Chapter 6

Systems of Open Sentences

Solving Systems of Linear Equations in Two Variables

OBJECTIVES for Sections 6-1 through 6-4:
1. *To determine whether a system of two linear equations in two variables is consistent or inconsistent.*
2. *To solve a system of two linear equations in two variables by using graphs, addition or subtraction, linear combinations, or substitution.*

6–1 Using Graphs

In Chapter 5, you learned how to solve linear equations. In this chapter you will learn methods for solving *systems of linear equations*.

Two or more linear equations in the same variables, such as

$$x + y = 3 \qquad \text{and} \qquad x - 2y = 0,$$

together form a **system of linear equations** in two variables, or a set of **simultaneous linear equations.** To *solve* a system of linear equations *in two variables*, you find all ordered pairs of numbers that satisfy each equation in the system. Each such ordered pair is called a **solution of the system**; the set of all solutions is called the **solution set of the system.**

One method for solving a system of linear equations in two variables is to graph the equations on the same coordinate plane and find the points that are common to the graphs. Recall that the graph of a linear equation on a coordinate plane is a line. Thus, when two linear equations in two variables are graphed on the same coordinate plane, their graphs will be related in *exactly one* of the following three ways.

1. The graphs will be *parallel lines.*
2. The graphs will be *coincident lines.* (Two lines that have *all* their points in common are called **coincident lines.**)
3. The graphs will intersect in exactly *one* point. (The one point that is common to both lines is called their **point of intersection.**)

EXAMPLE 1 Solve each system of linear equations using graphs.

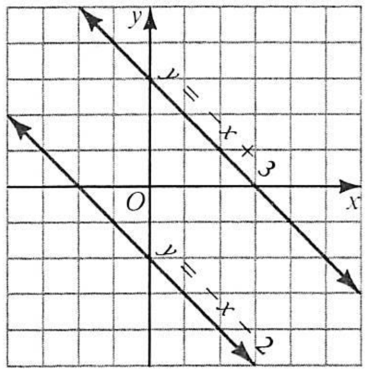
a. $y = -x + 3$
$\quad y = -x - 2$

b. $x + y = 3$
$\quad 2x + 2y = 6$

c. $x + y = 3$
$\quad x - 2y = 0$

SOLUTION Graph each equation in the system on the same coordinate plane.

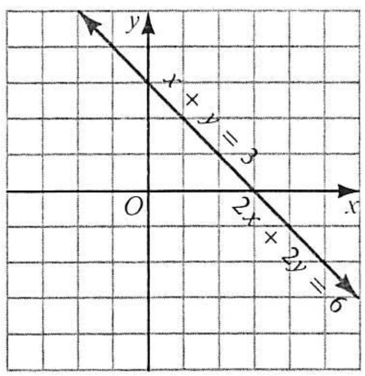
a.



The graphs of these linear equations are parallel lines. The graphs have no points in common, and so the system has no solution.

∴ the solution set is the empty set, $\emptyset$.

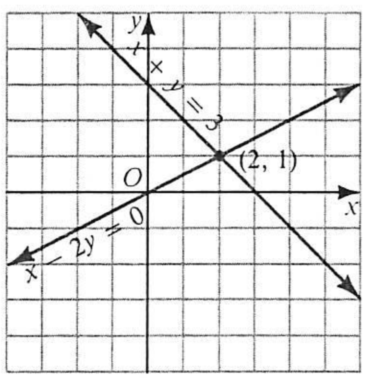
b.



The graphs of these linear equations coincide. Thus, the two equations represent the same line. The graphs have all their points in common, and so all the points on the line are solutions of the system.

∴ the solution set is the infinite set $\{(x, y): x + y = 3\}$.

c.



The graphs of these linear equations intersect in exactly one point, (2, 1). The graphs have one point in common, and so the ordered pair (2, 1) is the only solution of the system.

Check: Does (2, 1) satisfy both of the original equations?

$x + y = 3$	$x - 2y = 0$
$2 + 1 \stackrel{?}{=} 3$	$2 - 2(1) \stackrel{?}{=} 0$
$3 = 3 \quad \surd$	$0 = 0 \quad \surd$

∴ the solution set is $\{(2, 1)\}$.

Notice in part (b) of Example 1 that the solution set is written in set-builder notation and is read as, "the set of all ordered pairs x, y such that x plus y equals 3."

In part (c) of Example 1, the proposed solution (2, 1) was checked in *both* of the given equations. This is essential. Because any linear equation has an infinite number of solutions, you cannot conclude that you have found the solution of the *system* of equations until you have shown that the proposed solution satisfies *each* equation in the system.

The graphing method for solving a system of linear equations in two variables can be summarized as follows.

The Graphing Method

To solve a system of linear equations in two variables graphically:

1. Graph the equations on the same coordinate plane.

2. Determine the coordinates of all points common to the graphs.

A system of equations that has at least one solution is called a **consistent** system. A system of equations whose solution set is the empty set is called an **inconsistent** system. You can tell whether a system of linear equations in two variables representing *nonvertical* lines is consistent or inconsistent by comparing the slope-intercept forms of the equations of the system.

EXAMPLE 2 Determine whether each system of equations in Example 1 is consistent or inconsistent.

SOLUTION

a. $\left.\begin{array}{l} y = -x + 3 \\ y = -x - 2 \end{array}\right\}$ same slope but different y-intercepts

The graphs are parallel lines.
The solution set is the empty set.

$\therefore$ the system is inconsistent.

b. $\begin{array}{rl} x + y = 3 & \longrightarrow \\ 2x + 2y = 6 & \longrightarrow \end{array} \left.\begin{array}{l} y = -x + 3 \\ y = -x + 3 \end{array}\right\}$ same slope and same y-intercept

The graphs are coincident lines.
The solution set is an infinite set.

$\therefore$ the system is consistent.

c. $\begin{array}{rl} x + y = 3 & \longrightarrow \\ x - 2y = 0 & \longrightarrow \end{array} \left.\begin{array}{l} y = -x + 3 \\ y = \frac{1}{2}x \end{array}\right\}$ different slopes

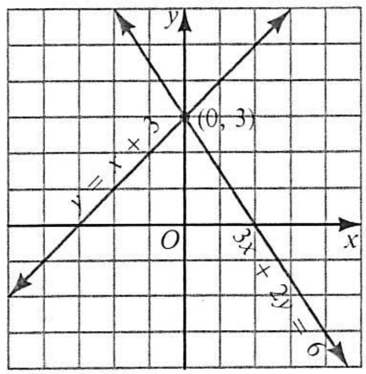
The graphs are lines that intersect in exactly one point.
The solution set has exactly one member.

$\therefore$ the system is consistent.

EXAMPLE 3 Find the point on the graph of $3x + 2y = 6$ where the ordinate is 3 more than the abscissa.

SOLUTION The ordinate is 3 more than the abscissa for every point on the line $y = x + 3$. Thus, you can find the required point by solving the system

$$3x + 2y = 6 \qquad \text{and} \qquad y = x + 3.$$



The graphs intersect at (0, 3).

Check:

Is (0, 3) on the graph of $3x + 2y = 6$?

$$3x + 2y = 6$$
$$3(0) + 2(3) \stackrel{?}{=} 6$$
$$6 = 6 \quad \checkmark$$

Is the ordinate 3 more than the abscissa?

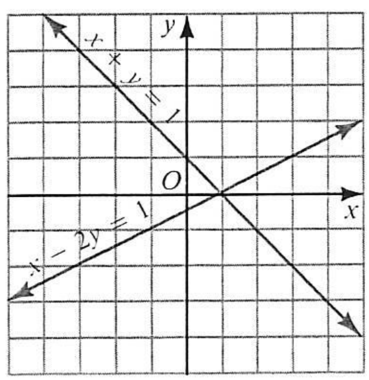
$$y = x + 3$$
$$3 \stackrel{?}{=} 0 + 3$$
$$3 = 3 \quad \checkmark$$

$\therefore$ the solution is the point (0, 3).

Oral Exercises

Each graph represents a system of linear equations. Determine the solution of each system.

1.



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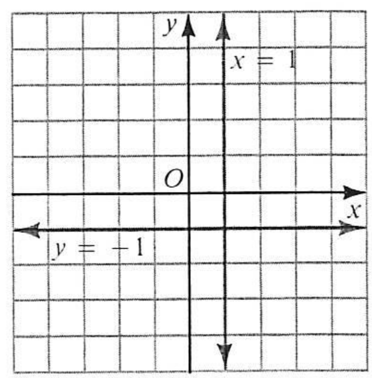

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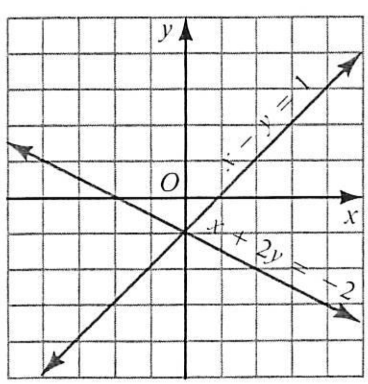

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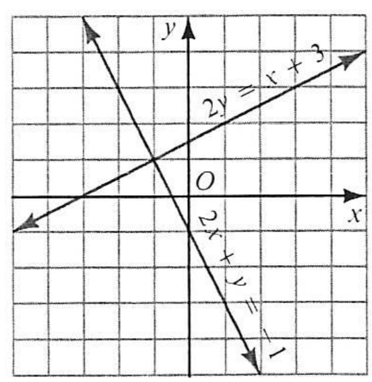


Tell whether or not (−3, −5) is a solution of the given system.

5. $x - y = 2$
$2x + y = -11$

6. $x - 2y = 7$
$2x - 3y = 8$

7. $x = 3y + 12$
$2y = 5x + 5$

8. $x + 2y = -13$
$y = -5$

9. $3x + y = -14$
$x = -3$

10. $x + 3 = 0$
$y + 5 = 0$

The following sentences refer to lines on the same coordinate plane. Replace each __?__ with one of the words or phrases in (a), (b), (c), or (d) to make a true statement.

11. Lines that have the same slope are either coincident or __?__.
a. intersecting **b.** vertical **c.** parallel **d.** none of these

12. Nonvertical lines that intersect in exactly one point have __?__.
a. the same slope **b.** no slope **c.** different slopes **d.** none of these

13. Parallel lines either have the same slope or are __?__.
a. coincident lines **b.** vertical lines **c.** horizontal lines **d.** none of these

14. If a vertical line intersects another line in exactly one point, the second line cannot be __?__.
a. vertical **b.** horizontal **c.** of negative slope **d.** of positive slope

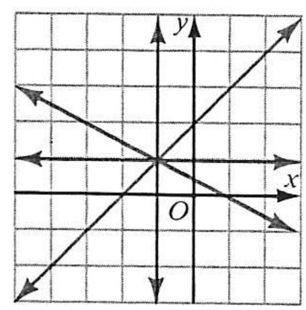
15. If the solution set of a system of two linear equations in x and y is the empty set, the graphs of the equations are __?__.
a. coincident **b.** parallel **c.** intersecting **d.** none of these

16. If a system of two linear equations in x and y has at least two solutions, the graphs of the equations are __?__.
a. parallel lines **b.** coincident lines **c.** intersecting lines. **d.** none of these

17. In the figure at the right you see the graphs of the equations of the system:

$$x = -1$$
$$y = 1$$
$$y = x + 2$$
$$2y + x = 1$$

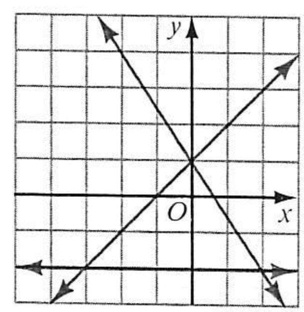
What is the solution set of the system?



18. In the figure at the right you see the graphs of the equations of the system:

$$y = x + 1$$
$$3x + 2y = 2$$
$$y = -2$$

What is the solution set of the system?



Written Exercises

Solve each system of equations using graphs.

A

1. $x + 7 = 0$
   $y - 2 = 0$
2. $y - 3 = -8$
   $x + 1 = -2$
3. $y - x = 0$
   $y + x = 2$
4. $y + x = 0$
   $y - x = -4$
5. $y = -3x + 2$
   $y = -3x - 4$
6. $y = -x + 4$
   $3y = -3x + 12$
7. $y = 2x + 3$
   $2y = 4x + 6$
8. $y = 5x - 2$
   $y = 5x + 1$
9. $y = 2x + 2$
   $3y = -x - 15$
10. $x = 2 - 2y$
    $2x = -2 - y$
11. $x = 4y - 12$
    $2x = 8y - 24$
12. $2x = 3y + 4$
    $6x = 9y - 1$

Determine whether the solution set of the given system is the empty set, an infinite set, or a set with exactly one member. Then tell whether the system is consistent or inconsistent.

13. $y = 3x + 1$
    $y = 2x + 2$
14. $y = -3x + 2$
    $y = -3x - 1$
15. $y + 5x = -1$
    $2y + 10x = -2$
16. $y - 4x = 5$
    $y + 4x = -3$
17. $3y - 2x = -7$
    $3y + 2x = 1$
18. $6y - 9x = -3$
    $2y - 3x = -1$
19. $2y - x = 2$
    $3x + 6 = 6y$
20. $2x + y = -5$
    $4x + 1 = y$
21. $x + 3 = 0$
    $x - 5 = 0$
22. $y - 7 = 4$
    $y + 1 = -3$
23. $3y + 2x - 6 = 0$
    $3 + 2x + 3y = 0$
24. $3y - 2x - 3 = 0$
    $3x - 2y - 8 = 0$

Solve each system of equations using graphs. Estimate the coordinates of the point of intersection to the nearest half unit.

B

25. $x + y = 5$
    $x - y = 2$
26. $x - 3y = 5$
    $x + 3y = 5$
27. $4y + 2 = -3x$
    $4y - 28 = 7x$
28. $-16y + 4 = 4x$
    $4y + 9 = 3x$

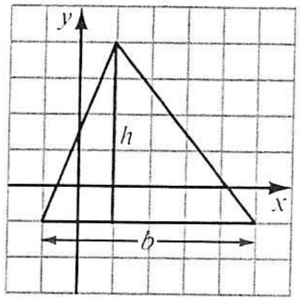
Solve.

29. Find the point on the graph of $x + 2y = 6$ where the ordinate is equal to the abscissa.
30. Find the point on the graph of $3y - x = 15$ where the ordinate is twice the abscissa.
31. Find the point on the graph of $3x + 5y = -10$ where the abscissa is the opposite of the ordinate.
32. Find the point on the graph of $2x - 5y = -13$ where the ordinate is one more than two thirds the abscissa.

In the following diagrams, a base b and the corresponding height h are indicated on a triangle and on a parallelogram. In Exercises 33–38 use the given formulas for area.

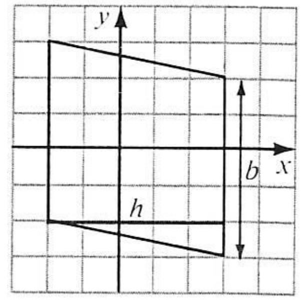
Area $= \frac{1}{2} \times$ base $\times$ height

$A = \frac{1}{2}bh$



Area = base × height

$A = bh$



The points of intersection of the graphs of the given equations are the vertices of a triangle.

a. Find the vertices of the triangle. **b. Find the area of the triangle.**

C

33. $y - x = 1$
$y + x = -3$
$x = 3$

34. $3x + 2y = 8$
$2x - 3y = 1$
$y = -5$

35. $x + 4y = 29$
$y = 4x + 3$
$x = -3$

The points of intersection of the graphs of the given equations are the vertices of a parallelogram.

a. Find the vertices of the parallelogram. **b. Find the area of the parallelogram.**

36. $y = 3x + 3$
$y = 6$
$y = 3x - 3$
$y = 3$

37. $3y - x = 6$
$x = 6$
$3y - x = -18$
$x = -3$

38. $2x + y = 7$
$y = -1$
$2y + 8 = -4x$
$y = -7$

Computer Exercises

For students with computer experience

Write a program that will determine if an ordered pair (x, y) is a solution of a system of the form

$$ax + by = c$$
$$dx + ey = f$$

when you input values for x and y as well as a, b, c, d, e, and f. RUN the program to determine if $(3, -1)$ is a solution of each of the following.

1. $2x + 5y = 1$
$2x - 4y = 1$

2. $x - y = 4$
$x + y = 3$

3. $x + 5y = -2$
$y = -1$

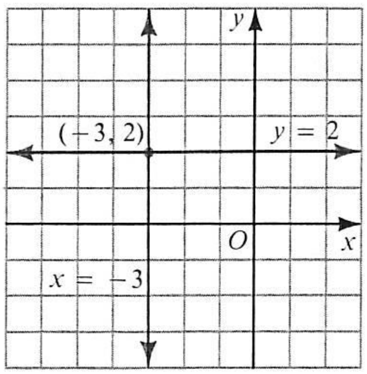
4. $2x = 2y + 8$
$6y = -2x$

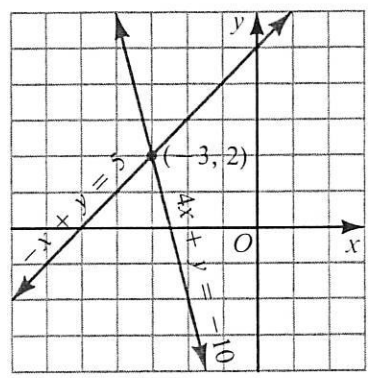
6–2 Using Addition or Subtraction

Often it is difficult to determine the exact solution of a system of linear equations by the graphing method. Therefore, it is important to learn algebraic methods that will enable you to find the exact solution set of any system of linear equations.

To solve a system of linear equations algebraically, you use number properties to find an *equivalent system* that can be solved by inspection. **Equivalent systems** of equations over a given domain are systems that have the same solution set over that domain. For example, consider the following systems of equations.

$$x = -3 \qquad\qquad -x + y = 5$$
$$y = 2 \qquad\qquad 4x + y = -10$$





The solution set is $\{(-3, 2)\}$. The solution set is $\{(-3, 2)\}$.

Since the two systems have the same solution set, they are equivalent systems of equations.

The following two examples show how you can use addition or subtraction to solve a system of equations in two variables in which a coefficient in one equation has the same absolute value as the corresponding coefficient in the other equation.

EXAMPLE 1 Solve the system: $2x - 3y = -6$
$2x - y = 2$

SOLUTION

1. Subtract corresponding terms of the two equations.
$$\begin{array}{r} 2x - 3y = -6 \\ \underline{2x - y = 2} \\ -2y = -8 \end{array}$$
2. Solve for y. $\quad y = 4$
3. Replace one of the original equations with the equation $y = 4$. This system of equations is equivalent to the original system.
$$\begin{array}{r} 2x - 3y = -6 \\ y = 4 \end{array}$$

4. Substitute 4 for y in the first equation of Step 3 and solve for x.

$$\begin{aligned} 2x - 3y &= -6 \\ 2x - 3(4) &= -6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

5. The resulting system is equivalent to the original system and can be solved by inspection.

$$\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}$$

6. Check that (3, 4) satisfies both of the *original* equations.

$$\begin{aligned} 2x - 3y &= -6 \\ 2(3) - 3(4) &\stackrel{?}{=} -6 \\ -6 &= -6 \quad \checkmark \end{aligned} \qquad \begin{aligned} 2x - y &= 2 \\ 2(3) - 4 &\stackrel{?}{=} 2 \\ 2 &= 2 \quad \checkmark \end{aligned}$$

$\therefore$ the solution set of the given system is $\{(3, 4)\}$.

In the first step of Example 1, each term of the second equation was subtracted from the corresponding term of the first equation. Since the coefficients of the variable x were identical, this produced an equation in which the coefficient of x was zero. You can say that the variable x was *eliminated* from the system.

In solving a system of equations similar to the one in Example 1, it is usual to omit some of the steps.

EXAMPLE 2 Solve the system:

$$\begin{aligned} 3s - 2t &= -9 \\ 5s + 2t &= 1 \end{aligned}$$

SOLUTION

1. Add corresponding terms of the two equations to eliminate the variable t.

$$\begin{aligned} 3s - 2t &= -9 \\ 5s + 2t &= 1 \\ \hline 8s \quad\quad &= -8 \end{aligned}$$

2. Solve for s.

$$s = -1$$

3. Substitute -1 for s in either of the original equations.

$$\begin{aligned} 3s - 2t &= -9 \\ 3(-1) - 2t &= -9 \\ -2t &= -6 \\ t &= 3 \end{aligned}$$

4. Check that $(-1, 3)$ satisfies both of the original equations.

$$\begin{aligned} 3s - 2t &= -9 \\ 3(-1) - 2(3) &\stackrel{?}{=} -9 \\ -9 &= -9 \quad \checkmark \end{aligned} \qquad \begin{aligned} 5s + 2t &= 1 \\ 5(-1) + 2(3) &\stackrel{?}{=} 1 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

$\therefore$ the solution set of the given system is $\{(-1, 3)\}$.

In working with equations in two variables, the order of the numbers in the ordered pair representing the solution should be the same as the alphabetical order of the corresponding variables. Thus the solution in the preceding example is given as $(-1, 3)$ to correspond to (s, t).

Oral Exercises

Add or subtract the equations in each system to eliminate one of the variables. Then solve for the other variable.

1. $x + 2y = 12$
   $7x - 2y = 4$

2. $5a + b = -2$
   $2a + b = 1$

3. $5r + 2s = 19$
   $5r - s = 17$

4. $5m - 4n = -6$
   $-3m + 4n = -10$

5. $4e = 6 - 3f$
   $4e = -2 - 7f$

6. $9w = 6 + 4z$
   $-5w = 2 - 4z$

Written Exercises

Solve each system of equations using addition or subtraction.

A

1. $3x + y = 7$
   $4x + y = 13$

2. $4a - 3b = -5$
   $a + 3b = -5$

3. $3s - 2t = -9$
   $-3s + 5t = 21$

4. $4r + 3s = 11$
   $4r - s = 23$

5. $4u + v = 0$
   $4u - 3v = 8$

6. $8c + 5d = 0$
   $11c - 5d = 0$

7. $6w = 1 - 3x$
   $6w = 7 + 6x$

8. $3d = 4e - 16$
   $3d = 12e - 18$

9. $8c = 3b + 18$
   $4c = -3b - 9$

10. $10k = -19 - 8j$
    $-20k = 18 + 8j$

11. $2a - 3b = 8$
    $-2a + 3b = 8$

12. $-7x + 3y = -4$
    $7x - 3y = 4$

B

13. $6p - 5t = -4$
    $5t + 9p = -6$

14. $4t - 3u = -1$
    $9u + 4t = -5$

15. $7f = -38 - 5e$
    $5e = 3f - 23$

16. $8v = 5u - 22$
    $10u = -10 - 8v$

17. $6m = 8 + 10n$
    $11 = 6m - 5n$

18. $13 = 3c + 4d$
    $4d = 7c + 63$

19. $2w + 2x = -2 - 4w$
    $7w + 27 = 3x + w$

20. $5c - 3f = 31 + f$
    $2f - 8 = -2f - 8c$

21. $3(c + d) = 6$
    $3(c - d) = -36$

22. $4(r + s) = -20$
    $4(r - s) = -12$

23. $3(b - 3) - 2c = 0$
    $2(b - c) = -b - 3$

24. $3(j - 1) - 4k = 0$
    $4(j - k) = j + 3$

Solve each system of equations for *a* and *b* in terms of *c*.

C

25. $4a + 3b + c = 18$
    $3a + 3b + c = 17$

26. $3a - 2b + 4c = -28$
    $4a + 2b - 4c = -7$

27. If $a \neq 0$ and $b \neq 0$, under what conditions will the system of equations at the right have no solution?
    $ax + by = c$
    $bx + ay = c$

28. If $a \neq 0$ and $b \neq 0$, solve the system of equations at the right for x and y in terms of a, b, c, and d.
    $ax + by = c$
    $ax - by = d$

Computer Exercises For students with computer experience

Write a program that will solve a system of the form

$$ax + by = c$$
$$dx + ey = f$$

when you input values for a, b, c, d, e, and f as well as two integral replacement sets for x and y. RUN the program to solve the following systems of linear equations.

1. $x - 3y = 1$, $x + y = 13$ $\quad x, y \in \{1, 2, \ldots, 10\}$
2. $2x - 4y = 8$, $2y - x = -4$ $\quad x, y \in \{-5, -4, \ldots, 5\}$
3. $2x + 4y = 6$, $3x + 2y = 3$ $\quad x \in \{0, 1, \ldots, 8\}; \; y \in \{-8, -7, \ldots, 0\}$

6–3 Using Linear Combinations

You can use the multiplicative property of equality to transform one or both of the given equations in a system of equations. This can be done to any given system of equations to produce an equivalent system in which a coefficient in one equation has the same absolute value as the corresponding coefficient in the other. You can then use the addition and subtraction method discussed in the preceding section to find the solution set of the given system.

EXAMPLE 1 Solve the system: $5x - 2y = -25$, $3x + y = -4$

SOLUTION

1. To obtain equations in which the coefficients of the y-terms have the same absolute value, multiply both sides of the second equation by 2.
   $$5x - 2y = -25$$
   $$6x + 2y = -8$$
2. Add the equations.
   $$11x = -33$$
3. Solve for x.
   $$x = -3$$
4. Substitute -3 for x in one of the given equations and solve for y.
   $$3x + y = -4$$
   $$3(-3) + y = -4$$
   $$y = 5$$
5. Checking that $(-3, 5)$ satisfies both of the original equations is left to you.

   $\therefore$ the solution set of the given system is $\{(-3, 5)\}$.

The equation that you obtain by multiplying one equation of a system by a nonzero constant and another equation of the system by another nonzero constant and adding or subtracting the two resulting equations is called a **linear combination** of the given equations. Thus, the equation in Step 2 of Example 1,

$$11x = -33,$$

is a linear combination of the equations in the given system of equations. The method for solving a system of linear equations illustrated in Example 1 is often referred to as the *linear-combination method.*

EXAMPLE 2 Solve the system: $6a + 5b = 9$
$5a + 2b = 14$

SOLUTION

1. Multiply both sides of the first equation by 2. Multiply both sides of the second equation by 5.

$$\begin{aligned} 12a + 10b &= 18 \\ \underline{25a + 10b} &\underline{= 70} \end{aligned}$$

2. Subtract the resulting equations. $\quad -13a = -52$

3. Solve for a. $\quad a = 4$

4. Substitute 4 for a in one of the given equations and solve for b.

$$\begin{aligned} 6a + 5b &= 9 \\ 6(4) + 5b &= 9 \\ 5b &= -15 \\ b &= -3 \end{aligned}$$

5. Checking that $(4, -3)$ satisfies both of the original equations is left to you.

$\therefore$ the solution set of the given system is $\{(4, -3)\}$.

There are several ways to transform each equation in a given system of equations to produce an equivalent system in which a coefficient in one equation has the same absolute value as the corresponding coefficient in the other. For example, in the first step of the preceding solution, you might choose to multiply both sides of the first equation by 2 and to multiply both sides of the second equation by -5.

$$\begin{aligned} 2(6a + 5b) = 2(9) &\longrightarrow 12a + 10b = 18 \\ -5(5a + 2b) = -5(14) &\longrightarrow -25a - 10b = -70 \end{aligned}$$

You would then add the resulting equations to eliminate the variable b.

You could also choose to multiply both sides of the first equation by 5 and to multiply both sides of the second equation by 6.

$$\begin{aligned} 5(6a + 5b) = 5(9) &\longrightarrow 30a + 25b = 45 \\ 6(5a + 2b) = 6(14) &\longrightarrow 30a + 12b = 84 \end{aligned}$$

In this case you would then subtract the resulting equations to eliminate the variable a.

Oral Exercises

In solving each of the following systems of equations by the linear-combination method, explain how you can eliminate one of the variables by answering the following questions.

a. Which variable would you eliminate?

b. Which equation(s) would you transform by multiplication?

c. By what number(s) would you multiply?

d. Would you then add or subtract the equations?

1. $r - 3s = -1$; $2r - 2s = 3$
2. $3y - 4z = 5$; $2y + z = 1$
3. $a + b = 8$; $5a + 8b = -1$
4. $4c + 2d = -3$; $12c - 5d = -2$
5. $9s - 6t = -7$; $18s + 12t = 1$
6. $8w - 20x = 4$; $2w - 5x = 7$
7. $11d - 3e = 4$; $6d + 2e = -5$
8. $4p + 7q = -5$; $5p - 9q = 0$
9. $9f - 2g = -3$; $12f + 7g = 6$
10. $4j - 5k = 7$; $3j + 9k = -8$
11. $5r - 5s = 14$; $8r + 12s = -3$
12. $4c - 9d = 5$; $9c + 4d = -7$

Written Exercises

Solve each system of equations using the linear-combination method.

A

1. $2d - 5e = -3$; $3d + 2e = 5$
2. $2x + 3y = 8$; $6x + 5y = 0$
3. $3p - 4q = 12$; $5p + 8q = 20$
4. $2a + 5b = 3$; $3a - 2b = -5$
5. $3j - 5k = 15$; $4j - 7k = 21$
6. $3w - 4x = -4$; $5w - 7x = -9$
7. $3c + 6d = -7$; $11c + 9d = -30$
8. $8r + 6s = 0$; $6r + 9s = -3$
9. $7x - 2y = 0$; $-14x + 4y = 3$
10. $6a + 15b = -3$; $2a - 5b = -11$
11. $4d - 5e = -10$; $6d + 8e = 16$
12. $8s - 3t = -5$; $-16s + 6t = 10$

Solve each system of equations for *x* and *y* in terms of the other variable(s).

B

13. $5x - n = 2y$; $y = 3n - x$
14. $2x = 5y - 7n$; $-3x + 4y = 7n$
15. $9x + k + 8y = 0$; $12y + k = -15x$
16. $8x - 3cy = 9c$; $12x - 7cy = -39c$
17. $3ax + 2by = -5ab$; $4ax - 5by = 24ab$
18. $8bx + 12ab = -7ay$; $4ay + 9ab = -5bx$

C

19. If a, b, and c are positive integers such that $a = b$ and $c = 3a$, find the vertices of the triangle formed by the x-axis and the graphs of the lines

$$ax + by = c \quad \text{and} \quad by - ax = c.$$

20. a. Solve the system: $9x - 8y = 1$
$6x + 12y = 5$

b. Show that for all nonzero real numbers e and f, the solution of the system in part (a) satisfies the equation

$$e(9x - 8y - 1) + f(6x + 12y - 5) = 0.$$

(This shows that the solution of the system in part (a) is a solution of every linear combination of the equations in that system.)

6–4 Using Substitution

A method that is sometimes easier to use than the linear-combination method in solving a system of linear equations is the *substitution method.* Given two equations in x and y, you can transform one of them to express one variable in terms of the other. You can then use the substitution principle to replace one of the given equations by a third equation involving only one variable. The following example illustrates the substitution method.

EXAMPLE Solve the system: $3x + 7y = -6$
$x - 2y = 11$

SOLUTION

1. Solve the second equation for x in terms of y.

$$\begin{aligned} x - 2y &= 11 \\ x &= 2y + 11 \end{aligned}$$

2. Substitute this expression for x in the first equation, and solve for y.

$$\begin{aligned} 3x + 7y &= -6 \\ 3(2y + 11) + 7y &= -6 \\ 6y + 33 + 7y &= -6 \\ 13y &= -39 \\ y &= -3 \end{aligned}$$

3. Substitute this value of y in the second equation in Step 1, and solve for x.

$$\begin{aligned} x &= 2y + 11 \\ x &= 2(-3) + 11 \\ x &= 5 \end{aligned}$$

4. Checking that $(5, -3)$ satisfies both of the original equations is left to you.

$\therefore$ the solution of the given system of equations is $\{(5, -3)\}$.

In the first step of the preceding example, you could have solved for either variable in either equation. However, it is usually most convenient to solve for a variable that has a coefficient of one.

The following is a summary of the substitution method for solving a system of linear equations in two variables.

The Substitution Method

To solve a system of linear equations in two variables:

1. Solve one equation for one of the variables.
2. Substitute this expression in the other equation and solve.
3. Find the corresponding value of the other variable.
4. Check the solution in both of the original equations.

The transformations used in solving systems of linear equations algebraically are summarized below.

Transformations That Produce an Equivalent System of Linear Equations

1. Replacing any equation of the system with an equivalent equation in the same variables.
2. Replacing any equation with a linear combination of itself and another equation of the system.
3. Substituting for one variable in any equation either
   (a) the actual value of the variable, or
   (b) an equivalent expression for that variable obtained from another equation of the system.

Oral Exercises

To use the substitution method, first select one equation. Using that equation, solve for one variable in terms of the other. In Exercises 1–6:

a. Which equation would you select?

b. Which variable would you solve for?

1. $2a + b = 7$
   $5a - 3b = 1$

2. $2u - 3v = -2$
   $2u - v = 15$

3. $4d - 3e = 1$
   $d - 7e = -4$

4. $w + 8x = -3$
   $4w - 3x = 7$

5. $2m - n = 5$
   $2m + 5n = -2$

6. $3 - r = 2s$
   $5 + 7r = 5s$

Solve each system of equations using the substitution method.

7. $c = 3d$
   $c + d = 16$

8. $s = 5r$
   $r + s = -6$

9. $v = u - 3$
   $u + v = 13$

10. $f = e + 7$
    $f - 2e = 4$

11. $m - 2n = 7$
    $m = n + 2$

12. $8 - p = q$
    $q = 2p - 10$

Written Exercises

Solve each system of equations using the substitution method.

A

1. $e + 2f = 8$
   $2e - 3f = -19$
2. $v + 4w = 5$
   $5v - 7w = -2$
3. $p - 6q = 2$
   $2p - 7q = 9$
4. $x - 8y = 4$
   $3x + 5y = 12$
5. $2j - 15k = 7$
   $j - 6k = 4$
6. $6d - 2e = 2$
   $e + 7d = 3$
7. $2m - n = 13$
   $3m - 8n = 13$
8. $2a - b = 7$
   $4a - 3b = 9$
9. $8c - 7d = 6$
   $5c - d = -3$
10. $3v + 9w = 6$
    $v + 3w = 2$
11. $8b + 2c = 4$
    $4b + c = 1$
12. $2s + 3t = 15$
    $9s - t = -5$
13. $8w = x - 1$
    $2x = 10w + 5$
14. $3f = g - 11$
    $3g = 2f + 19$
15. $n = \frac{2}{3}m$
    $2m + 7n = 4$
16. $s = \frac{3}{4}r$
    $9r - 4s = -2$
17. $2x + 6y = 12$
    $3x + 4y = 13$
18. $4a + 12b = -8$
    $3a - 5b = 8$

Solve each system of equations by any method. (You may transform any equation in a system of equations into an equivalent equation before you begin to solve the system.)

B

19. $3x + 5y = -9$
    $5x - 2y = 16$
20. $6p - 7q = 11$
    $8p + 3q = -47$
21. $4(a + b) = 6(b - 1)$
    $8(a + 1) = b + 1$
22. $4(e + f) = 8(f - 4)$
    $2(e - 1) = f - 15$
23. $7(t + u) = 5 + 6t$
    $8(t - 1) = 2(1 + 3u)$
24. $2x - 3y = 8$
    $2(2x - 8) = 6y$
25. $5a + 2b = -5$
    $4(a + b) = 6(2 - a)$
26. $5(p + q) = 9q - 1$
    $2(2q + 3) = 5(p + 1)$
27. $\frac{y}{7} + z = \frac{-11}{7}$
    $y - \frac{5z}{3} = 2$
28. $\frac{h}{2} + k = \frac{1}{2}$
    $\frac{h}{7} + 1 = \frac{4k}{7}$
29. $\frac{x}{2} + \frac{y}{3} = 6$
    $3x - 2y = 12$
30. $\frac{s}{5} - \frac{t}{3} = -2$
    $5s - 3t = 14$

In Exercises 31 and 32 two equivalent systems of linear equations are given. Find the values of *a* and *b*.

C

31. $-2x + 3y = -13$ $\quad -2ax + by = 1$
    $4x + y = 5$ $\quad ax + by = 4$
32. $3x - 4y = -11$ $\quad ax + 4by = 19$
    $5x + 2y = -1$ $\quad ax + 2by = 7$
33. The graphs of the equations $2ax + by = -20$ and $-ax + 2by = -10$ intersect at $(-2, 2)$. Find a and b.
34. The graphs of the equations $-3ax + 5by = 14$ and $2ax - 4by = -4$ intersect at $(\frac{1}{3}, -1)$. Find a and b.

35. Show that if $m_1 = m_2$, but $b_1 \neq b_2$, then the following system has no solution.

$$y = m_1x + b_1$$
$$y = m_2x + b_2$$

36. Show that any solution of the system

$$ax + by + c = 0$$
$$y = mx + k$$

is also a solution of the equation $ax + b(mx + k) + c = 0$.

Self-Test 1

VOCABULARY

system of linear equations, or simultaneous linear equations (p. 267)
solution of a system (p. 267)
coincident lines (p. 268)
point of intersection (p. 268)
consistent system of equations (p. 269)
inconsistent system of equations (p. 269)
equivalent systems (p. 274)
linear combination (p. 278)
substitution method (p. 281)

1. Tell whether the given system is consistent or inconsistent. *Obj. 1, p. 267*

$$3y + 2x = 6$$
$$6y - 6 = -4x$$

2. Solve the given system of equations using graphs. *Obj. 2, p. 267*

$$3y = 2x + 5$$
$$3x + y = 9$$

3. Solve the given system of equations using addition or subtraction.

$$3m - 3n = 9$$
$$3n + 2m = 1$$

4. Solve the given system of equations using the linear-combination method.

$$5a + 3b = -3$$
$$4a + 5b = 8$$

5. Solve the given system of equations using the substitution method.

$$c + 4d = 1$$
$$2c + 7d = 3$$

Check your answers with those at the back of the book.

Solving Problems

OBJECTIVES for Sections 6-5 through 6-7:
1. To use two variables to solve problems.
2. To solve wind and water current problems.
3. To solve digit problems.

6–5 Using Two Variables to Solve Problems

In Chapter 4 you used equations in one variable to solve problems. Now that you know how to solve a system of linear equations, you may find it more convenient to use two variables when solving certain problems. When you use two variables, you ordinarily need to form two equations. The next example illustrates how a problem may be solved using either one variable or two variables.

EXAMPLE A realtor has two homes for sale. Ten years ago the older home was four times as old as was the newer home. Thirty years from now, the older home will be only twice as old as the newer home will be. How old are the homes now?

SOLUTION 1 (Using two variables)

Step 1 The problem asks for the ages of the homes now.

Step 2 Let x = the present age in years of the older home.
Let y = the present age in years of the newer home.

	Age in years 10 years ago	*Age in years now*	*Age in years 30 years from now*
Older home	$x - 10$	x	$x + 30$
Newer home	$y - 10$	y	$y + 30$

Step 3 Ten years ago, older home was four times as old as newer home.

$$\underbrace{\text{older home}}_{x - 10} \ \text{was} \ \underbrace{\text{four times as old as newer home}}_{4(y - 10)}$$

$$x - 10 \overset{\downarrow}{=} 4(y - 10)$$

Thirty years from now, older home will be twice as old as newer home.

$$x + 30 \overset{\downarrow}{=} 2(y + 30)$$

Step 4 You can find x and y by solving the following system of equations.

$$\begin{aligned} x - 10 = 4(y - 10) &\longrightarrow & x - 4y &= -30 \\ x + 30 = 2(y + 30) &\longrightarrow & x - 2y &= 30 \\ & & -2y &= -60 \\ & & y &= 30 \end{aligned}$$

$$\begin{aligned} x - 2y &= 30 \\ x - 2(30) &= 30 \\ x &= 90 \end{aligned}$$

Step 5 Ten years ago, was the older home four times as old as the newer one?

$$90 - 10 \stackrel{?}{=} 4(30 - 10)$$
$$80 = 80 \quad \checkmark$$

Thirty years from now, will the older home be twice as old as the newer one?

$$90 + 30 \stackrel{?}{=} 2(30 + 30)$$
$$120 = 120 \quad \checkmark$$

$\therefore$ the older home is 90 years old, and the newer home is 30 years old.

SOLUTION 2 (Using one variable)

Step 1 The problem asks for the ages of the homes now.

Step 2 Let x = the age in years of the newer home 10 years ago.
Then $4x$ = the age in years of the older home 10 years ago.

	Age in years 10 years ago	*Age in years now*	*Age in years 30 years from now*
Older home	$4x$	$4x + 10$	$4x + 40$
Newer home	x	$x + 10$	$x + 40$

Step 3 Thirty years from now, $\underbrace{\text{older home}}_{4x + 40}$ $\underset{=}{\text{will be}}$ $\underbrace{\text{twice as old as newer home}}_{2(x + 40)}$.

Step 4

$$\begin{aligned} 4x + 40 &= 2(x + 40) \\ 4x + 40 &= 2x + 80 \\ 2x &= 40 \\ x &= 20 \end{aligned}$$

present age of older home = $4x + 10 = 4(20) + 10 = 90$ years
present age of newer home = $x + 10 = 20 + 10 = 30$ years

Step 5 The check is left to you.

$\therefore$ the older home is 90 years old, and the newer home is 30 years old.

Oral Exercises

Solve each problem by using the following:

a. two equations in two variables

b. one equation in one variable

1. The sum of two numbers is 24. One number is three times the other. What are the two numbers?
2. One number is six less than another. The sum of the numbers is 17. What are the two numbers?
3. The sum of two numbers is 61. Their difference is five. What are the two numbers?
4. A wire that is 16 m long is cut into two pieces. One piece is three times the length of the other. What is the length of the shorter piece of wire?
5. Les has 20 coins in his pocket. Some of them are nickels and the rest are dimes. The nickels and dimes altogether are worth $1.40. How many dimes does Les have?
6. The perimeter of a rectangle is 108 cm. If the length of the rectangle is five times the width, what are the dimensions of the rectangle?

Problems

Solve each problem using two equations in two variables.

A

1. Susan has seven more fish than Tammy. They have 43 fish altogether. How many fish does each have?
2. Bob is three years older than his brother. The sum of their ages is 33. How old is Bob?
3. Two angles are supplementary. The measure of one angle is 30° more than the measure of the other. What is the measure of the larger angle?
4. Two angles are complementary. The sum of the measures of the larger angle and three times the measure of the smaller angle is 114°. What is the measure of each angle?
5. Two games and five puzzles cost $33.25 altogether. Three games and two puzzles cost $32.00. If every game costs the same amount, and every puzzle costs the same amount, what is the cost of each puzzle?
6. The difference between three times one number and a lesser one is 37. The sum of the greater number and twice the lesser number is 38. Find the numbers.
7. The length of a rectangular garden is three times the width. If the perimeter is 32 m, what are the dimensions of the garden?

8. Two sides of a triangle have the same length. The remaining side is one third as long as each of the other sides. If the perimeter of the triangle is 315 cm, what is the length of each side?

9. During the year, Charles read five more books than twice the number of books read by Frank. If together they read a total of 29 books, how many books did each read?

10. Two angles are supplementary. The measure of one of these angles is 12° less than one third the measure of the other. What is the measure of each angle?

11. Kim has 40 coins worth a total of \$8.80. Some of the coins are nickels and the rest are quarters. How many of each kind of coin does Kim have?

12. Lou has \$8.60 in dimes and quarters. If he has twelve more quarters than dimes, how many dimes does he have?

13. Jessica purchased some 20¢ stamps and some 25¢ stamps at the post office. If she paid \$7.75 for 35 stamps, how many of each kind did she purchase?

14. The charge for admission to the zoo is \$3.25 for each adult and \$1.50 for each child. On a day when 500 people paid to visit the zoo, the receipts totalled \$1275. Find the number of adult tickets purchased that day.

15. The ages of Lee's mother and aunt total 83 years. If her mother's age were doubled, then her mother would be 58 years older than her aunt. How old are Lee's mother and aunt?

16. In four years Cathy's cat Byte will be three fourths as old as Cathy will be. Four years ago, Byte was only half as old as Cathy was. How old are Cathy and her cat now?

Solve.

B 17. Bath towels sell for \$13.25 each, while hand towels sell for \$4.50 each. Theresa buys some of each type of towel for a total of \$62.25. If she spends \$17.25 more on bath towels than she spends on hand towels, how many of each type does she buy?

18. It costs \$5.40 to ship a radial saw and \$6.50 to ship a table saw to Ohio. On Monday an order consisting of table saws and radial saws was sent to Ohio for a total shipping cost of \$140.70. If nine more table saws had been sent, the number of table saws in the order would have been three times the number of radial saws. How many saws of each type were sent in the order?

19. Sarah purchased two picture frames. The perimeter of the larger frame is 240 cm, and the perimeter of the smaller frame is 140 cm. The height of the smaller frame is the same as the width of the larger frame, and the width of the smaller frame is 10 cm less than the height of the smaller frame. Find the dimensions of the larger frame.

20. A video disk rental company charges a fixed amount for the first two days of a rental and an additional charge for each day thereafter. Juan paid \$19.50 for a disk that he kept six days, and Tom paid \$27.00 for a disk that he kept for nine days. How much would it cost to rent a video disk for ten days from the same company?

C **21.** Erik has 50 coins, all nickels, dimes, and quarters. They are worth \$3.60 altogether. There are two more dimes than quarters, and four times as many nickels as the combined number of dimes and quarters. How many of each kind of coin does Erik have?

22. The sum of the present ages of Pam, Sue, and Jim is 63 years. The age of Pam three years ago is the same as Jim's age two years from now. The sum of Pam's present age and the ages of Sue and Jim three years ago is one more than twice the age that Pam will be in two years. How old are Pam, Sue, and Jim now?

ON THE CALCULATOR

Using either the substitution or linear-combination method, you can verify that the solution of the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

is

$$\left(\frac{ce - bf}{ae - bd}, \frac{af - cd}{ae - bd}\right),$$

provided that $ae - bd \neq 0$. Sometimes using these expressions can simplify the process of solving a system of equations. In particular, when the values of a, b, c, d, e, and f are decimals, it may be easier to solve the system by using a calculator to evaluate these expressions.

Solve each system using a calculator.

1. $2.3x + 4.2y = 0.24$
$1.2x + 1.4y = 0.6$

2. $3.4x + 5.6y = 39$
$0.7x + 0.93y = 5.8$

3. $4.9x - 8.3y = 0.4$
$3.2x + 5.6y = 21.2$

4. $4.3x + 3.8y = 9.8$
$1.8x - 7.2y = 25.2$

6–6 Wind and Water Current Problems

In solving motion problems about airplanes flying with or against the wind, you may need to know the meaning of the following terms.

air speed the speed of the airplane in still air

wind speed the speed of the wind

tail wind a wind blowing in the same direction as the one in which the airplane is heading

head wind a wind blowing in the direction opposite to the one in which the airplane is heading

ground speed the speed of the airplane relative to the ground

with a tail wind: ground speed = air speed + wind speed

with a head wind: ground speed = air speed − wind speed

EXAMPLE With a tail wind, a light plane can fly 720 km in 2 h. Going against the wind, the plane can fly the same distance in 3 h. What are the wind speed and the air speed of the plane?

SOLUTION

Step 1 The problem asks for the wind speed and the air speed of the plane.

Step 2 Let x = the air speed of the plane in kilometers per hour.
Let y = the wind speed in kilometers per hour.

	ground speed r (km/h)	time t (h)	distance $d = rt$ (km)
With a tail wind	$x + y$	2	$2(x + y)$
With a head wind	$x - y$	3	$3(x - y)$

Step 3 The plane can fly 720 km in 2 h with a tail wind.

$$2(x + y) = 720$$

The plane can fly 720 km in 3 h with a head wind.

$$3(x - y) = 720$$

Step 4 You can find x and y by solving the following system of equations.

$$2(x + y) = 720 \longrightarrow x + y = 360$$
$$3(x - y) = 720 \longrightarrow x - y = 240$$

Completing Step 4 and checking your results (Step 5) are left to you. You should find that the air speed of the plane is 300 km/h and that the wind speed is 60 km/h.

The method of solution used in the example on the preceding page can also be used to solve problems involving boats moving in a current.

Oral Exercises

In Exercises 1–4, Sharon rows at the rate 5 km/h in still water, and the rate of the current is 3 km/h.

1. How fast does Sharon move rowing upstream?
2. How fast does Sharon move rowing downstream?
3. How long would it take her to row 8 km upstream and 8 km back?
4. What would happen if she tried to row upstream in a current flowing at 5.1 km/h?

In Exercises 5–8, the plane's air speed is 500 km/h.

5. What is the plane's ground speed on a windless day?
6. What is its ground speed if it has a 25 km/h tail wind?
7. What is its ground speed when it encounters head winds of 50 km/h?
8. a. How long will the plane take to fly 1200 km with a 100 km/h tail wind?
   b. How long will the plane take to fly 1200 km with a 100 km/h head wind?

Problems

Solve.

A

1. Flying against a head wind, a plane could fly 3000 km in 6 h. The plane would require only 5 h for the return trip with no change in the wind. Find the wind speed and the air speed of the plane.
2. The air speed of a plane was 132 km/h. Flying with the wind, the plane traveled twice the distance in 5 h as it traveled against the wind in 3 h. What was the wind speed?
3. With a tail wind, a plane flew 240 km in 45 min. With no change in the wind, the return trip took 48 min. Find the wind speed and the air speed of the plane.
4. A boat travels 4 km in 20 min with the current. The return trip takes 24 min. Find the speed of the current and the speed of the boat in still water.
5. Rose took a half hour to row 3 km with the current. When she returned, she took 90 min. Find her rowing rate and the speed of the current.

6. A sea lion swims 18 km from one feeding ground to another in 27 min. The return trip against the ocean current takes 36 min. How fast does the sea lion swim in still water?

7. Steve flew his experimental plane 56.25 km with the wind in 45 min. The return trip took 75 min with no change in the wind. What was the wind speed?

8. Walking down a long moving escalator, Phil covered the 75 m distance in 25 s. Walking back up against the motion of the escalator, the distance was covered in 75 s. What was the speed of the escalator?

B 9. When Anthony went bass fishing, he rowed 4 km against the current in 2 h. On his return trip, he still had 1 km to go to his starting point after he had been traveling for 0.5 h. Find Anthony's rate of rowing in still water and the speed of the current.

10. Jennifer can paddle a certain distance with the current in 2.5 h. To cover the same distance against the current, she takes 5 h. How many times faster is her rate of paddling in still water than the speed of the current?

11. A sightseeing cruiser takes the same time to sail a certain distance up a river as it takes to sail three times that distance down the river. If the speed of the cruiser is s and that of the current is c, find the relationship between s and c.

12. Jack's plane will normally cover 52 km in 12 min in still air. On one trip, he flew with the wind for 2.5 h. Returning, he still had 100 km to go after 3.5 h. What was the wind speed?

C 13. Ann regularly swam 0.4 km in 20 min at the school pool. Swimming in a river against the current, she swam 0.25 km in the same time that she swam 0.75 km with the current. Find the speed of the current and the time it took Ann to swim 0.75 km downstream.

14. Walking at 4 km/h, Bruce can make the round trip between his campsite and Lookout Point in 2.5 h. Rowing on Crooked River, he can row upstream from the campsite to Lookout Point in 1 h and can row back again in 40 min. Find Bruce's rate of rowing in still water and the speed of the current in Crooked River.

15. Flying against the wind, Lisa made the flight to Reno in 3 h with a steady air speed of 400 km/h. Returning later with a tail wind that had doubled in magnitude, she landed 225 km beyond the starting point in 3 h. What was the original wind speed? How far away from Reno did she begin the trip?

6–7 Digit Problems

Using a system of two equations in two variables is often a good way to solve problems involving two-digit numbers.

You can write a two-digit decimal number like 83 in expanded form as follows.

$$\underset{\text{tens' digit}}{10 \cdot 8} + \underset{\text{units' digit}}{3}$$

In fact, any two-digit decimal number can be written in the expanded form

$$\underset{\text{tens' digit}}{10t} + \underset{\text{units' digit}}{u},$$

where $t \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $u \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

To represent a number with the same digits in reverse order, you write

$$10u + t.$$

In each case, the sum of the values of the digits is represented by $t + u$.

EXAMPLE 1 A catalog clerk mistakenly reversed the two digits in the price of a radio fuse and overcharged the customer 36¢. If the sum of the digits was 14, what was the correct price?

SOLUTION

Step 1 The problem asks for the correct price of the fuse.

Step 2 Let t = the tens' digit of the correct price.
Let u = the units' digit of the correct price. Then:
$10t + u$ = the correct price;
$10u + t$ = the price mistakenly charged

Step 3 The price charged was 36¢ more than the correct price.

$$10u + t = 10t + u + 36$$
$$9u - 9t = 36$$
$$u - t = 4$$

Transform the first equation into an equivalent equation in the same variables.

The sum of the digits was 14.

$$u + t = 14$$

Step 4 You can find u and t by solving the following system of equations.

$$\begin{aligned} u - t &= 4 \\ u + t &= 14 \\ \hline 2u &= 18 \\ u &= 9 \end{aligned} \qquad \text{Then: } \begin{aligned} u + t &= 14 \\ 9 + t &= 14 \\ t &= 5 \end{aligned}$$

Step 5 When you reverse the digits of 59, is the result 36 more than 59?

$$95 \stackrel{?}{=} 59 + 36$$
$$95 = 95 \quad \checkmark$$

$\therefore$ the correct price of the fuse was 59¢.

In Section 6-5, you saw that the same problem could be solved using either two variables and two equations or one variable and one equation. In the following example, you will see a problem that can be solved using either three variables and three equations or two variables and two equations.

EXAMPLE 2 The sum of the digits of a three-digit number is 15. The units' digit is one more than four times the hundreds' digit. The tens' digit is two times the hundreds' digit. Find the number.

SOLUTION 1 (Using three variables)

Step 1 The problem asks for a three-digit number the sum of whose digits is 15.

Step 2 Let h = the hundreds' digit.
Let t = the tens' digit.
Let u = the units' digit.

Step 3 The sum of the digits is 15.

$$\underbrace{\text{The sum of the digits}}_{h + t + u} \; \underbrace{\text{is}}_{=} \; \underbrace{15}_{15}$$

The units' digit is one more than four times the hundreds' digit.

$$\underbrace{\text{The units' digit}}_{u} \; \underbrace{\text{is}}_{\stackrel{?}{=}} \; \underbrace{\text{one more than four times the hundreds' digit}}_{1 + 4h}$$

The tens' digit is two times the hundreds' digit.

$$\underbrace{\text{The tens' digit}}_{t} \; \underbrace{\text{is}}_{\stackrel{?}{=}} \; \underbrace{\text{two times the hundreds' digit}}_{2h}$$

Step 4 Solve the system: $h + t + u = 15$
$u = 1 + 4h$
$t = 2h$

In this case, the expressions for t and u can be substituted into the first equation.

$$h + t + u = 15$$
$$h + (2h) + (1 + 4h) = 15 \qquad \text{Then: } u = 1 + 4h \quad \text{and} \quad t = 2h$$
$$7h + 1 = 15 \qquad u = 1 + 4(2) \qquad t = 2(2)$$
$$h = 2 \qquad u = 9 \qquad t = 4$$

Step 5 Checking that 249 is the solution is left to you.
$\therefore$ the number is 249.

On the following page, the same problem is solved using two variables and two equations.

SOLUTION 2 (Using two variables)

Step 1 The problem asks for a three-digit number the sum of whose digits is 15.

Step 2 Let h = the hundreds' digit.
Let t = the tens' digit.
Then $15 - (h + t)$ = the units' digit.

Step 3 The units' digit is one more than four times the hundreds' digit.

$$\underbrace{15 - (h + t)}_{\text{The units' digit}} \stackrel{?}{=} \underbrace{1 + 4h}_{\text{one more than four times the hundreds' digit}}$$

The tens' digit is two times the hundreds' digit.

$$\underbrace{t}_{\text{The tens' digit}} \stackrel{?}{=} \underbrace{2h}_{\text{two times the hundreds' digit}}$$

Step 4 Solve the system:

$$15 - (h + t) = 1 + 4h \longrightarrow 5h + t = 14$$
$$t = 2h \longrightarrow -2h + t = 0$$
$$7h = 14$$
$$h = 2$$
$$-2h + t = 0$$
$$-2(2) + t = 0$$
$$t = 4$$

The units' digit is $15 - (h + t) = 15 - (2 + 4) = 9$.

Step 5 Checking that 249 is the solution is left to you.
$\therefore$ the number is 249.

You will learn to solve more complicated systems of equations in three variables in Section 6-10. In general, the problems in this section should be attempted using two variables.

The following table lists some *divisibility tests* that you may have learned in an earlier course.

Divisibility by	*Test*
2	Units' digit of number must be 0, 2, 4, 6, or 8.
3	Sum of digits must be divisible by 3.
4	Number formed by last two digits must be divisible by 4.
5	Units' digit of number must be 0 or 5.
6	Number must be divisible by both 2 and 3.
8	Number formed by last three digits must be divisible by 8.
9	Sum of digits must be divisible by 9.

These divisibility tests have their basis in the structure of the decimal system of numeration. They can be demonstrated by using some basic properties of the real numbers, such as the commutative, distributive, and associative axioms.

EXAMPLE 3 Show that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

SOLUTION Let h = the hundreds' digit.
Let t = the tens' digit.
Let u = the units' digit.

Then $100h + 10t + u$ represents a three-digit number.

Notice that the expression $100h + 10t + u$ can be rewritten as follows.

$$\begin{aligned} 100h + 10t + u &= (99h + h) + (9t + t) + u \\ &= (99h + 9t) + (h + t + u) \\ &= 3(33h + 3t) + (h + t + u). \end{aligned}$$

Since $3(33h + 3t)$ is divisible by 3, the entire right side is divisible by 3 if and only if the sum of the digits, $h + t + u$, is divisible by 3.

$\therefore$ a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Oral Exercises

A two-digit number has tens' digit *t* and units' digit *u*. Express the following in terms of *t* and *u*.

1. the sum of the digits
2. the value of the two-digit number
3. the value of the two-digit number obtained by reversing the digits

A three-digit number has hundreds' digit *h*, tens' digit *t*, and units' digit *u*. Express the following in terms of *h*, *t*, and *u*.

4. the sum of the digits
5. the value of the three-digit number
6. the value of the three-digit number obtained by reversing the digits

Use the divisibility tests given on page 294 to answer the following.

7. Is 4270 divisible by 2?
8. Is 3130 divisible by 3?
9. Is 39,495 divisible by 3?
10. Is 587,646 divisible by 4?
11. Is 38,750 divisible by 5?
12. Is 45,678 divisible by 6?
13. Is 469,152 divisible by 8?
14. Is 363,927 divisible by 9?

Problems

Solve.

A

1. The sum of the digits of a two-digit number is 10. The value of the number is 16 times the units' digit. Find the number.
2. The sum of the digits of a two-digit number is 12. The number with the digits reversed is 15 times the original tens' digit. Find the original number.
3. The sum of the digits of a two-digit number is 4. If the order of the digits is reversed, the result is a number that exceeds the original number by 18. Find the original number.
4. The sum of the digits of a two-digit number is 11. The number obtained by reversing the order of the digits is 27 less than the original number. Find the original number.
5. The tens' digit of a two-digit number exceeds the units' digit by 2. The sum of the tens' digit and twice the units' digit is 17. Find the number.
6. The units' digit of a two-digit number exceeds three times the tens' digit by 3. If the tens' digit is subtracted from the units' digit, the difference is 7. Find the number.
7. If the tens' digit of a two-digit number is subtracted from the units' digit, the difference is 8. The number with the digits reversed is 10 more than nine times the units' digit of the original number. Find the number.
8. The sum of the digits of a two-digit number is 15. The number with the digits reversed is 30 more than six times the tens' digit of the original number. Find the number.
9. The sum of the digits of a three-digit number is 17. The units' digit is 6. If the order of the digits is reversed, the result is a number that is 297 more than the original number. What is the original number?
10. The tens' digit of a three-digit number is 5. The units' digit is twice the hundreds' digit. If the order of the digits is reversed, the new number is 396 more than the original number. What is the original number?

B

11. The sum of the digits of a three-digit number is 14. The hundreds' digit is four times the tens' digit and twice the units' digit. Find the number.
12. The sum of the digits of a three-digit number is 13. The hundreds' digit is two more than the tens' digit, and the tens' digit is four more than the units' digit. Find the number.
13. Show that a three-digit number is divisible by 9 if and only if the sum of the digits is divisible by 9.

14. Show that a four-digit number is divisible by 4 if and only if the number formed by the last two digits is divisible by 4.

15. Show that the sum of a two-digit number and the number with the order of the digits reversed is always divisible by 11.

16. Show that the difference between a three-digit number and the number with the order of the digits reversed is always divisible by 99.

C 17. The hundreds' digit of a three-digit number is 2 more than twice the tens' digit. The tens' digit is 1 less than twice the units' digit. If the order of the digits is reversed, the number obtained is 594 less than the original number. Find the original number.

18. The hundreds' digit of a three-digit number is one more than twice the units' digit. The units' digit is three more than the tens' digit. If the order of the digits is reversed, the number obtained is 396 less than the original number. Find the original number.

19. Show that the difference between a two-digit number and the number obtained when the tens' digit is decreased by 1 and the units' digit is increased by 1 is always 9.

20. Find all three-digit numbers, if any, that satisfy *all* of the requirements (a) through (c).

    a. The units' digit is one half the tens' digit.

    b. The hundreds' digit is 1 less than 3 times the units' digit.

    c. The difference between the number and the number obtained when the order of its digits is reversed is 495.

Self-Test 2

Solve.

1. Five years ago, Jake was five times as old as Jenny was at that time. Four years from now, Jake will be twice as old as Jenny will be. Find Jenny's age now. *Obj. 1, p. 284*

2. Bridget needed 1 h to row 4 km upstream, but only 30 min to row the same distance back downstream. Find Bridget's rowing rate in still water and the speed of the current. *Obj. 2, p. 284*

3. The sum of the digits of a two-digit number is 9. When the digits are reversed, the new number is 63 less than the original number. Find the original number. *Obj. 3, p. 284*

Check your answers with those at the back of the book.

Systems of Inequalities; Linear Equations in Three Variables

OBJECTIVES for Sections 6-8 through 6-10:
1. To graph the solution set of a system of linear inequalities in two variables.
2. To solve linear programming problems that involve two variables.
3. To solve a system of three linear equations in three variables.

6–8 Graphs of Systems of Linear Inequalities

You can use graphs to determine the solution set of a *system of linear inequalities* in two variables.

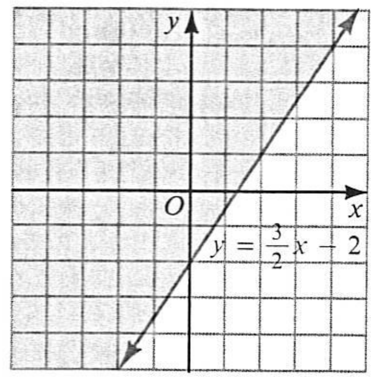
EXAMPLE 1 Graph the solution set of the following system of inequalities.

$$3x - 2y \leq 4$$
$$x + 2y < 2$$

SOLUTION

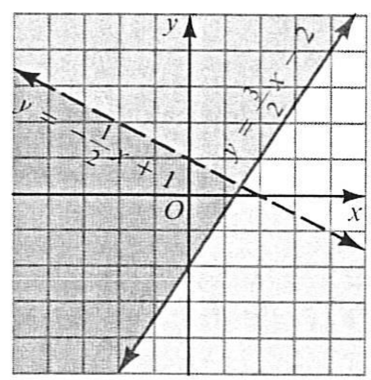
1. Transform each inequality into an equivalent inequality with y alone as one side.

$$3x - 2y \leq 4 \longrightarrow y \geq \tfrac{3}{2}x - 2$$
$$x + 2y < 2 \longrightarrow y < -\tfrac{1}{2}x + 1$$



2. Draw the graph of $y = \frac{3}{2}x - 2$. The graph of $y \geq \frac{3}{2}x - 2$ is the *closed* half-plane *on and above* the line $y = \frac{3}{2}x - 2$.

3. Draw the graph of $y = -\frac{1}{2}x + 1$. The graph of $y < -\frac{1}{2}x + 1$ is the *open* half-plane *below* the line $y = -\frac{1}{2}x + 1$.



4. The intersection of the half-planes found in Steps 2 and 3 (double shading) is the graph of the given system:

$$3x - 2y \leq 4$$
$$x + 2y < 2$$

5. Check your work by selecting any point within the double-shaded region. A convenient point to use is (0, 0). This ordered pair should satisfy each of the original inequalities.

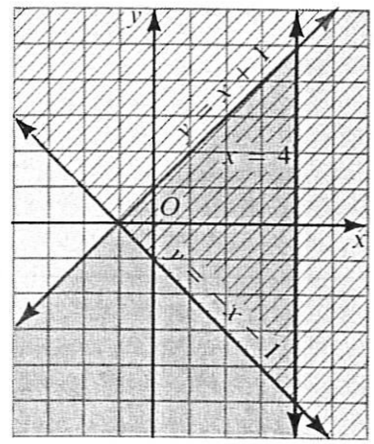
$$3x - 2y \leq 4 \qquad\qquad x + 2y < 2$$
$$3(0) - 2(0) \overset{?}{\leq} 4 \qquad\qquad 0 + 2(0) \overset{?}{<} 2$$
$$0 \leq 4 \; \checkmark \qquad\qquad 0 < 2 \; \checkmark$$

EXAMPLE 2 Graph the solution set of the following system of inequalities.

$$y - x \leq 1$$
$$y + x \geq -1$$
$$x - 4 \leq 0$$

SOLUTION

1. The graph of $y - x \leq 1$ consists of the points on and below the line $y = x + 1$.
2. The graph of $y + x \geq -1$ consists of the points on and above the line $y = -x - 1$.
3. The graph of $x - 4 \leq 0$ consists of the points on and to the left of the vertical line $x = 4$.
4. The graph of the solution set of the given system is the intersection of the three shaded regions. This intersection is represented by all the points on the sides and in the interior of the triangle formed by the three lines.



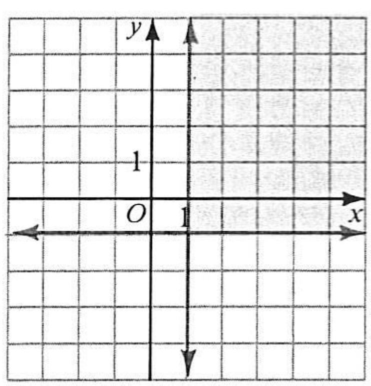
5. *Check:* Since (0, 0) is in the region representing the solution set, the ordered pair (0, 0) should satisfy each of the original inequalities.

$$y - x \overset{?}{\leq} 1 \qquad\qquad y + x \overset{?}{\geq} -1 \qquad\qquad x - 4 \overset{?}{\leq} 0$$
$$0 - 0 \overset{?}{\leq} 1 \qquad\qquad 0 + 0 \overset{?}{\geq} -1 \qquad\qquad 0 - 4 \overset{?}{\leq} 0$$
$$0 \leq 1 \; \checkmark \qquad\qquad 0 \geq -1 \; \checkmark \qquad\qquad -4 \leq 0 \; \checkmark$$

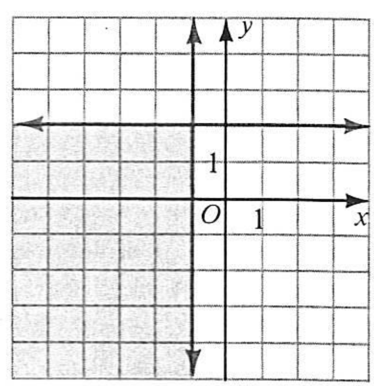
Oral Exercises

Name a system of two linear inequalities whose solution set is shown by the shaded region in each graph.

1.

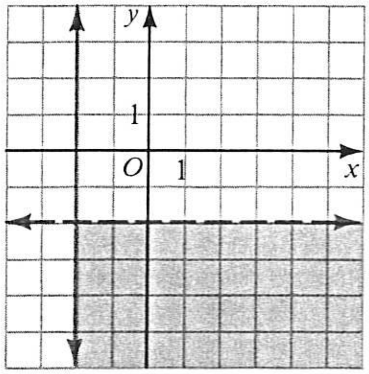


2.

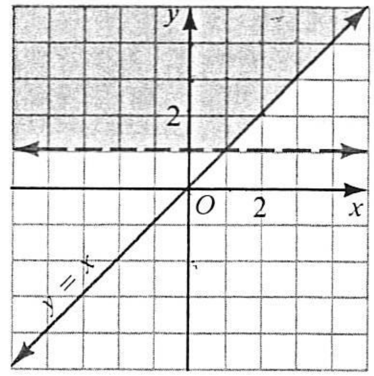


Name a system of two linear inequalities whose solution set is shown by the shaded region in each graph.

3.



4.



Tell which of the systems in Exercises 1–4, if any, have solution sets that contain the given ordered pair.

5. (3, 1) 6. (−1, 3) 7. (−3, 2) 8. (1, −2) 9. (2, 4) 10. (−2, −3)

Tell whether or not each point belongs to the graph of the system.

$x + 2 > 0$ and $y \geq x - 2$

11. (4, 1) 12. (0, −2) 13. (−3, 0) 14. (−2, 2) 15. (3, 1) 16. (−1, −1)

Written Exercises

Graph the solution set of each system of inequalities.

A

1. $y > 0$, $x \leq 0$
2. $y \leq 0$, $x > 0$
3. $y < 1$, $x \geq -1$
4. $y \geq -3$, $x \leq 2$
5. $y < x$, $x \geq -1$
6. $y > x$, $y \leq 3$
7. $y \geq x + 1$, $y \leq -x + 2$
8. $y \leq x - 2$, $y > -x - 1$
9. $y + 1 \leq 2x$, $y + x > -2$
10. $y - 2 > 2x$, $y - x > 3$
11. $2y - x > -2$, $-2x - y > -1$
12. $2y + x > 2$, $3x - y \geq -3$

B

13. $y > -1$, $x \leq 3$, $y < x$
14. $x \leq 5$, $y < 4$, $y \geq -x$
15. $y + x \geq 0$, $y - x \leq 1$, $x - 2 \leq 0$
16. $y + x \leq 1$, $y - x \leq -1$, $y + 2 \geq 0$
17. $y - 1 < 2x$, $y + x > -1$, $x < 0$
18. $y + 3 > 3x$, $y + x > -2$, $y < 3$
19. $3 - y > 0$, $y + x \geq -2$, $y + 3 \geq 3x$
20. $y + 2 > 0$, $x - y \geq -3$, $y - 2 \leq -5x$

C

21. $y - 2 < 0$, $x + 3 > 0$, $2y + x < 2$, $3y + 3 > 2x$
22. $2y - 6 \leq x$, $2y + 3x \geq -6$, $5y + 15 \geq 2x$, $3y + 5x \leq 15$

PROGRAMMING IN BASIC

Using the program listed on page 247, you were able to print a graph of an open sentence in two variables. By making the following changes in that program, you will now be able to print a graph of *two* open sentences in two variables. In this program, the capital letter "O" is used to identify the point or region that is the graph of the solution set of the system formed by the two open sentences.

```
10   PRINT "TO GRAPH TWO OPEN SENTENCES"
20   PRINT "IN TWO VARIABLES"
30   PRINT "(SENTENCES ARE IN LINES 125-135):"
40  }
 :  }  from program on page 247
120 }
125  IF Y = X + 2 AND Y = 4 - X THEN 258
130  IF Y = X + 2 THEN 250
135  IF Y = 4 - X THEN 254
140 }
 :  }  from program on page 247
250 }
252  GOTO 260
254  PRINT ".";
256  GOTO 260
258  PRINT "O";
260 }
 :  }  from program on page 247
340 }
```

Exercises

1. Type in and RUN the revised program.

Change lines 125–135 as necessary to RUN the program for each of the following systems of open sentences. In each case, INPUT 9 for the extent of the graph.

2. $x = -3$
$y = 4$

3. $y = 4$
$2x + 3y = 6$

4. $x = -3$
$2x - 3y = -18$

5. $2x + 3y = 6$
$2x - 3y = -18$

6. $x > -3$
$y < 4$

7. $y > x + 2$
$y < 4 - x$

8. $x > -3$
$2x - 3y < -18$

9. $2x + 3y > 6$
$2x - 3y < -18$

6–9 Linear Programming

Many practical problems in business, science, and industry can be solved using the techniques of *linear programming*. **Linear programming** is a method of solving problems in which a quantity represented by a linear equation, often profit or cost, is to be maximized or minimized subject to conditions expressed by a system of linear inequalities. The following example illustrates this method.

During an illness Gregory supplements his daily diet with vitamin pills. Each day he needs at least 4 mg of thiamine, 6 mg of riboflavin, and 80 mg of niacin. To meet these needs, he can buy either Brand X pills at 3¢ apiece or Brand Y pills at 4¢ apiece. The following table shows the amount of vitamins in one pill of each brand.

	Brand X	*Brand Y*
Thiamine	1 mg	4 mg
Riboflavin	3 mg	5 mg
Niacin	30 mg	40 mg

What combination of pills will provide his minimum daily needs for the three vitamins at the lowest cost?

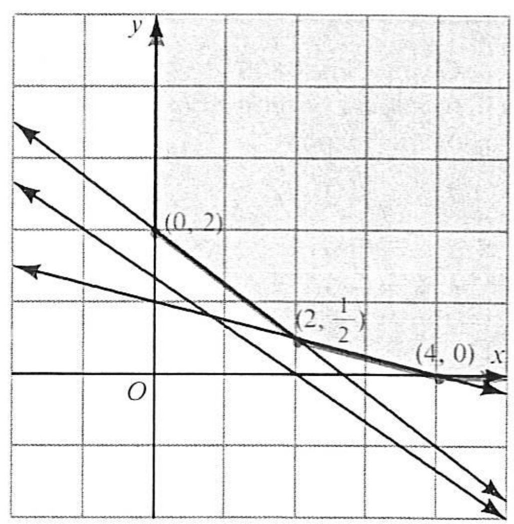
Let x = the number of Brand X pills used daily.
Let y = the number of Brand Y pills used daily.
Let C = the daily cost in cents for the pills.

Then: $$C = 3x + 4y$$

Gregory wants to minimize the cost C subject to the restrictions, or *constraints*, described by the following inequalities.

$$\left.\begin{aligned} x + 4y &\geq 4 \\ 3x + 5y &\geq 6 \\ 30x + 40y &\geq 80 \end{aligned}\right\}$$ The total daily amount of each vitamin must equal at least the daily need.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \end{aligned}\right\}$$ He cannot use a negative amount of either pill.

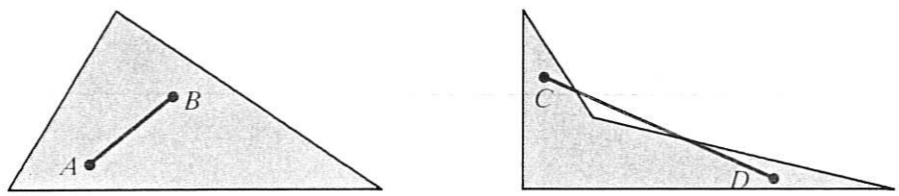


The graph of the solution set of this system of inequalities is indicated by shading in the diagram. Notice that $3x + 5y \geq 6$ has no effect on the result.

The shaded region of the graph on the preceding page is called the *feasible region*. A feasible region has the following characteristics.

1. Where it is bounded, the boundary is determined by straight lines. The points of the region where such boundary lines intersect are called **corner points.** In this case the corner points are (0, 2), $(2, \frac{1}{2})$, and (4, 0).
2. Every point on the boundary is a part of the region.
3. The region is *convex*.

A region is said to be **convex** if, whenever you choose any two points in the region and draw the segment joining them, the segment is contained in the region. For example, in the following diagram the shaded region at the left is convex, but the shaded region at the right is not.



Any plane region that is the intersection of a finite number of closed half-planes has the three characteristics listed above and is called a **convex polygonal region.**

A remarkable result, not proved here, is the following:

> Over a convex polygonal region, any maximum or minimum values of a *linear expression* $ax + by$, where a and b are real numbers, occur for the coordinates of a corner point of the region.

It is possible that maximum or minimum values may also occur at other points besides the corner points. What is important is that by testing values at the corner points, you can, in a finite number of steps, find the greatest and least values, if they exist, of any linear expression over any closed polygonal region.

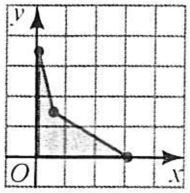
Thus the minimum value of C can be found by evaluating $3x + 4y$ at each corner point.

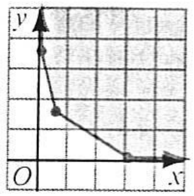
$$\begin{aligned} (0, 2)&: \quad 3(0) + 4(2) = 8 \\ (2, \tfrac{1}{2})&: \quad 3(2) + 4(\tfrac{1}{2}) = 8 \\ (4, 0)&: \quad 3(4) + 4(0) = 12 \end{aligned}$$

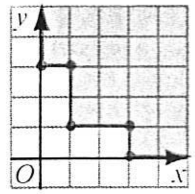
In this case both (0, 2) and $(2, \frac{1}{2})$ yield the minimum value, 8; however, it is inconvenient to take half a pill. Therefore the most economical and practical choice for Gregory is 2 Brand Y pills and no Brand X pills.

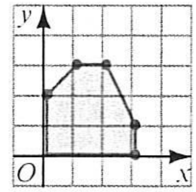
Oral Exercises

Tell whether or not each shaded region is convex.

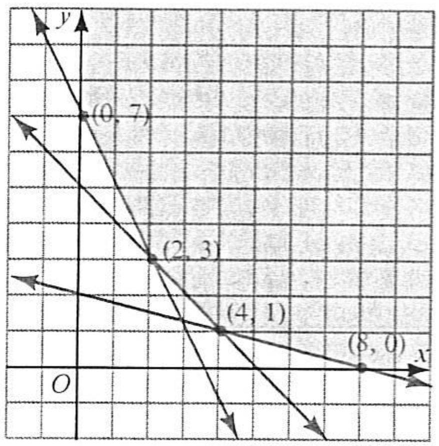
1. 

2. 

3. 

4. 

5. Evaluate the linear expression $3x + 2y$ at each corner point of the convex polygonal region shown in the graph below.



6. What is the minimum value of $3x + 2y$ over the region shown in the graph above?

Written Exercises

Exercises 1–3 refer to the following system of inequalities.

$$\begin{aligned} y &\geq 1 \\ x &\geq 2 \\ 3y + x &\geq 9 \\ y + 2x &\geq 8 \end{aligned}$$

A

1. Graph the solution set of the system.
2. Determine the corner points of the region that represents the solution set.
3. Find the minimum value of each of the following linear expressions over the region graphed in Exercise 1 by evaluating the expression at each of the corner points.

   a. $x + 5y$ b. $5x + y$ c. $x + y$

Exercises 4–6 refer to the following system of inequalities.

$$x \geq 1$$
$$y \leq 6$$
$$y \geq -x + 4$$
$$2y \geq x - 1$$
$$3y \leq -x + 21$$

4. Graph the solution set of the system and label the corner points with their coordinates.
5. Find the minimum value of the expression $x + 5y$ over the region graphed in Exercise 4.
6. Find the maximum value of the expression $4x - 3y$ over the region graphed in Exercise 4.

In Exercises 7–10:

a. Graph the solution set of the system of inequalities and label the corner points with their coordinates.

b. Find the minimum value of the expression $2x + 3y$ over the region graphed.

7. $x \geq 0$
   $y \geq -3x + 5$
   $y \geq -\frac{2}{3}x + \frac{8}{3}$
   $y \geq 0$

8. $3y + 4x \geq 18$
   $3y + x \geq 9$
   $x - 1 \geq 0$
   $y \geq 0$

9. $y \geq 0$
   $x \geq 0$
   $y + 4x \geq 7$
   $y + x \geq 4$
   $3y + x \geq 6$

10. $y \geq 0$
    $x \geq 0$
    $y + 5x \geq 13$
    $y + 2x \geq 10$
    $3y + 2x \geq 18$

In Exercises 11–14, use the following information: The owners of a pet store plan to purchase some hamsters and some rabbits. They decide to spend at least \$100 and purchase at least eight animals. Each hamster costs \$5 and each rabbit costs \$25.

B 11. Introduce variables for the number of hamsters and the number of rabbits to be purchased. Write four inequalities that express the conditions of the problem. (Remember that the number of each kind of animal must be at least zero.)

12. Graph the solution set of the system of inequalities. Determine the coordinates of the corner points.

13. Suppose that it costs 13¢ per day to feed each hamster and 30¢ per day to feed each rabbit. If daily feeding costs are to be held to a minimum, how many hamsters and rabbits should be purchased?

14. Suppose that the cost of feeding each animal changes before the owners have purchased the animals. It now costs 14¢ per day to feed each hamster and 70¢ per day to feed each rabbit. The owners are surprised to find there are now several ways to make their selection while still meeting the original conditions and still keeping the daily feeding costs to a minimum. Give four possible choices that the owners have in making their selection.

C 15. A manufacturer has two plastic molding shops and must produce at least 1500 model cars and 900 model planes each day. Shop A can make 120 cars and 60 planes per hour. Shop B can make 150 cars and 100 planes per hour. The cost of running the machine in Shop A is $2.55 per hour, and the cost of running the machine in Shop B is $3.20 per hour. How many hours should each machine be used every day to minimize the costs?

16. Each day Central Post Office handles at least 52,800 letters and 16,800 advertising flyers. The cancelling machine now in use can handle 5500 letters and 1600 flyers per hour at an hourly cost of $10.05. The manufacturer of the machine is about to release an improved model which can do 6600 letters and 2400 flyers each hour at a cost of $12.75 per hour. Neither machine can be operated for more than 12 hours due to internal heating.

a. How would the expense of handling the mail be affected if the post office could use both machines?

b. Would the use of both machines be practical if the post office increased the cost of operating the old machine by 50¢ an hour?

6–10 Systems of Linear Equations in Three Variables

You have seen that a solution of an equation in *one* variable, such as

$$2x + 1 = 0,$$

is a *single* real number, and that a solution of an equation in *two* variables, such as

$$3x - 5y = 3,$$

is an *ordered pair* of real numbers.

A **solution** of an equation in *three* variables, such as

$$x + 2y - 4z = 9,$$

is an **ordered triple** (x, y, z) of real-number values of the variables for which the equation is a true statement. The set of all such ordered triples is the **solution set** of the equation.

EXAMPLE 1 Which of the following ordered triples are solutions of the equation $2x - y + 3z = 11$?

a. (3, 1, 2) **b.** (2, 2, 3) **c.** (1, 3, 2)

SOLUTION Check each ordered triple in the original equation.

a. $2(3) - 1 + 3(2) \stackrel{?}{=} 11$

$11 = 11 \quad \checkmark$

b. $2(2) - 2 + 3(3) \stackrel{?}{=} 11$

$11 = 11 \quad \checkmark$

c. $2(1) - 3 + 3(2) \stackrel{?}{=} 11$

$5 \neq 11$

$\therefore$ (3, 1, 2) and (2, 2, 3) are solutions.

Any equation of the form

$$ax + by + cz = d,$$

where a, b, c, and d are real numbers and a, b, and c are not all zero, is called a **linear equation in three variables.** (The graph of such an equation is a plane in space.) You can find as many solutions as you wish for such equations by choosing values for two of the variables and determining the corresponding value of the third variable.

EXAMPLE 2 Find three solutions of the equation $3x - 2y + z = 10$.

SOLUTION 1. Transform the given equation into an equivalent equation with z alone as one side.

$$3x - 2y + z = 10 \longrightarrow z = 10 - 3x + 2y$$

2. Substitute any values for x and y into the equation. Then solve the equation for z.

x	y	$z = 10 - 3x + 2y$	z
0	0	$z = 10 - 3(0) + 2(0)$	10
1	0	$z = 10 - 3(1) + 2(0)$	7
0	1	$z = 10 - 3(0) + 2(1)$	12

$\therefore$ three possible solutions are (0, 0, 10), (1, 0, 7), and (0, 1, 12).

In Sections 6-2 through 6-4 you learned to solve systems of two linear equations in two variables using algebraic methods. Similar methods can be used to solve systems of three linear equations in three variables, since the transformations listed on page 281 are also valid for these systems. A **solution of a system** of three equations in three variables is an ordered triple that satisfies each equation in the system. The **solution set of the system** is the set of all such triples.

EXAMPLE 3 Solve the system:

$$\begin{aligned} 2x - y + z &= -8 \quad &(1)\\ 2x + 3y - 2z &= 7 \quad &(2)\\ x + 2y + 3z &= 1 \quad &(3) \end{aligned}$$

SOLUTION

1. Subtract equation (2) from equation (1) to eliminate x.

$$\begin{array}{r} 2x - y + z = -8 \\ 2x + 3y - 2z = 7 \\ \hline -4y + 3z = -15 \quad (4) \end{array}$$

2. Multiply equation (3) by 2 and subtract the result from equation (1) to eliminate x. Simplify the new equation when possible.

$$\begin{array}{rcr} 2x - y + z = -8 & \longrightarrow & 2x - y + z = -8 \\ x + 2y + 3z = 1 & \longrightarrow & 2x + 4y + 6z = 2 \\ \cline{3-3} & & -5y - 5z = -10 \\ & & y + z = 2 \quad (5) \end{array}$$

3. Equation (4) and equation (5) form a system of equations in two variables that can be solved for y and z.

$$\begin{array}{rcr} -4y + 3z = -15 & \longrightarrow & -4y + 3z = -15 \\ y + z = 2 & \longrightarrow & 3y + 3z = 6 \\ \cline{3-3} & & -7y = -21 \\ & & y = 3 \end{array}$$

$$\begin{aligned} y + z &= 2 \\ 3 + z &= 2 \\ z &= -1 \end{aligned}$$

4. Substitute 3 for y and -1 for z in one of the original equations. Then solve for x.

$$\begin{aligned} 2x - y + z &= -8 \\ 2x - 3 + (-1) &= -8 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

5. Checking that $(-2, 3, -1)$ satisfies each of the three original equations is left to you.
   $\therefore$ the solution set is $\{(-2, 3, -1)\}$.

The method for solving a system of three equations in three variables illustrated in Example 3 involves working with pairs of equations. There are three ways to combine the original three equations into pairs: (1) and (2), (2) and (3), and (1) and (3). The same variable is eliminated from any two of these pairs. In Step 1 of Example 3, x is eliminated from equations (1) and (2). In Step 2, x is eliminated from equations (1) and (3). This results in two new equations, (4) and (5), in two variables that are solved for y and z.

Oral Exercises

Tell whether or not each ordered triple is a solution of $x - 2y - 5z = 3$.

1. $(-5, 2, -3)$
2. $(4, -2, 1)$
3. $(3, 0, 0)$
4. $(-4, -1, -1)$

Tell whether or not each ordered triple is a solution of $4x - y + 2z = 7$.

5. $(\frac{3}{4}, 6, -1)$
6. $(0, 1, 4)$
7. $(0, -5, \frac{3}{2})$
8. $(\frac{1}{2}, 3, 4)$

Find three solutions of each equation.

9. $2x - y - z = 2$
10. $x - 3y + 2z = 5$
11. $3x + y - 2z = 8$
12. $3x + 2y - z = -3$

Written Exercises

Solve each system of equations.

A

1. $x + y + z = 2$
   $x + 2y - z = -3$
   $2x - 2y - z = 2$
2. $x + y + z = 4$
   $x - y + z = 0$
   $2x - 3y + z = 2$
3. $x + y + z = 6$
   $x + y - z = 4$
   $2x - y = 1$
4. $2x + y - z = 1$
   $x + 2y + z = 5$
   $y - z = -3$
5. $x + y - 2z = 3$
   $3x + 2y - z = 2$
   $2x + 3y + z = -7$
6. $4x + y + 3z = 2$
   $5x - y - 2z = -3$
   $3x + 2y + 5z = 5$

B

7. $3(2a - b) = 4 + 5c$
   $4a = 5b + 3c - 7$
   $2(a - 3b) = 7c - 3$
8. $3(r - 3q) = -4(1 + p)$
   $2(p - 3q + 4r) = -9$
   $3(2p + q) + 5r = -1$
9. $a + b - c = 1$
   $\frac{1}{2}a - b + \frac{1}{3}c = 3$
   $\frac{1}{3}a + \frac{2}{3}b - \frac{1}{6}c = 2$
10. $r + 2s - t = 12$
    $\frac{3}{2}r + \frac{2}{3}s - \frac{1}{2}t = -7$
    $\frac{1}{4}r + \frac{2}{3}s - \frac{1}{2}t = 1$
11. $5x - 2y - 5z = -1$
    $-x + 2y - 5z = -9$
    $-10x + 4y + 10z = -2$
12. $5x - 2y + 3z = 6$
    $3x + 2y - z = 5$
    $-10x + 4y - 6z = -6$

Find *a*, *b*, and *c* so that the following ordered triples will be solutions of the given equation.

C

13. $ax + by + cz = 8$; $(0, -2, 2)$, $(1, 1, 7)$, $(2, -1, -3)$
14. $ax + 2y - bz = c$; $(1, 5, 3)$, $(3, 3, 5)$, $(0, -3, -4)$

Find *a*, *b*, and *c* so that the following ordered pairs will be solutions of the given equation.

15. $y = ax^2 + bx + c$; $(0, 4), (1, 5), (-1, 9)$
16. $y = ax^2 + bx + c$; $(0, -3), (-1, 4), (2, -5)$
17. $y = ax^2 + bx + c$; $(0, 3), (1, 3), (-1, 5)$
18. $y = ax^2 + bx + c$; $(-1, -1), (0, -2), (-2, 0)$

Problems

Solve.

A

1. The sum of the length, width, and height of a rectangular box is 17 cm. The length is one third the height. The sum of the length and height exceeds twice the width by 2 cm. Find the length, width, and height of the box.
2. The sum of three numbers is 12. One of the numbers is twice the sum of the other two numbers. This same number is also equal to one of the other numbers decreased by triple the third number. What are the three numbers?
3. Marie has \$1.95 in her purse, consisting entirely of nickels, dimes, and quarters. The number of quarters is one more than the sum of the number of nickels and the number of dimes. The number of dimes is two more than twice the number of nickels. How many of each kind of coin does Marie have?
4. Tino has one-dollar, five-dollar, and ten-dollar bills that total \$153. He has as many five-dollar bills as one-dollar and ten-dollar bills combined. The sum of four times the number of one-dollar bills and twice the number of ten-dollar bills is one more than three times the number of five-dollar bills. How many of each kind of bill does he have?

Find the three-digit number that satisfies *all* the conditions (a) through (c).

B

5. a. The sum of the three digits is 15.
   b. The hundreds' digit is half the sum of the tens' digit and the units' digit.
   c. If the hundreds' digit is subtracted from the tens' digit, the difference is one fourth as great as the units' digit.
6. a. The sum of the hundreds' digit and the tens' digit is 9.
   b. The sum of the tens' digit and the units' digit is five times as great as the hundreds' digit.
   c. The tens' digit is three less than twice the sum of the hundreds' digit and the units' digit.

Self-Test 3

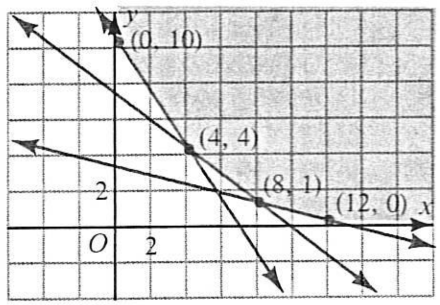
VOCABULARY

linear programming (p. 302)
corner point (p. 303)
convex region (p. 303)
convex polygonal region (p. 303)
solution of an equation in three variables (p. 306)
ordered triple (p. 306)
linear equation in three variables (p. 307)
solution of a system of three equations in three variables (p. 307)

1. Graph the solution set of the following system of inequalities. *Obj. 1, p. 298*

$$y - 2x \leq 1$$
$$2y + x > 4$$
$$x - 5 \leq 0$$

2. Find the minimum value of the linear expression $50x + 80y$ over the convex polygonal region that is shaded in the diagram below. *Obj. 2, p. 298*



3. Calgary Shipping Company needs at least 500 barrels of diesel fuel, 600 barrels of gasoline, and 42 barrels of oil to keep its trucks in operation. Each order placed with Provincial Oil will provide 250 barrels of diesel fuel, 75 barrels of gasoline, and 7 barrels of oil for a total cost of $16,700. Each order placed with Rocky Mountain Oil will provide 50 barrels of diesel fuel, 150 barrels of gasoline, and 7 barrels of oil for a total cost of $10,900. How many orders should the Calgary Shipping Company place with each of the two oil companies if the shipping company wishes to keep all its trucks in operation at the minimum cost?

4. Solve the following system of equations. *Obj. 3, p. 298*

$$2x + 3y - 2z = -6$$
$$2x + 5y + 6z = 2$$
$$6x + 7y - 2z = -2$$

Check your answers with those at the back of the book.

Careers

Aerospace Engineering

Aerospace engineers develop rockets, space exploration vehicles, satellites, and various aircraft. They are involved in projects from the earliest stages of research through the final stages of testing the finished product.

The early development stage of a project involves research, cost analysis, and testing. During this stage, a model of the proposed craft is built, and the design is tested and modified using this model. In the next stage, production, engineers supervise the manufacture of the craft in order to assure that each part conforms to the specifications developed in working with the model. In the final, or testing, stage, the performance of the manufactured craft is tested in actual flight.

EXAMPLE Two rockets are launched vertically at the same time. The acceleration of the first rocket is three times that of the second, and after 4 s the first rocket is 96 m higher than the second. The formula $d = \frac{1}{2}at^2$ gives the distance d in meters that is traveled in t seconds by an object whose acceleration is a m/s^2 (meters per second squared). Determine the acceleration of the first rocket.

SOLUTION Let h = height in meters of the second rocket, and
a = acceleration in m/s^2 of the second rocket.

Then: $h + 96$ = height in meters of the first rocket, and
$3a$ = acceleration in m/s^2 of the first rocket.

Substitute these expressions into the given formula.

Second rocket:

$$d = \tfrac{1}{2}at^2$$
$$h = \tfrac{1}{2}a(4)^2$$
$$h = 8a$$

First rocket:

$$d = \tfrac{1}{2}at^2$$
$$h + 96 = \tfrac{1}{2}(3a)(4)^2$$
$$h + 96 = 24a$$

The equations $h = 8a$ and $h + 96 = 24a$ form a system of two equations in two variables. Solving the system, the value of a is found to be 6, and so the value of $3a$ is 3(6), or 18.

$\therefore$ the acceleration of the first rocket is 18 m/s^2.

EXTRA

Matrices

You can find the solution of a system such as

$$\begin{aligned} x + 4y &= 9 \\ 2x + y &= 4 \end{aligned}$$

by working only with the coefficients of x and y and the constant terms. To do this, you represent the coefficients and constants by means of an ordered array of numbers called a **matrix** (plural: *matrices*). Parentheses or brackets are used to group the *elements* of a matrix. Thus

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

is the *coefficient matrix* of the given system, and

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 4 \end{bmatrix}$$

is the *augmented matrix* of the system.

Now, compare the sequence of steps used to solve the given system by linear combinations, as shown at the left, with the corresponding sequence of matrices shown at the right.

$$\begin{aligned} x + 4y &= 9 \\ 2x + y &= 4 \end{aligned} \qquad \begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 4 \end{bmatrix}$$

Multiplying each member of the first equation by -2 and adding the result to the second equation produces the equivalent system:

$$\begin{aligned} x + 4y &= 9 \\ 0x + (-7)y &= -14 \end{aligned} \qquad \begin{bmatrix} 1 & 4 & 9 \\ 0 & -7 & -14 \end{bmatrix}$$

Dividing the second equation by -7, or multiplying by $-\frac{1}{7}$, produces the equivalent system:

$$\begin{aligned} x + 4y &= 9 \\ 0x + y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 4 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

Adding -4 times the second equation to the first equation yields the equivalent system:

$$\begin{aligned} x + 0y &= 1 \\ 0x + y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The only solution of the system, (1, 2), is evident from this last set of equations and from the right-hand column of the corresponding matrix. At each step, the matrix shown is obtained from the preceding matrix in the sequence in exactly the same way as the corresponding system of equations is obtained from its preceding system.

Each of the matrices in the example on the preceding page is said to be *row-equivalent* to each of the other matrices in the example. In general:

> Two matrices are **row-equivalent** if one can be obtained from the other by means of one or more of the following *row transformations*:
> 1. Interchanging two rows.
> 2. Multiplying each entry in a row by the same nonzero real number.
> 3. Multiplying each entry in a row by a nonzero real number and adding the resulting product to the corresponding entry in another row.

As illustrated by the example, you can use the concept of equivalent matrices to solve a system of linear equations in two variables. The following is a summary of this method.

> *Steps in Solving Systems by Matrices*
> 1. Represent the system by its augmented matrix.
> 2. Use row transformations to obtain an equivalent matrix of the form
> $$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}.$$
> 3. Read the solution, (p, q), from this matrix.

EXAMPLE Solve the system: $3x - 4y = 5$
$2x + 5y = -12$

SOLUTION $\begin{bmatrix} 3 & -4 & 5 \\ 2 & 5 & -12 \end{bmatrix}$

$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3} \\ 2 & 5 & -12 \end{bmatrix}$ ⟵ $\frac{1}{3} \times$ row 1

$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3} \\ 0 & \frac{23}{3} & -\frac{46}{3} \end{bmatrix}$ ⟵ row 2 + [(−2) × row 1]

$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$ ⟵ $\frac{1}{23} \times$ row 2

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$ ⟵ row 1 + (4 × row 2)

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$ ⟵ 3 × row 2

∴ the solution set is $\{(-1, -2)\}$.

If, in the solution process, you can obtain a matrix of the form:

$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ a & b & c \end{bmatrix}$, the two equations in the system are equivalent;

$\begin{bmatrix} a & b & c \\ 0 & 0 & d \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & d \\ a & b & c \end{bmatrix}$, $d \neq 0$, the two equations in the system are inconsistent.

Exercises

Solve each system using matrices.

1. $2x - y = 0$
$x + 3y = 7$

2. $2x - 3y = -5$
$5x - 8y = -13$

3. $-2x + y = -6$
$x - 3y = -2$

4. $4x - 3y = -15$
$x + 2y = -1$

5. $3x - 5y = 17$
$-5x + y = -21$

6. $x - 3y = 5$
$2x - 6y = 7$

Chapter Summary

1. The *solution set* of a *system of linear equations in two variables* is the set of all ordered pairs of numbers that satisfy each equation in the system.
2. The graphs of two linear equations in two variables on the same coordinate plane are either *parallel lines, coincident lines,* or *lines that intersect in exactly one point.*
3. A system of equations having one or more solutions is called a *consistent system.* A system whose solution set is the empty set is called *inconsistent.*
4. To solve a system of linear equations in two variables graphically, graph the equations on the same coordinate plane and determine the coordinates of all points common to the graphs.
5. The *linear-combination method* and the *substitution method* are algebraic methods that may be used to solve a system of linear equations.
6. A *system of linear inequalities* may be solved graphically by finding the intersection of the half-planes that are the graphs of the inequalities.
7. *Linear programming* is a method that may be used to solve problems in which a quantity that can be represented by a linear equation is to be maximized or minimized subject to conditions that can be represented by a system of linear inequalities.
8. The *solution set* of a *system of linear equations in three variables* is the set of all *ordered triples* of numbers that satisfy each equation in the system. Systems of three or more equations in *three or more variables* can be solved using the algebraic methods developed for systems of two equations in two variables.

Chapter Review

Write the letter of the correct answer.

1. If the system $\begin{matrix} 2x - 3y = 4 \\ nx - y = 4 \end{matrix}$ is consistent, which of the following statements must be true? 6-1

   a. $n = 2$ b. $n = \frac{2}{3}$ c. $n \neq 0$ d. $n \neq \frac{2}{3}$

2. Given the system $\begin{matrix} y - 3x = -2 \\ 3x - y = 2 \end{matrix}$, how are the graphs of the two equations in the system related?

   a. The graphs coincide.
   b. The graphs are parallel lines.
   c. The graphs intersect at exactly one point, (2, 4).
   d. None of the above statements is true.

3. The solution set of the system $\begin{matrix} 3x - y = 10 \\ x - 3y = -2 \end{matrix}$ is $\{(4, 2)\}$. Which of the following is *not* an equivalent system? 6-2

   a. $\begin{matrix} x = 4 \\ y = 2 \end{matrix}$ b. $\begin{matrix} x + y = 6 \\ x - y = 2 \end{matrix}$

   c. $\begin{matrix} x = y + 2 \\ y = \frac{1}{2}x \end{matrix}$ d. $\begin{matrix} x + y = 6 \\ 2x + 2y = 12 \end{matrix}$

4. Use addition or subtraction to solve the system: $\begin{matrix} 4a = 5b - 9 \\ -2b = 4a - 5 \end{matrix}$

   a. $\{(2, \frac{1}{4})\}$ b. $\{(4, 2)\}$ c. $\{(\frac{1}{4}, 2)\}$ d. $\emptyset$

5. To solve the system $\begin{matrix} 5x - 3y = 7 \\ 3x + 2y = 9 \end{matrix}$ using linear combinations, the first equation is multiplied by a and the second equation is multiplied by b. Which values of a and b will eliminate one of the variables when the resulting equations are added? 6-3

   a. $a = 3, b = 5$ b. $a = -2, b = 3$

   c. $a = -3, b = 5$ d. $a = 2, b = -3$

6. Use linear combinations to solve the system: $\begin{matrix} 3x + 2y = 5 \\ 2x - 3y = 12 \end{matrix}$

   a. $\{(3, -2)\}$ b. $\{(-2, 3)\}$ c. $\{(3, 2)\}$ d. $\{(3, -\frac{1}{2})\}$

7. To solve the system $\begin{matrix} 2x - y = 1 \\ 2x + 3y = 7 \end{matrix}$ by substitution, which expression can be substituted for y in the *second* equation? 6-4

   a. $2x - y$ b. $1 - 2x$ c. $2x - 1$ d. $\frac{7}{3} - \frac{2}{3}x$

8. Use substitution to solve the system: $\begin{aligned} 2x + 3y &= 7 \\ x - 4y &= -13 \end{aligned}$

a. $\{(2, 1)\}$ b. $\{(-5, 2)\}$ c. $\{(-1, 3)\}$ d. $\{(\frac{9}{5}, \frac{19}{5})\}$

9. A collection of coins contains a total of fourteen dimes and quarters and is worth $1.85 in all. How many dimes are in the collection? 6-5

a. 8 b. 11 c. 3 d. 6

10. With a tail wind, a plane travels 300 km in 1 h. With no change in the wind, the return trip takes $1\frac{1}{2}$ h. Find the speed of the wind. 6-6

a. 100 km/h b. 250 km/h c. 50 km/h d. 30 km/h

11. The units' digit of a two-digit number is twice the tens' digit. If the digits of the number were reversed, the resulting number would be six less than twice the original number. Find the original number. 6-7

a. 12 b. 24 c. 36 d. 48

12. Which point belongs to the graph of the following system? 6-8

$$\begin{aligned} x + y &\geq 1 \\ x - y &\leq 3 \\ x - 2 &\leq 0 \end{aligned}$$

a. (0, 0) b. (1, 2) c. (−1, 0) d. (2, −3)

13. Find the coordinates of the corner points of the solution set of the following system. 6-9

$$\begin{aligned} x &\geq 0 \\ x &\leq 2 \\ y &\geq 0 \\ x + y &\leq 4 \end{aligned}$$

a. (0, 0), (4, 0), (2, 2), (0, 4)
b. (0, 0), (0, 2), (2, 0), (2, 2)
c. (0, 0), (2, 0), (0, 2), (4, 0)
d. (0, 0), (2, 0), (2, 2), (0, 4)

14. Find the minimum value of the expression $4x + 3y$ over the region that has corner points (1, 0), (3, 0), (3, 2), and (1, 4).

a. 18 b. 12 c. 4 d. 16

15. Solve the system: $\begin{aligned} 2x + y - z &= -5 \\ -5x - 3y + 2z &= 7 \\ x + 4y - 3z &= 0 \end{aligned}$ 6-10

a. $\{(-2, 5, 6)\}$ b. $\{(3, -4, -7)\}$ c. $\{(4, 3, -2)\}$ d. $\{(-1, 0, 5)\}$

Chapter Test

1. Solve the system $\begin{array}{l} x + 2y = 6 \\ x = 2 - 2y \end{array}$ using graphs. 6-1

2. Is the system $\begin{array}{r} 3x + 3y = -9 \\ -4x - 4y = 12 \end{array}$ consistent or inconsistent?

Solve each system using addition or subtraction. 6-2

3. $\begin{array}{r} 32 = 5a - 3b \\ -8 = 5a + 7b \end{array}$

4. $\begin{array}{r} 3r + 3s = -5 - 2r \\ 6r + 3 = 7s + r \end{array}$

Solve each system using the linear-combination method. 6-3

5. $\begin{array}{r} 7c + 5d = 2 \\ 8c - 9d = 17 \end{array}$

6. $\begin{array}{r} 5x + 4y = 11 \\ -7x + 3y = 19 \end{array}$

Solve each system using the substitution method. 6-4

7. $\begin{array}{r} 3m - 4n = 5 \\ m + 7n = 10 \end{array}$

8. $\begin{array}{r} 2(x + y) = 4(x + 1) \\ 4(x + 2) = y - 3 \end{array}$

Solve.

9. Four years ago Sylvia was two thirds as old as Alex was then. Four years from now, she will be four fifths as old as Alex will be. How old are Sylvia and Alex now? 6-5

10. Donna took 3 h to row 18 km upstream. Her return trip downstream took 1 h less. Find her rate of rowing in still water and the rate of the current. 6-6

11. The hundreds' digit of a three-digit number is two less than the tens' digit, and the tens' digit is one more than the units' digit. The sum of the digits is 12. Find the number. 6-7

12. Graph the solution set of the system: $\begin{array}{r} 2y - 2 \geq x \\ 2y + x \leq 8 \\ x \geq 0 \end{array}$ 6-8

13. Graph the solution set of the system at the right and label the corner points with their coordinates. $\begin{array}{r} 2x + y \geq 4 \\ x + y \geq 3 \\ x \geq 0 \\ y \geq 1 \end{array}$ 6-9

14. Find the minimum value of the expression $2x + 3y$ over the region graphed in Exercise 13.

15. Solve the system: $\begin{array}{r} x + y + z = 3 \\ x + 2y - z = -1 \\ -2x - y + 3z = 11 \end{array}$ 6-10

Mixed Review

Simplify.

1. $-(-3 + 2) + 7$
2. $2|3 - 6| - |-5 + 8|$
3. $(3.2 + 6.7) - (12.7 + 4.2)$
4. $-25\frac{1}{3} + 8\frac{2}{3} - 10\frac{1}{3}$
5. $\frac{1}{2}[24 \div 2(6)] \div [12(3) \div 6]$
6. $[(11 - 3)(7 - 4)] - [18 \div (6 \div 2)]$
7. $(28 \div 2 + 20 \div 5) \div 9(2)^2$
8. $(2 + 3)^2 - [9 \div (8 - 5)^2]$
9. $2x + 5y + 2x + 5y$
10. $3q^2 - 2q - 13q^2 + 8q$
11. $(6 + 4a) - (7 - 3a)$
12. $-3[x - (6 - x)]$
13. $2(4m + 5) + 3(m + 8)$
14. $9(r + 3r^2) + 7(r^2 + 4r)$
15. $3(c - 2d) - 6(3c + d)$
16. $4(-x + 2y) - 3(x - 5y)$
17. $-4p(-3q)(5r)$
18. $x(-x)(14x)$

Graph on a number line.

19. {the natural numbers between -2 and 5}
20. {the positive even integers less than or equal to 6}
21. {the real numbers between -3 and 4}
22. {the positive real numbers}
23. $\{x: |x| \geq 2\}$
24. $\{y: |y| < 4\}$

Graph on a coordinate plane.

25. $f: x \longrightarrow x + 3;\ D = \{-2, -1, 0, 1, 2\}$
26. $g: x \longrightarrow -3;\ D = \{-4, -2, 0, 2, 4\}$
27. $x = -1$
28. $y = 3$
29. $2x - y = 4$
30. $2x + 4y = 3$
31. $x - y \geq 3$
32. $6x + 3y > 2$

Evaluate each expression when $r = 4$, $s = -2$, and $t = \frac{1}{2}$.

33. $rt + s$
34. $r^2 - st$
35. $\frac{r - s}{-t}$
36. $4(r - s) + 2t$

Write an equation of the line that satisfies the given requirements. The equation should be in the form $ax + by = c$, where a, b, and c are integers.

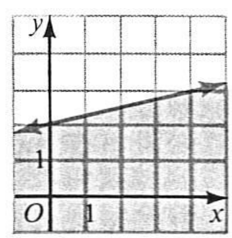
37. has slope $-\frac{4}{5}$ and y-intercept $\frac{7}{4}$
38. has slope $\frac{2}{3}$ and passes through the point (1, 2)
39. is parallel to the x-axis and passes through the point $(-5, 6)$
40. passes through the points $(4, -1)$ and $(-2, 3)$

PREPARING FOR COLLEGE ENTRANCE EXAMS

Strategy for Success: If you cannot solve a certain problem within a short period of time, leave that problem and go on to others that may be easier for you. Working on the easier problems may give you an idea that will help in solving the harder problem. Then, if time permits, you can go back and attempt the harder problem again.

Decide which is the best of the choices and write the corresponding letter on your answer sheet.

1. Determine an equation of the line that intersects the y-axis at the same point as the line containing (3, 2) and (−1, −2) and that is parallel to the line containing (4, 5) and (10, 1).
   (A) $3y - 2x = 3$ (B) $2x - 3y = 3$ (C) $2x - 3y = 1$
   (D) $2x + 3y = 1$ (E) $2x + 3y = -3$
2. Which of the following points lies on the line that has slope $-\frac{1}{2}$ and that passes through the point (−2, 3)?
   I. $(-1, \frac{5}{2})$ II. $(-3, \frac{1}{4})$ III. (2, 3)
   (A) I only (B) II only (C) III only (D) I and II only (E) I and III only
3. The domain of the function $f\colon x \longrightarrow \dfrac{1}{|x|}$ is the set of all real numbers except 0. Which of the following statements is true?
   (A) There are two values of x such that $f(x) = 0$.
   (B) The range of f is the set of all nonnegative real numbers.
   (C) The graph of f on a coordinate plane is a line.
   (D) The ordered pair $(-\frac{1}{2}, 2)$ satisfies the function.
   (E) None of the above statements is true.
4. Solve the system: $4x - \dfrac{y}{4} = -\dfrac{3}{2}$ and $4x + 7y = -4$
   (A) $\{(-8, 4)\}$ (B) $\{(4, -8)\}$ (C) $\emptyset$
   (D) $\{(x, y)\colon x = -\frac{7}{4}y - 1\}$ (E) none of these
5. The sum of the digits of a three-digit number is 15. The units' digit is half the tens' digit, and the hundreds' digit is one less than the units' digit. Find the number.
   (A) 438 (B) 843 (C) 348 (D) 384 (E) 483
6. Identify the inequality that is graphed on the coordinate plane at the right.
   (A) $4x - y \geq 8$ (B) $x - 4y \leq -8$
   (C) $x - 4y \leq 8$ (D) $x - 4y \geq -8$
   (E) $x - 4y \geq 8$



Contest Problems

1. The coordinates of two opposite vertices of a rectangle are $(-1, -2)$ and $(2, 3)$. If the sides of the rectangle are parallel to the coordinate axes, what is the area of the rectangle in square units?

2. The sum of the ages of a family of six persons is 160. If their ages range from six to fifty, what was the sum of their ages four years ago?

3. If $9x = 5x - 8c + 6$, what is the value of $6x + 5c$ in terms of c?

4. A straight line passes through the points $(2k + 1, -2)$, $(-1 - 3k, 8)$, and $(1, 3)$. Through which of the following points does the line also pass?

   a. $(-7, -6)$ **b.** $(-7, -8)$ **c.** $(9, 13)$ **d.** $(9, 15)$

5. How many solid cubes that measure 2 cm on each edge are needed to completely fill an empty cube that measures 6 cm on each edge?

6. Determine the least number of steps necessary to measure exactly 6 L of water if you have only a 4 L pail and a 9 L pail. If a represents the contents of the 4 L pail and b represents the contents of the 9 L pail, use an ordered pair (a, b) to describe the contents of each pail at each step.