2) across the line $y = -x$?

Chapter Summary

1. A *monomial* is a numeral, a variable, or an indicated product of a numeral and one or more variables. A sum of monomials is called a *polynomial*.
2. If a and b are real numbers and m and n are positive integers, the following *laws of exponents* for multiplication are true:

$$a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$$

3. To find the product of two polynomials, multiply each term of one polynomial by each term of the other using the laws of exponents, then simplify the result by adding similar terms.
4. The *prime factorization* of a positive integer is the expression of the integer as the product of primes. Prime factorization can be used to find the *greatest common factor (GCF)* and the *least common multiple (LCM)* of two or more integers.
5. To factor a polynomial, you express it as the product of polynomials that are members of a specified factor set. A factorization of a polynomial is *complete* when each of the factors is either a monomial or a polynomial whose greatest monomial factor is 1.
6. The following three factor patterns occur frequently.

difference of squares: $a^2 - b^2 = (a + b)(a - b)$

trinomial squares: $a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

7. A *quadratic equation* is an equation that can be written equivalently in the form $ax^2 + bx + c = 0, a \neq 0$. Many quadratic equations can be solved by the use of factoring and the *zero-product property*.

Chapter Review

Write the letter of the correct answer.

1. What is the degree of the polynomial $7x^2y^3 - xy^3 + 9x^3y$? 7-1

a. 2 b. 3 c. 4 d. 5

Simplify.

2. $(5m - 7mn - 10n) - (-6m + 8mn + 12n)$

a. $-m + mn + 2n$ b. $-m + 15mn - 2n$

c. $11m - 15mn - 22n$ d. $-11m + mn + 2n$

3. $(6m^3)(-2m^3)^2$ 7-2

a. $4m^8$ b. $-12m^8$ c. $-12m^9$ d. $24m^9$

4. $(5x^3)(3x^3) + (2x^3)^2$

a. $17x^{12}$ **b.** $17x^6$ **c.** $19x^{14}$ **d.** $19x^6$

5. $-a^3(4 - 6a - 4a^3)$ *7-3*

a. $4a^3 + 6a^4 - 4a^6$ **b.** $-4a^3 + 6a^4 + 4a^9$

c. $-4a^3 + 6a^4 + 4a^6$ **d.** $2a^3 + 4a^9$

6. $(b + 4)(b^2 - 2b + 9)$

a. $4b^3 - 8b^2 + 36b$ **b.** $b^2 + b + 13$

c. $b^3 - 2b^2 + 9b + 36$ **d.** $b^3 + 2b^2 + b + 36$

7. $(6 - 5q)(8 + 3q)$ *7-4*

a. $48 + 22q - 15q^2$ **b.** $14 - 58q - 15q^2$

c. $14 - 22q - 15q^2$ **d.** $48 - 22q - 15q^2$

8. $(2r - 5)^3$

a. $8r^3 - 125$ **b.** $8r^3 - 60r^2 + 150r - 125$

c. $6r^3 - 15$ **d.** $4r^3 - 10r^2 + 50r - 50$

9. The difference of the squares of two consecutive positive odd integers is 64. Find the integers.

a. 31, 33 **b.** 9, 11 **c.** 25, 27 **d.** 15, 17

10. Factor 126 over the set of prime numbers. *7-5*

a. $2 \cdot 63$ **b.** $3 \cdot 42$ **c.** $2 \cdot 7 \cdot 9$ **d.** $2 \cdot 3^2 \cdot 7$

11. What is the GCF of 160 and 240?

a. 16 **b.** 24 **c.** 32 **d.** 80

12. What is the LCM of $51x^2z^3$ and $34xz^5$?

a. $17xz^3$ **b.** $102x^2z^5$ **c.** $1734xz^3$ **d.** $1734x^2z^5$

Factor completely.

13. $18r^3s - 24rs^2 + 48r^2s^2$ *7-6*

a. $6rs^2(3r^2 - 4 + 8r)$ **b.** $6r^2s(3r - 4 + 4s)$

c. $6(3r^3s - 4rs^2 + 8r^2s^2)$ **d.** $6rs(3r^2 - 4s + 8rs)$

14. $t(t - 5) + 9(5 - t)$

a. $(t + 9)(t - 5)$ **b.** $(t + 9)(5 - t)$

c. $(t - 9)(t + 5)$ **d.** $(t - 9)(t - 5)$

15. $9w^2 - 24w + 16$ *7-7*

a. $(3w - 4)(3w + 4)$ **b.** $(9w + 4)(w + 4)$

c. $(3w - 4)^2$ **d.** $(3w + 4)^2$

16. $x^2 + zx - xy - zy$

a. $(x - y)(x + z)$ **b.** $(x - z)(x + y)$

c. $x(x + z - y) - zy$ **d.** $x^2 + x(z - y) - zy$

Factor completely.

17. $m^2 - 5m - 24$ 7-8

a. $(m + 8)(m - 3)$ b. $(m - 6)(m + 4)$
c. $(m - 12)(m + 2)$ d. $(m - 8)(m + 3)$

18. $2x^4 - 10x^3 - 12x^2$

a. $2x^2(x - 6)(x + 1)$ b. $2x^2(x - 3)(x - 2)$
c. $2(x^4 - 5x^3 - 6x^2)$ d. $2x^2(x^2 - 5x - 6)$

19. $5n^2 - 18n - 56$ 7-9

a. $(5n - 7)(n + 8)$ b. $(5n - 28)(n + 2)$
c. $(5n - 8)(n + 7)$ d. $(5n + 28)(n - 2)$

20. $12p^6 - 27p^4$

a. $(6p^5 + p^4)(2p - 3)$ b. $(6p^5 - 9p^4)(2p + 3)$
c. $3p^4(2p + 3)(2p - 3)$ d. $(4p^6 - 3)(3 + 9p^4)$

Solve.

21. $a(3a - 1)(a + 4) = 0$ 7-10

a. $\{\frac{1}{3}, -4\}$ b. $\{-\frac{1}{3}, 4\}$ c. $\{0, \frac{1}{3}, -4\}$ d. $\{0, -\frac{1}{3}, 4\}$

22. $13z - 63 = -6z^2$

a. $\{-\frac{7}{3}, \frac{9}{2}\}$ b. $\{\frac{7}{6}, -9\}$ c. $\{\frac{7}{3}, -\frac{9}{2}\}$ d. $\{-\frac{7}{6}, 9\}$

23. A rectangular garden measures 10 m by 8 m. Surrounding it is a brick walk of uniform width. The combined area of the garden and the walk is 120 m². Find the width of the walk. 7-11

a. 10 m b. 2 m c. 1 m d. 8 m

Chapter Test

1. What is the degree of the polynomial $8p^5q - 5p^3q^2 + 7p^2q^3$? 7-1

Simplify.

2. $(-9ab + 3b - 7) + (2ab - 6a + 7)$

3. $(-3g^2h)^2(-2g^3h^2)^2$ 4. $(2k)(4k^5) - (2k^2)^3$ 7-2

5. $-2v^2(v^2 + 3v - 1)$ 6. $(w + 3)(2w^2 - w - 2)$ 7-3

7. $(6c + 7d)(5c - 2d)$ 8. $(9g + 11h)(9g - 11h)$ 7-4

9. A certain rectangle is 4 cm longer than it is wide. A second rectangle is 2 cm longer and 1 cm wider than the first, and its area is 12 cm² greater than the area of the first. Find the dimensions of the first rectangle.

10. Factor 216 over the set of prime numbers. 7-5

11. Find the GCF and the LCM of $120a^3b^2c^2$ and $40a^2c^2$.

Factor completely.

12. $27m^4n^3 + 45m^3n^2 - 18m^2$

13. $q(q - 7) - 2(7 - q)$ 7-6

14. $4x^2 - 28x + 49$

15. $8y^3 + 27$ 7-7

16. $j^2 + 11j + 18$

17. $3k^3 - 21k^2 + 24k$ 7-8

18. $6a^2 - a - 40$

19. $16b^3 - 28b^2 - 30b$ 7-9

Solve.

20. $-4t(5t + 3)(3t - 5) = 0$

21. $3s - 14 = 20s - 6s^2$ 7-10

22. Find two consecutive negative integers such that the difference of their squares is 63. 7-11

Cumulative Review

Chapter 2

Tell whether each statement is true or false.

1. For any natural number a, $a^3 > a$.
2. There exists a whole number w such that $4w + 1 = 15$.
3. The absolute value of any real number is greater than zero.
4. The opposite of any real number is equal to the absolute value of that number.

Name the axiom that is illustrated by each statement.

5. $3(4a) = (3 \cdot 4)a$
6. $(3a - 2b)7 = 7(3a - 2b)$
7. For any real number a, $a + 0 = a$.
8. If $5 \cdot 3 = 15$ and $15 = 45 \div 3$, then $5 \cdot 3 = 45 \div 3$.

Simplify.

9. $\frac{2}{3} \times 25 \times 18 \times \frac{1}{5}$

10. $[3.2 + (-9.7)] + (-10.2 + 4.7)$

11. $(9x - 13) - (-7x - 15)$

12. $-(7y + 2y^2) - (16y^2 - 5y^3) - 6y^3$

13. $9\left(\frac{1}{3}x + \frac{1}{3}y\right) - 10\left(\frac{1}{5}x - \frac{1}{2}y\right)$

14. $\frac{1}{3}(-45v - 9z) + \frac{1}{2}(44v - 32z)$

15. $\frac{3}{4}cd \div \left(-\frac{1}{8}\right)$

16. $\frac{26ab}{-13a}$, $a \neq 0$

Chapter 5

Find the range of each relation. Then tell whether or not the relation is a function.

17. $\{(-2, 1), (-2, 3), (6, 1), (-4, 5)\}$

18. $\{(4, -2), (3, 4), (2, -2), (1, 0)\}$

19. $y = |x + 1|$; $D = \{-2, -1, 0, 1, 2\}$

20. $y = |x| + 1$; $D = \{-2, -1, 0, 1, 2\}$

Given $f: x \longrightarrow 3x^2 - 2x - 1$. Compute the following.

21. $f(0)$

22. $f(-1)$

23. $f(1) - f(3)$

24. $2f(2) - f(-2)$

Graph on a coordinate plane.

25. $x = -2$

26. $x + 2y = 4$

27. $x - y \geq 2$

28. $3x - 2y < 4$

Determine an equation of the line that satisfies the given requirements.

29. has slope $\frac{1}{4}$ and y-intercept 1

30. passes through the point $(-3, 4)$ and has slope $\frac{2}{3}$

31. passes through the point $(-2, -1)$ and is parallel to the y-axis

32. passes through the points $(-2, 4)$ and $(-5, 5)$

Chapter 6

Solve each system by any method.

33. $2y = 21 - 5x$
$7x - 2y = 39$

34. $7y + 10z = 17$
$8y + 15z = 23$

35. $m + 3n = 2$
$2m + 3n = 7$

36. $4a + 3b = 14$
$9a - 14 = 2b$

37. $x + 2y = 6$
$x = 2 - 2y$

38. $9x = 6y - 12$
$y = \frac{3}{2}x + 2$

39. $2r - 3s = 3$
$r - \frac{s}{2} = 2$

40. $2x - 3y = -14$
$\frac{x}{2} - \frac{y}{4} = \frac{9}{4}$

41. $2(a - b) = -5(b - 1)$
$3a - b = 9(a + 1)$

Graph each system on a coordinate plane.

42. $x < -2$
$y \geq 3$

43. $y > -x + 4$
$y \leq 2x - 3$

44. $3x - 5y > 10$
$3y + 2x \geq 6$

Solve.

45. Mike and Lisa live twenty blocks apart in opposite directions from their school. Mike lives one block less than twice as far from the school as Lisa does. How many blocks from the school does Mike live?

46. Flying against the wind, Francine flew her plane 1200 km in 6 h. With no change in the wind, she made the return trip in 4 h. Find the wind speed and the speed of the plane.

47. The sum of the digits of a three-digit number is nine. The units' digit is three times the hundreds' digit and is seven less than twice the tens' digit. Find the number.

48. Frank has 65 coins worth a total of $9.20. Some of the coins are dimes and the rest are quarters. How many of each type of coin does Frank have?

Contest Problems

1. Factor $(x^2 - 5x - 1)^2 - 25$ into a product of binomials of the form $(x - r_1)(x - r_2) \ldots (x - r_n)$. What is the value of $r_1 \cdot r_2 \cdot \ldots \cdot r_n$?

2. Solve the following system:

$$|x - 2| < 3$$
$$x - 1 \geq 0$$
$$2x + 1 \leq 3$$

3. What is the units' digit of the simplified form of 1987^{1987}?

4. Simplify the following expression, given that n is a whole number.

$$[(-1)^n]^2 - [(+1)^n]^3$$

5. Central Junior High School has exactly 1000 students, and there are exactly 1000 lockers in its locker room. On a certain day, the lockers were opened and closed by the students in the following manner. The first student to enter the locker room opened all the lockers. The second student closed each even-numbered locker. The third student *changed* every *third* locker, opening each one that was closed and closing each one that was open. The fourth student changed every fourth locker, the fifth student changed every fifth locker, and so on. After all 1000 students had passed through the locker room, how many lockers were open?