Chapter 5

Graphs and Functions

Ordered Pairs and Functions

OBJECTIVES for Sections 5-1 through 5-3:

1. *To graph ordered pairs of numbers on a coordinate plane and to find the coordinates of any point on the plane.*
2. *To identify the domain and range of a relation specified by a set of ordered pairs and to draw a mapping diagram for the relation.*
3. *To determine whether a relation with a given finite domain is a function.*
4. *To graph relations and functions with given finite domains.*

5–1 The Coordinate Plane

In Section 1-1 you learned how to graph real numbers as points on a number line. In this section, you will learn how to graph *ordered pairs* of real numbers as points on a *number plane*. The following steps outline a general procedure for constructing a number plane.

1. Draw two perpendicular number lines that intersect at the origin of each. These two number lines are called **axes.** As shown in Figure 1, one axis is usually horizontal and the other, vertical. The point of intersection of these axes is called the **origin** of the number plane and is labeled O.

2. Indicate the positive direction on each axis by an arrowhead. The positive direction is usually to the right on the horizontal axis and upward on the vertical axis, as shown in Figure 1.

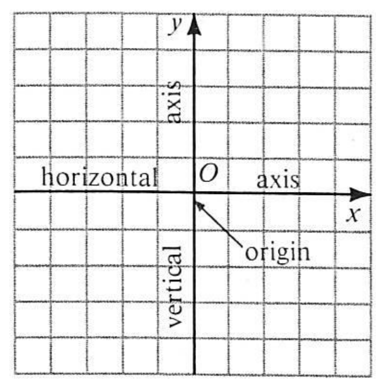


Figure 1

On a number plane, the horizontal axis is usually labeled with an x and is referred to as the **x-axis.** Similarly, the vertical axis is labeled with a y and is referred to as the **y-axis.** The x- and y-axes together are called **coordinate axes,** and the number plane is called a **coordinate plane.** The coordinate axes separate a coordinate plane into four **quadrants,** which are numbered as shown in Figure 2. Points on the coordinate axes are not considered to be in any quadrant.

Each point on a coordinate plane can be assigned to a pair of real numbers. These numbers are called the **coordinates** of the point. You can find the coordinates of a given point P as follows.

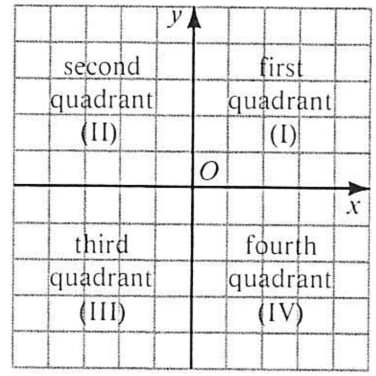


Figure 2

1. Draw a *vertical line* from P to the x-axis. The coordinate of the point at which this line intersects the x-axis is called the **x-coordinate,** or **abscissa,** of P. In Figure 3 below, the abscissa of P is 3.
2. Next, draw a *horizontal line* from P to the y-axis. The coordinate of the point at which this line intersects the y-axis is called the **y-coordinate,** or **ordinate,** of P. In Figure 3, the ordinate of P is 2.
3. Name the coordinates of P as an **ordered pair** of real numbers in the form (*abscissa*, *ordinate*). In Figure 3, you refer to P as the point (3, 2) and write $P(3, 2)$.

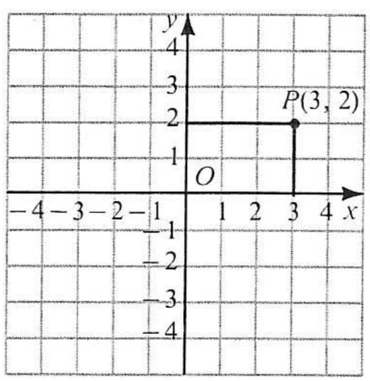


Figure 3

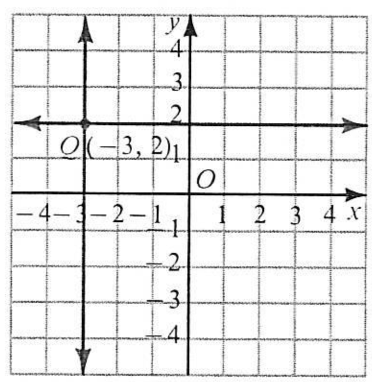


Figure 4

By reversing the process just described, each ordered pair of real numbers can be assigned to a point on a coordinate plane. When you find the point assigned to an ordered pair, you **graph the ordered pair,** or **plot the point** that corresponds to the ordered pair. For example, referring to Figure 4 above, the ordered pair (−3, 2) is graphed as follows.

1. Draw a *vertical line* through the graph of −3 on the x-axis.
2. Next, draw a *horizontal line* through the graph of 2 on the y-axis.
3. Locate the point of intersection of the vertical and horizontal lines. In Figure 4, the graph of (−3, 2) is the point Q.

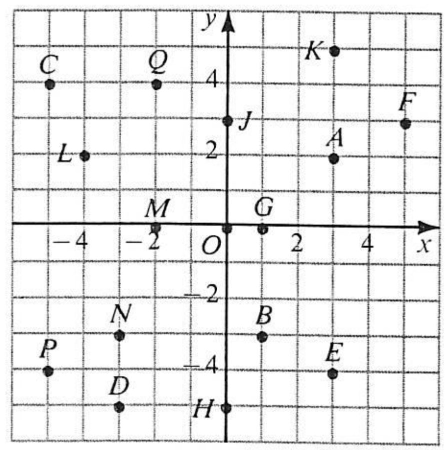
In working with a coordinate plane, you take the following facts for granted.

1. Each ordered pair of real numbers corresponds to exactly one point on a coordinate plane.
2. Each point on a coordinate plane corresponds to exactly one ordered pair of real numbers.

This one-to-one correspondence between the set of all points on a coordinate plane and the set of all ordered pairs of real numbers is called a **plane rectangular,** or **Cartesian, coordinate system.** (The Cartesian coordinate system is named after René Descartes, a seventeenth century French mathematician and philosopher who developed the early ideas about the system.)

Oral Exercises

Exercises 1–16 refer to the coordinate plane at the right. Name the coordinates of each point.



Exs. 1-16

1. *A* 2. *B*
3. *C* 4. *D*
5. *G* 6. *H*
7. *N* 8. *O*

Name the point that has the given coordinates.

9. (3, 5) 10. (5, 3)
11. (−2, 4) 12. (−4, 2)
13. (3, −4) 14. (−5, −4)
15. (−2, 0) 16. (0, 3)

Name all the quadrants, if any, that contain points satisfying the given requirements.

17. The ordinate is 2.
18. The ordinate is −3.
19. The abscissa is −2.
20. The abscissa is 3.
21. The ordinate is positive.
22. The ordinate is negative.
23. The abscissa is positive.
24. The abscissa is negative.
25. The ordinate is zero.
26. The abscissa is zero.
27. The ordinate equals the abscissa.
28. The ordinate equals the opposite of the abscissa.
29. Name the ordinate of every point on the x-axis.
30. Name the abscissa of every point on the y-axis.

Written Exercises

Plot the given points on a coordinate plane.

A
1. $A(3, 4)$
2. $B(-2, -1)$
3. $C(3, -1)$
4. $D(-5, 2)$
5. $E(4, 0)$
6. $F(0, -3)$
7. $G(4, -2)$
8. $H(-1, -3)$
9. $L(4, 4)$
10. $M(-1, -1)$
11. $N(0, -2)$
12. $O(0, 0)$

Given a point with coordinates (x, y), name all the quadrants or axes in which the point might lie under the specified conditions.

13. $x > 0$ and $y > 0$
14. $x < 0$ and $y < 0$
15. $x > 0$ and $y < 0$
16. $x < 0$ and $y > 0$
17. $x = 0$
18. $y = 0$
19. $xy > 0$
20. $xy < 0$
21. $xy = 0$
22. $\frac{x}{y} = 0$
23. $x - y = 0$
24. $x + y = 0$

Plot four points that are in at least two different quadrants and whose coordinates are integers that satisfy the given requirements. Construct a different set of axes for each exercise.

B
25. $y = x$
26. $y = -x$
27. $y = 4x$
28. $y = \frac{1}{4}x$
29. $y = x + 1$
30. $y = x - 1$
31. $y = |x|$
32. $y = -|x|$

In Exercises 33–38, each set of ordered pairs lists the coordinates of three vertices of a rectangle on a coordinate plane. Graph the ordered pairs and sketch the rectangle. Then determine the coordinates of the fourth vertex.

33. $(-1, 1), (5, -3), (-1, -3)$
34. $(2, 3), (4, 3), (2, -4)$
35. $(1, 2), (-4, 2), (1, -5)$
36. $(-5, 0), (0, -2), (-5, -2)$
37. $(\frac{1}{2}, 1), (3, 1), (3, 6)$
38. $(-4, -\frac{3}{2}), (2, \frac{3}{2}), (-4, \frac{3}{2})$

For any three given points not all on one line, there are three possible ways of choosing a fourth point so that the four points are the vertices of a parallelogram. In each of Exercises 39 and 40, the coordinates of three vertices of a parallelogram are given. What are the coordinates of the three possible fourth vertices?

C
39. $(2, -3), (6, -2), (5, 1)$
40. $(-2, -1), (1, 0), (-1, 4)$
41. Three vertices of an isosceles trapezoid are the points with coordinates $(0, 0)$, $(4, 0)$, and $(6, -2)$. Find the coordinates of two possible points for the fourth vertex.
42. The base of an isosceles triangle has endpoints $(-6, 0)$ and $(0, -4)$. Using only integers, name the coordinates of six possible points for the third vertex of the triangle.
43. What is the length of the segment that joins $P(a, b)$ and $Q(a, c)$?
44. What is the length of the segment that joins $R(a, c)$ and $S(b, c)$?

5–2 Relations and Functions

In mathematics, any set of ordered pairs is called a **relation.** For example, the set

$$\{(0, 2), (-4, 3), (-3, -2), (2, -1)\}$$

is a relation. The set D of all the *first coordinates* of the ordered pairs is called the **domain** of the relation, while the set R of all the *second coordinates* is called the **range** of the relation. Thus, for the relation just specified,

$$D = \{0, -4, -3, 2\} \qquad \text{and} \qquad R = \{2, 3, -2, -1\}.$$

There are various ways to picture a relation. One of these is to use a table such as that shown in Figure 5, which pictures the relation in the preceding example. Another way is to use a *mapping diagram* such as that shown in Figure 6, which pictures this same relation.

D	R
0	2
−4	3
−3	−2
2	−1

Figure 5

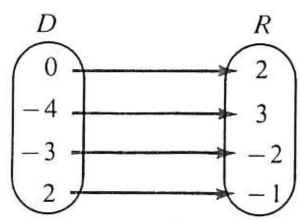


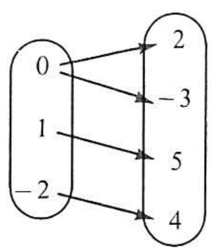
Figure 6

If a relation is such that no two ordered pairs have the same first coordinate, the relation is called a **function.**

EXAMPLE 1 Draw a mapping diagram of each relation. Then tell whether or not the relation is a function.

a. $\{(0, 2), (0, -3), (1, 5), (-2, 4)\}$ **b.** $\{(-2, -1), (1, 2), (5, 2), (-1, 0)\}$

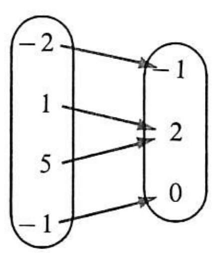
SOLUTION **a.**



The mapping diagram clearly shows that two of the ordered pairs, $(0, 2)$ and $(0, -3)$, have the same first coordinate, 0.

∴ the relation is not a function.

b.



The mapping diagram clearly shows that no two of the ordered pairs have the same first coordinate.

∴ the relation is a function.

Notice in part (b) of Example 1 that a relation may be a function even if two ordered pairs of the relation have the same *second* coordinate.

Since a relation is a set of ordered pairs, another way to picture a relation is to graph the ordered pairs on a coordinate plane. For any given relation, the set of points on a coordinate plane that correspond to the ordered pairs is called the **graph of the relation.** For example, the relation

$$\{(-4, 1), (-1, 4), (3, 2), (3, -2)\}$$

can be graphed as shown in Figure 7. Notice that numbers in the *domain* of the relation are associated with the x-axis, while numbers in the *range* are associated with the y-axis.

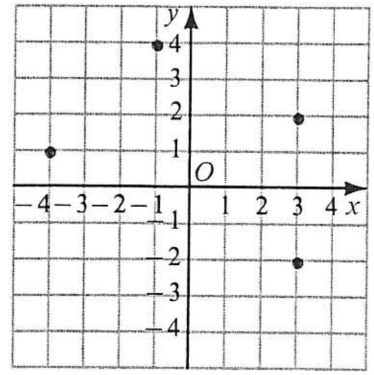


Figure 7

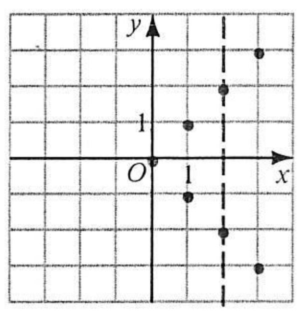
You know that the relation graphed in Figure 7 is *not* a function because two of the ordered pairs, (3, 2) and (3, −2), have the same first coordinate. On the graph, the points that correspond to these two ordered pairs lie directly above one another. This suggests the following "vertical-line test" to determine whether a given graph represents a function: *A relation is a function if and only if no vertical line can be drawn that intersects the graph of the relation in more than one point.*

EXAMPLE 2 Graph each relation. Then use the vertical-line test to determine whether or not the relation is a function.

a. $\{(0, 0), (1, 1), (1, -1), (2, 2), (2, -2), (3, 3), (3, -3)\}$

b. $\{(0, 0), (1, 1), (-1, 1), (2, 2), (-2, 2), (3, 3), (-3, 3)\}$

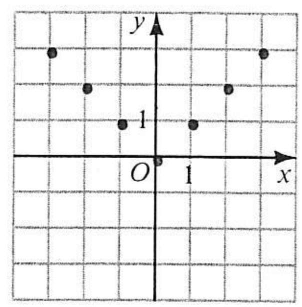
SOLUTION **a.**



The vertical line drawn through two points indicates that two ordered pairs have the same first coordinate.

∴ the relation is not a function.

b.



It is not possible to draw a vertical line that intersects two points of the graph, and so no two ordered pairs have the same first coordinate.

∴ the relation is a function.

In general, then, a relation is a pairing between two sets, the domain and the range. A function is a special type of relation in which no member of the domain is paired with more than one member of the range. Thus, the following is an alternative definition of a function.

A **function** is a pairing that assigns to each member of one set, called the domain, *exactly one* member of a second set, called the range. Each member of the range is assigned to *at least one* member of the domain.

Oral Exercises

State the domain and range of each relation. Then tell whether or not the relation is a function.

1. {(1, 1), (2, 2), (3, 3)}
2. {(−4, 4), (−2, 2), (−1, 1)}
3. {(1, 1), (1, 2), (1, 3)}
4. {(1, 1), (2, 1), (3, 1)}
5. {(−2, 4), (−1, 1), (1, 1), (2, 4)}
6. {(4, −2), (1, −1), (1, 1), (4, 2)}

Tell whether or not each mapping diagram represents a function.

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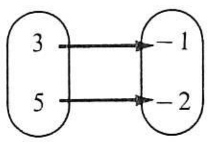

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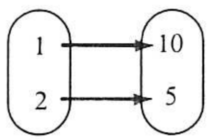

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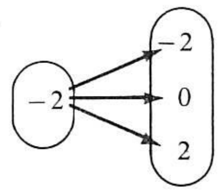

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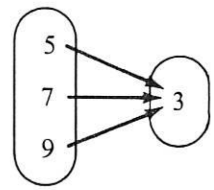

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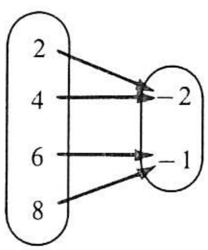

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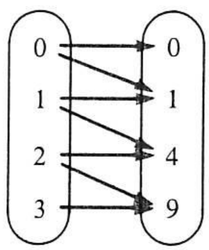

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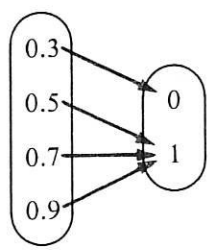

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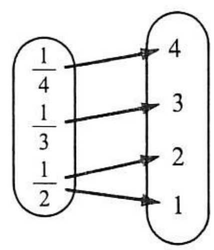

Tell whether or not each graph represents a function.

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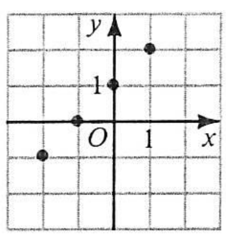

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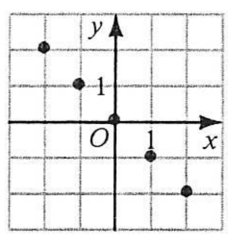

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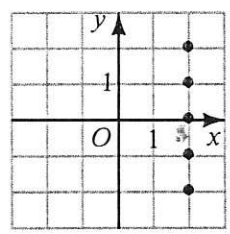

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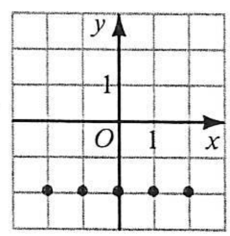

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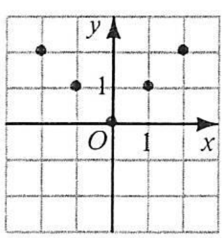

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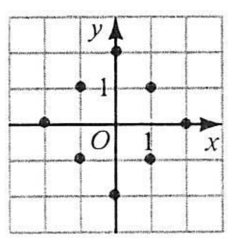

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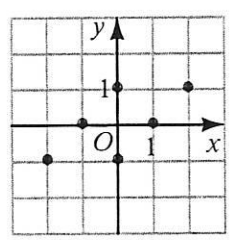

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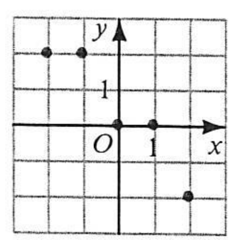
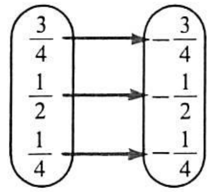
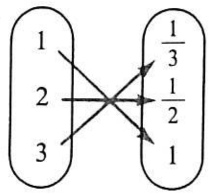


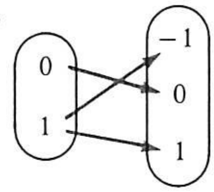
Written Exercises

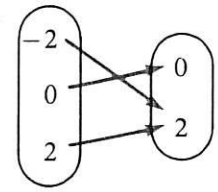
Write the set of ordered pairs in the relation that is represented by each mapping diagram.

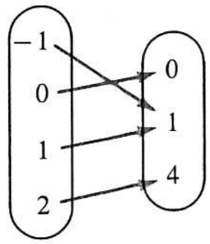
A

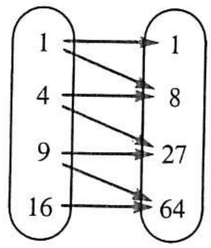
1. 

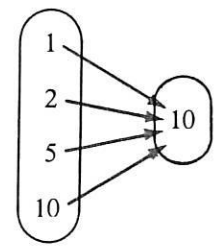
2. 

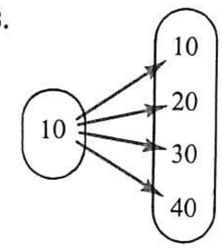
3. 

4. 

5. 

6. 

7. 

8. 

In Exercises 9–16:

a. Graph the relation whose ordered pairs are shown in the given table.

b. Tell whether or not the relation is a function.

9.

D	*R*
1	3
2	5
3	1
4	4
5	2

10.

D	*R*
1	3
2	5
3	1
4	3
5	5

11.

D	*R*
−1	1
−1	2
0	−3
3	4
5	−2

12.

D	*R*
−3	−1
−2	4
0	−1
1	5
4	−1

13.

D	*R*
0	0
1	2
2	4
3	6
4	8

14.

D	*R*
−6	−2
−3	−1
0	0
3	1
6	2

15.

D	*R*
4	−4
2	−2
0	0
2	2
4	4

16.

D	*R*
−4	4
−2	2
0	0
2	2
4	4

Draw a mapping diagram that represents each relation. Then tell whether or not the relation is a function.

B 17. {(0, 5), (1, 6), (2, 7), (3, 8)}

18. {(1, 3), (2, 5), (3, 5), (4, 7)}

19. {(−10, −1), (0, −1), (1, 3), (5, 3)}

20. {(1, 1), (1, −1), (3, 5), (3, −5)}

21. {(2, −7), (2, −3), (2, 0), (2, 4)}

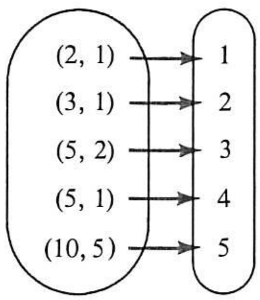
22. {(−6, 1), (−5, 1), (3, 1), (7, 1)}

23. {(−2, 4), (−1, 1), (1, 1), (2, 4)}

24. {(9, −3), (1, −1), (1, 1), (9, 3)}

In each of Exercises 25 and 26 there is a mapping diagram of a relation. Write the ordered pairs in the relation and tell whether or not it is a function. Then state a rule that can be used to calculate the number in the range that is associated with any given ordered pair in the domain. (*Note*: The first coordinate of an ordered pair may itself be an ordered pair.)

25.



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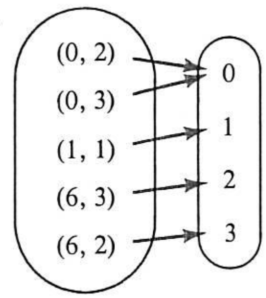


Exercises 27–30 refer to the mapping diagrams in Written Exercises 1–8. Tell which of these mapping diagrams represents a *function* that might be described by the given "rule."

C 27. reciprocal

28. additive inverse

29. square

30. absolute value

31. A certain mapping diagram shows a domain of three members being mapped onto a range of four members. Can this relation be a function? Why or why not?

Computer Exercises For students with computer experience

1. Write a program that will allow you to input the coordinates of any point and will display the name of the quadrant or axis (or axes) in which the point is located. RUN the program for the points given in Exercises 1–12 on page 218.
2. Write a program that will determine whether a relation that consists of *four* ordered pairs is a function. RUN the program for the relations given in Exercises 17–24 above.
3. Modify the program that you wrote for Exercise 2 so that it will determine whether a relation that consists of *any number* of ordered pairs is a function.

5–3 Defining Relations and Functions

Recall that in Section 1-2 you learned how to specify sets both by roster and by rule. Thus far, each of the relations that you have studied has been specified by a *roster* of its ordered pairs. Often, however, a relation is specified by a *rule* that describes exactly how the members of the domain and the range are related. Generally this rule is an open sentence. For example, consider the relation

$$\{(3, 1), (6, 2), (9, 3), (12, 4), (15, 5)\},$$

in which each member of the domain is the triple of the related member of the range. If x represents a member of the domain and y represents a member of the range, a rule for this relation is

$$x = 3y, \qquad \text{or} \qquad y = \frac{1}{3}x.$$

A rule for a relation is said to *define* the relation. That is, given a rule and a domain for a relation, it is possible to determine all the ordered pairs that form the relation. Therefore, it is also possible to determine the range of the relation.

EXAMPLE 1 Determine the range R of the relation defined by the rule $y = 2x + 6$, if the domain $D = \{-5, -3, 0, 4\}$.

SOLUTION Replace x in the expression $2x + 6$ with each member of D and find the corresponding member of R. A table may be helpful.

x	$2x + 6$
-5	$2(-5) + 6 = -10 + 6 = -4$
-3	$2(-3) + 6 = -6 + 6 = 0$
0	$2(0) + 6 = 0 + 6 = 6$
4	$2(4) + 6 = 8 + 6 = 14$

$\therefore R = \{-4, 0, 6, 14\}$

Notice that the relation specified in Example 1 is a function. Often a function is named by a single letter, such as f, F, or g. For example, the function defined by the rule $y = 2x + 6$ may be called f, and it may then be specified using *arrow notation*, as follows.

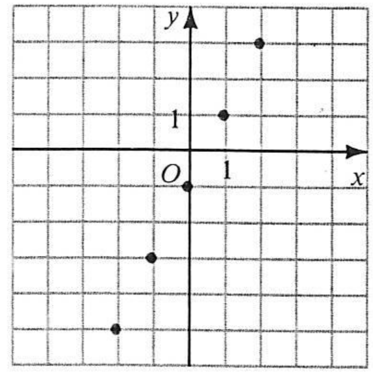
$$f\colon x \longrightarrow 2x + 6$$

This expression is read "the function f that pairs a number x with the number $2x + 6$."

EXAMPLE 2 Graph $f: x \longrightarrow 2x - 1$ if $D = \{-2, -1, 0, 1, 2\}$.

SOLUTION Replace x in $2x - 1$ with each member of D to find the corresponding value of y. Then graph the ordered pairs (x, y) on a coordinate plane.

x	$2x - 1 = y$	(x, y)
-2	$2(-2) - 1 = -5$	$(-2, -5)$
-1	$2(-1) - 1 = -3$	$(-1, -3)$
0	$2(0) - 1 = -1$	$(0, -1)$
1	$2(1) - 1 = 1$	$(1, 1)$
2	$2(2) - 1 = 3$	$(2, 3)$



Members of the range of a function are called the **values** of the function. Thus, the values of the function specified in Example 2 are -5, -3, -1, 1, and 3. The symbol $f(x)$, read "f of x," is used to denote the specific value of the function f that is paired with the number x. Therefore, for the function just specified:

$f(-2) = -5 \qquad f(-1) = -3 \qquad f(0) = -1 \qquad f(1) = 1 \qquad f(2) = 3$

Note that the symbol $f(x)$ does *not* denote the product of f and x.

Oral Exercises

State the domain and range of each relation. Then give a rule for the relation, letting x represent a member of the domain and y represent a member of the range.

1. $\{(2, -2), (1, -1), (1, 1), (2, 2)\}$
2. $\{(-5, 5), (-3, 3), (3, 3), (5, 5)\}$
3. $\{(-2, 4), (-1, 1), (1, 1), (2, 4)\}$
4. $\{(9, -3), (1, -1), (1, 1), (9, 3)\}$
5. $\{(0, 0), (1, 5), (2, 10), (3, 15)\}$
6. $\{(-6, -3), (-4, -2), (-2, -1), (0, 0)\}$
7. $\{(-1, 0), (0, 1), (1, 2), (2, 3)\}$
8. $\{(0, 2), (2, 4), (4, 6), (6, 8)\}$

Let $D = \{-2, -1, 0, 1, 2\}$. Determine the range R of the relation defined by each rule.

9. $y = x$
10. $y = -x$
11. $y = |x|$
12. $y = -|x|$
13. $y = x + 1$
14. $y = 5x$
15. $y = x^2$
16. $y = x^3$

Given the function $f: x \longrightarrow 2 - 3x$, find the following values of f.

17. $f(0)$
18. $f(1)$
19. $f(-1)$
20. $f(-2)$
21. $f(2)$
22. $f\left(\frac{1}{3}\right)$
23. $f\left(\frac{2}{3}\right)$
24. $f\left(-\frac{2}{3}\right)$

Written Exercises

Find the range of each function when $D = \{-2, -1, 0, 1, 2\}$.

A

1. $f: x \longrightarrow x + 3$
2. $g: x \longrightarrow x - 1$
3. $j: x \longrightarrow 2x$
4. $k: x \longrightarrow -\frac{1}{2}x$
5. $F: x \longrightarrow 3x - 1$
6. $G: x \longrightarrow 5 + 2x$
7. $h: x \longrightarrow |x|$
8. $H: x \longrightarrow -|x|$
9. $r: x \longrightarrow x^2$
10. $s: x \longrightarrow x^3$
11. $p: x \longrightarrow (x - 1)^2$
12. $q: x \longrightarrow x(x + 1)$

Graph each function.

13. $g: x \longrightarrow x;\ D = \{-3, -1, 1, 3\}$
14. $h: x \longrightarrow -x;\ D = \{-3, -1, 1, 3\}$
15. $J: x \longrightarrow 3x;\ D = \{-1, 0, 1, 2\}$
16. $K: x \longrightarrow \frac{x}{2};\ D = \{-4, 0, 2, 4\}$
17. $f: x \longrightarrow |x| - 1;\ D = \{-1, 0, 1, 2\}$
18. $F: x \longrightarrow |x| + 3;\ D = \{-2, -1, 0, 1\}$
19. $r: x \longrightarrow x^2 - 3;\ D = \{-3, -2, 0, 2\}$
20. $s: x \longrightarrow 5 - x^2;\ D = \{-2, 0, 2, 3\}$
21. $M: x \longrightarrow \frac{x - 1}{x + 1};\ D = \{-3, -2, 0, 1\}$
22. $N: x \longrightarrow \frac{3 - x}{1 - x};\ D = \{-1, 0, 2, 3\}$
23. $G: x \longrightarrow \frac{x^2 + 1}{2x + 1};\ D = \{-3, -1, 0, 2\}$
24. $H: x \longrightarrow \frac{1 - x^2}{3x + 1};\ D = \{-3, -1, 0, 1\}$

In Exercises 25–36, the function f is defined as given.
a. Find $f(0)$.
b. Find all values of x such that $f(x) = 0$.

B

25. $f: x \longrightarrow x + 5$
26. $f: x \longrightarrow x - 3$
27. $f: x \longrightarrow -5x$
28. $f: x \longrightarrow \frac{x}{7}$
29. $f: x \longrightarrow 2x + 6$
30. $f: x \longrightarrow 3 - 3x$
31. $f: x \longrightarrow |x| - 1$
32. $f: x \longrightarrow 2 - |x|$
33. $f: x \longrightarrow x^2 - 9$
34. $f: x \longrightarrow 1 - x^2$
35. $f: x \longrightarrow x^3 + x$
36. $f: x \longrightarrow x^2 + x$

In Exercises 37–44, let $f(x) = x^2$ and $g(x) = 3x - 1$. Find each of the following.

37. $f(2) + g(2)$
38. $f(2) \cdot g(2)$
39. $4f(3)$
40. $2g(5)$
41. $2f(3) + g(0)$
42. $f(2) - 2g(1)$
43. $\frac{5f(2)}{2g(1)}$
44. $\frac{f(3) + 1}{g(0)}$

In Exercises 45–52, let $f(x) = x^2$ and $g(x) = x - 1$. Find each of the following. (*Hint*: To find $g[f(x)]$, first find $f(x)$.)

C

45. $g[f(1)]$
46. $f[g(1)]$
47. $g[f(-1)]$
48. $f[g(-1)]$
49. $g[f(2)]$
50. $f[g(2)]$
51. $g[f(0)]$
52. $f[g(0)]$
53. If $f(x) = x - 2$ and $g[f(x)] = x$, what is $g(x)$?
54. If $g(x) = x - 2$ and $g[f(x)] = x$, what is $f(x)$?

The *greatest integer function* is denoted $f: x \longrightarrow [x]$, where the symbol $[x]$ is used to represent the greatest integer that is *less than or equal to* the real number x.

55. Simplify.
   a. $[3]$ b. $[3.5]$ c. $[-3]$ d. $[-3.5]$

56. Graph $f: x \longrightarrow [x]$ if $D = \{x: -4 \le x \le 4\}$.

Self-Test 1

VOCABULARY

axes (p. 215)
origin (p. 215)
x-axis (p. 216)
y-axis (p. 216)
coordinate axes (p. 216)
coordinate plane (p. 216)
quadrants (p. 216)
coordinates (p. 216)
x-coordinate (p. 216)
abscissa (p. 216)
y-coordinate (p. 216)
ordinate (p. 216)
ordered pair (p. 216)
graph of an ordered pair (p. 216)
plot a point (p. 216)
plane rectangular or Cartesian coordinate system (p. 217)
relation (p. 219)
domain (p. 219)
range (p. 219)
function (pp. 219, 221)
graph of a relation (p. 220)
value of a function (p. 225)

Plot the given points on a coordinate plane.

1. $A(2, -4)$ 2. $B(-3, -1)$ 3. $C(5, 0)$ *Obj. 1, p. 215*

State the domain and range of each relation. Then draw a mapping diagram that represents the relation.

4. $\{(0, 2), (-1, 3), (5, -2), (-3, 6)\}$ *Obj. 2, p. 215*
5. $\{(2, 1), (-1, 0), (0, 0), (5, 4)\}$

Tell whether or not each relation is a function.

6. $\{(-3, 3), (-1, 1), (0, 0), (1, 1), (3, 3)\}$ *Obj. 3, p. 215*
7. $\{(4, 2), (1, 1), (0, 0), (1, -1), (4, -2)\}$

Graph each of the following.

8. $\{(-3, 2), (0, 0), (4, -1), (-2, -3)\}$ *Obj. 4, p. 215*
9. $f: x \longrightarrow 3x - 1;\ D = \{-1, 0, 1, 2\}$
10. $g: x \longrightarrow |x| + 2;\ D = \{-4, -2, 0, 2, 4\}$

Check your answers with those at the back of the book.

Open Sentences in Two Variables

OBJECTIVES for Sections 5-4 through 5-6:
1. *To solve open sentences in two variables over given replacement sets of the variables.*
2. *To graph linear equations and inequalities in two variables on a coordinate plane.*

5–4 Solving Open Sentences in Two Variables

In Section 1-7 you learned that a solution of an open sentence in *one* variable is any value of the variable for which the open sentence is a true statement. For example, the equation $7x - 5 = 9$ has just one solution, namely 2.

A **solution of an open sentence in two variables,** x and y, is any ordered pair of numbers, (x, y), that *together* make the sentence a true statement. For example, consider the open sentence

$$2x + 3y = 11.$$

The ordered pair (4, 1) is a solution of this open sentence because

$$2(4) + 3(1) = 11,$$

but (5, 1) is *not* a solution because

$$2(5) + 3(1) \neq 11.$$

The set of *all* solutions of an open sentence in two variables is called the **solution set** of the open sentence. Any member of the solution set is said to **satisfy** the open sentence. You **solve** an open sentence in two variables when you determine its solution set.

EXAMPLE 1 Solve $3x + 2y = 14$ if the replacement set for x and y is the set of whole numbers.

SOLUTION 1. Solve the given equation for y.

$$3x + 2y = 14$$
$$2y = 14 - 3x$$
$$y = 7 - \tfrac{3}{2}x$$

2. Starting with 0, replace x in the expression $7 - \frac{3}{2}x$ with consecutive whole numbers in order to find the corresponding values of y. A table may be helpful.

x	$7 - \frac{3}{2}x$	y
0	$7 - \frac{3}{2}(0)$	7
1	$7 - \frac{3}{2}(1)$	$\frac{11}{2}$
2	$7 - \frac{3}{2}(2)$	4
3	$7 - \frac{3}{2}(3)$	$\frac{5}{2}$
4	$7 - \frac{3}{2}(4)$	1
5	$7 - \frac{3}{2}(5)$	$-\frac{1}{2}$
6	$7 - \frac{3}{2}(6)$	-2

3. If the value of y found in Step 2 is a whole number, the ordered pair (x, y) satisfies the open sentence.

As the table on the previous page indicates, values of x greater than 4 yield negative values of y, and so x cannot be greater than 4.

$\therefore$ the solution set is $\{(0, 7), (2, 4), (4, 1)\}$.

Notice that the solution set of the equation in Example 1 is a function with domain $\{0, 2, 4\}$ and range $\{7, 4, 1\}$. You can specify this function as

$$f: x \longrightarrow 7 - \tfrac{3}{2}x, \qquad x \in \{0, 2, 4\}.$$

The **graph of an open sentence in two variables** is the graph of its solution set.

EXAMPLE 2 Solve $y + 2x < 1$ if $x \in \{-1, 0, 1\}$ and $y \in \{-2, -1, 0, 1, 2\}$. Graph the solution set.

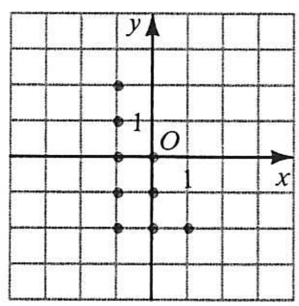
SOLUTION Solve the given inequality for y:

$$y + 2x < 1$$
$$y < 1 - 2x$$

x	$1 - 2x$	$y < 1 - 2x$	y
-1	$1 - 2(-1)$	$y < 3$	$-2, -1, 0, 1, 2$
0	$1 - 2(0)$	$y < 1$	$-2, -1, 0$
1	$1 - 2(1)$	$y < -1$	-2

$\therefore$ the solution set is $\{(-1, -2), (-1, -1), (-1, 0), (-1, 1), (-1, 2), (0, -2), (0, -1), (0, 0), (1, -2)\}$.

The graph is shown below.



Although the solution set of the inequality in Example 2 is *not* a function, it is a relation. In general, the solution set of *any* open sentence in two variables is a relation whose domain is the set of first coordinates and whose range is the set of second coordinates of the ordered pairs that satisfy the sentence.

Oral Exercises

Tell whether or not the given ordered pair is a solution of the open sentence.

1. $x - y = -4; (1, -5)$
2. $x + 5y = -2; (3, -1)$
3. $y + 2x = 2; (4, -6)$
4. $2y - 3x = 1; (-2, 2)$
5. $x < 3y - 4; (2, 2)$
6. $y \geq -4x + 1; (0, 2)$
7. $x - y^2 = 12; (4, 4)$
8. $x^2 + xy = 6; (-6, 5)$

Solve each open sentence for *y*.

9. $y + 2x = 3$
10. $3x - y = 7$
11. $2y = 4x - 6$
12. $12 = 4x + 3y$
13. $x + y \leq 3$
14. $2x - y > 7$
15. $8x - 10 > 2y$
16. $9x - 3y \leq 3$

Solve each equation if the replacement set for *x* and *y* is the set of whole numbers.

17. $x + y = 3$
18. $2x + y = 9$
19. $2xy = 16$
20. $x^2 + y^2 = 1$

Written Exercises

Complete each ordered pair to form a solution of the given equation.

A

1. $y = -x$; $(5, \underline{?})$, $(0, \underline{?})$, $(-3, \underline{?})$
2. $y = 4x$; $(4, \underline{?})$, $(0, \underline{?})$, $(-3, \underline{?})$
3. $y = x - 5$; $(9, \underline{?})$, $(5, \underline{?})$, $(-7, \underline{?})$
4. $y = 2x + 3$; $(6, \underline{?})$, $(0, \underline{?})$, $(-4, \underline{?})$
5. $y = 2x - 1$; $(\underline{?}, 1)$, $(\underline{?}, -1)$, $(\underline{?}, -7)$
6. $y = 5 + 3x$; $(\underline{?}, 8)$, $(\underline{?}, 5)$, $(\underline{?}, -7)$
7. $4x + y = 9$; $(2, \underline{?})$, $(0, \underline{?})$, $(-5, \underline{?})$
8. $x - 2y = 10$; $(\underline{?}, 10)$, $(\underline{?}, -1)$, $(\underline{?}, -10)$
9. $2x + 3y = 11$; $(\underline{?}, 3)$, $(\underline{?}, 0)$, $(\underline{?}, -5)$
10. $5x - 2y = 7$; $(3, \underline{?})$, $(0, \underline{?})$, $(-1, \underline{?})$
11. $x^2 - y = 5$; $(3, \underline{?})$, $(2, \underline{?})$, $(-2, \underline{?})$
12. $2x + y^2 = 18$; $(\underline{?}, 4)$, $(\underline{?}, 0)$, $(\underline{?}, -6)$

Solve each equation if $x \in \{-2, -1, 0, 1, 2\}$ and $y \in$ {the integers}. Graph the solution set.

13. $y = x$
14. $y = -x$
15. $y = 3x + 1$
16. $y = 6 - 2x$
17. $y + 2x = 5$
18. $3x - y = 2$
19. $6x - 2y = 4$
20. $9x + 3y = 9$
21. $3x - 2y = 1$
22. $2y - 5x = 3$
23. $6x = 1 - 2y$
24. $4x = 2y - 3$

Solve each inequality if $x \in \{-1, 0, 1\}$ and $y \in \{-2, -1, 0, 1, 2\}$. Graph the solution set.

B 25. $y \geq x$ 26. $y > -x$ 27. $y > x - 1$ 28. $y \leq 2x$

29. $x + y \leq 0$ 30. $y - x > 0$ 31. $y + 1 \leq 2x$ 32. $2x > y - 1$

33. $2x + y \geq 2y$ 34. $3y < y - 2x$ 35. $x - y \leq x + y$ 36. $x + y < y - x$

Solve each open sentence using the given replacement sets for the variables.

37. $x + y \leq 4;$ $x \in \{2, 3, 4\},$ $y \in \{\text{the positive integers}\}$

38. $x - y > 2;$ $x \in \{3, 4, 5\},$ $y \in \{\text{the positive integers}\}$

39. $y > 2x;$ $x \in \{-2, -1, 0\},$ $y \in \{\text{the negative integers}\}$

40. $y \leq -\frac{x}{2};$ $x \in \{-4, -2, 0\},$ $y \in \{\text{the whole numbers}\}$

41. $x + 2y \leq 5;$ $x \in \{3, 4, 5\},$ $y \in \{\text{the whole numbers}\}$

42. $3x + y > 0;$ $x \in \{0, 1, 2\},$ $y \in \{\text{the negative integers}\}$

C 43. $|x| + y = 5;$ $x, y \in \{-3, -2, -1, 0, 1, 2, 3\}$

44. $|x| + |y| = 6;$ $x, y \in \{-4, -2, -1, 0, 1, 2, 4\}$

45. $|x| - |y| = 2;$ $x, y \in \{-5, -3, -1, 0, 1, 3, 5\}$

46. $|x - y| = 3;$ $x, y \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

47. $x^2 - y = 1;$ $x, y \in \{-2, -1, 0, 1, 2\}$

48. $2x + y^2 = 4;$ $x, y \in \{-4, -2, -1, 0, 1, 2, 4\}$

Computer Exercises

For students with computer experience

Write a program that will allow you to input values for a, b, c, x, and y and will determine whether the ordered pair (x, y) is a solution of the equation $ax + by = c$. RUN the program to determine whether $(2, -3)$ is a solution of each of the following.

1. $5x + 2y = 4$ 2. $2x - 4y = 10$ 3. $3x - y = 6$ 4. $x + 4y = -10$

5. $-4x = 3y + 1$ 6. $x = 2$ 7. $y = -3$ 8. $3x = -2y$

9. Modify the program that you wrote for Exercises 1–8 so that, if (x, y) is *not* a solution of $ax + by = c$, the program will determine whether it is a solution of $ax + by < c$ or of $ax + by > c$.

Write a program that will solve the equation $2x + 3y = 24$ when you input two integral intervals as replacement sets for x and y, respectively. RUN the program for each of the following replacement sets.

10. $x, y \in \{1, 2, 3, \ldots, 12\}$ 11. $x, y \in \{-12, -11, -10, \ldots, 12\}$

12. $x \in \{-12, -11, -10, \ldots, 0\}$; $y \in \{0, 1, 2, \ldots, 12\}$

5–5 The Graph of a Linear Equation in Two Variables

Each of the following three figures shows a graph associated with the equation

$$x + y = 2.$$

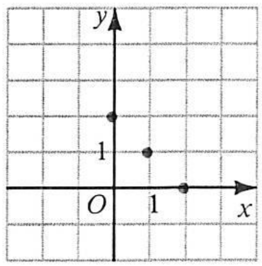


Figure 8

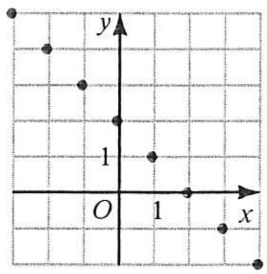


Figure 9

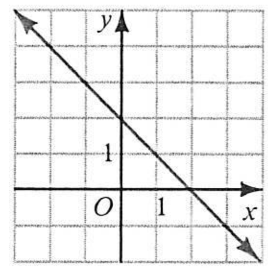


Figure 10

In Figure 8, the replacement set of the variables x and y is the set of whole numbers, W, and so the graph consists of just three points: (0, 2), (1, 1), and (2, 0). In Figure 9 you see only a partial graph; the replacement set of each variable is the set of integers, J, and the resulting graph is an infinite set of isolated points. In Figure 10, the replacement set of each variable is the set of real numbers, $\mathcal{R}$, and the graph is an infinite set of points that together form a line.

In fact, if the replacement set of each of the variables x and y is $\mathcal{R}$, each solution of the equation $x + y = 2$ gives the coordinates of a point on the line shown in Figure 10. Similarly, the coordinates of each point on this line satisfy the equation. Thus, the line shown in Figure 10 is the set of *all those points* and *only those points* whose coordinates satisfy the equation $x + y = 2$. This line is called the **graph of the equation** on the coordinate plane, and the equation is called an **equation of the line.**

In general, the graph of any equation of the form

$$ax + by = c,$$

where a, b, and c are real numbers and a and b are not both zero, is a line. The number a is called the *coefficient of* x, and b is called the *coefficient of* y. Any equation of this form is called a **linear equation in two variables,** x and y. Thus,

$$6x - 5y = 7 \quad \text{and} \quad 2y = 15$$

are linear equations in two variables, but

$$x^2 + y = 5, \quad xy = 6, \quad \text{and} \quad \frac{1}{x} + y = 3$$

are not.

The following facts about linear equations are taken for granted, provided that the variables represent real numbers.

1. The graph of each linear equation in two variables is a line on a coordinate plane.
2. Each line on a coordinate plane is the graph of a linear equation in two variables.

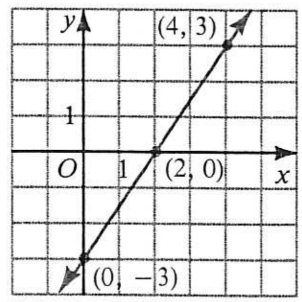
Throughout the rest of this book, the replacement set of each variable in a linear equation is $\Re$ unless otherwise specified.

Since two points determine a line, you need to find only two solutions of a linear equation in order to graph it. However, it is a good practice to find at least a third solution as a check on your work. Often the most convenient solutions to find are those where the line intersects the y-axis ($x = 0$) and where the line intersects the x-axis ($y = 0$).

EXAMPLE 1 Graph $3x - 2y = 6$ on a coordinate plane.

SOLUTION

Let $x = 0$.	Let $y = 0$.
$3(0) - 2y = 6$	$3x - 2(0) = 6$
$-2y = 6$	$3x = 6$
$y = -3$	$x = 2$
Solution: $(0, -3)$	*Solution:* $(2, 0)$



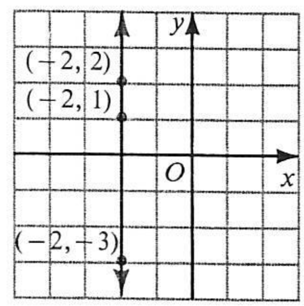
Any third solution, such as (4, 3), can be used as a check. The graph is shown at the right.

EXAMPLE 2 Graph each equation on a coordinate plane.

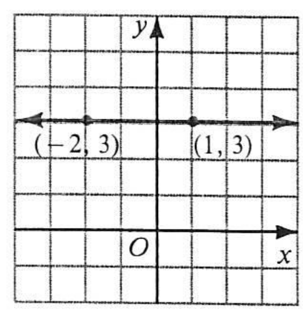
a. $x = -2$ **b.** $y = 3$

SOLUTION

a. The equation places no restriction on y, and so all points with abscissa -2 are graphs of solutions. Therefore, the graph of $x = -2$ is a vertical line two units to the left of the y-axis, as shown at the right.



b. The equation places no restriction on x, and so all points with ordinate 3 are graphs of solutions. Therefore, the graph of $y = 3$ is a horizontal line three units above the x-axis, as shown at the right.



Note that graphing on a number line and on a coordinate plane are different. For example, Figure 11 shows $x = 3$ considered as an equation in one variable. Its solution set is graphed as a single point on a number line. Figure 12 shows $x = 3$ considered as an equation in two variables, $x + 0y = 3$. Its solution set consists of all ordered pairs $(3, y)$ where the replacement set for y is $\mathcal{R}$, and its graph is a line on a coordinate plane.

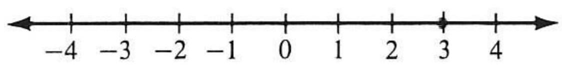


Figure 11

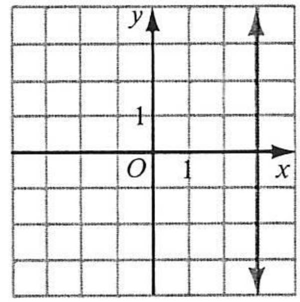


Figure 12

A function whose ordered pairs satisfy a linear equation is called a **linear function.** For example,

$$f\colon x \longrightarrow 2 - x, \quad x \in \mathcal{R}$$

is a linear function. Its graph is the line that is graphed on the coordinate plane in Figure 10 (page 232).

Oral Exercises

Tell whether or not each equation is a linear equation.

1. $x + y = 6$
2. $xy = 6$
3. $y = x$
4. $y = |x|$
5. $y = 2x$
6. $y = x^2$
7. $y = -2$
8. $x = 5$
9. $\frac{y}{3} = x$
10. $y = \frac{x}{3}$
11. $\frac{1}{3}x + \frac{1}{2}y = 7$
12. $\frac{3}{x} + \frac{2}{y} = 7$

Describe the graph of each equation.

13. a. $x = 1$ b. $x = -2$ c. $x = 0$
14. a. $y = 5$ b. $y = -4$ c. $y = 0$

Give the coordinates of the points where the graph of each equation intersects the x-axis and the y-axis.

15. $x + 3y = 6$
16. $2x - y = 4$
17. $2x + 5y = 10$
18. $2x - 3y = 12$
19. $2x = 8$
20. $y - 3 = 7$

State the relationship between the ordinate and the abscissa of each point on the graph of the given equation.

EXAMPLE $y = 3x + 1$

SOLUTION The ordinate is one more than three times the abscissa.

21. $y = 4x$ **22.** $y = x - 7$ **23.** $y = 5x - 2$

24. $y = \frac{1}{2}x + 1$ **25.** $y = |x| + 2$ **26.** $y = x^2 - 1$

Written Exercises

Transform each equation into an equivalent equation of the form $ax + by = c$, where a, b, and c are integers.

A **1.** $3x = 4y - 5$ **2.** $y = 2x + 7$ **3.** $5x - 8 = 3y$

4. $4y + 9 = 5x$ **5.** $\frac{1}{2}x + y = 3$ **6.** $2x + \frac{y}{3} = 5$

Graph each equation on a coordinate plane.

7. $x = -4$ **8.** $x = 3$ **9.** $y = 5$ **10.** $y = -1$

11. $x + y = 5$ **12.** $x - y = 2$ **13.** $x + 2y = -4$ **14.** $3x + y = 6$

15. $2x + 3y = 12$ **16.** $2x - 5y = 10$ **17.** $\frac{x}{3} + y = 2$ **18.** $x + \frac{y}{2} = -3$

19. $5y - 3x = 15$ **20.** $7y - 2x = 14$ **21.** $2y = 6 - 3x$ **22.** $4x = 20 - 5y$

Graph each function over $\mathcal{R}$.

B **23.** $f: x \longrightarrow x - 3$ **24.** $g: x \longrightarrow 3x + 1$ **25.** $j: x \longrightarrow \frac{x}{2} + 1$

26. $k: x \longrightarrow \frac{x + 1}{3}$ **27.** $R: x \longrightarrow \frac{1}{2}(x - 3)$ **28.** $S: x \longrightarrow -3(2 - x)$

In each of Exercises 29–32, graph the given equations on the same coordinate plane. Label each graph.

29. a. $y = x$ **b.** $y = x + 1$ **c.** $y = x + 2$

30. a. $y = x$ **b.** $y = x - 2$ **c.** $y = x - 3$

31. a. $y = x$ **b.** $y = 2x$ **c.** $y = 3x$

32. a. $y = x$ **b.** $y = \frac{1}{2}x$ **c.** $y = \frac{1}{3}x$

33. For any real number k, describe how the graph of $y = x + k$ is related to the graph of $y = x$.

34. For any real number k, describe how the graph of $y = kx$ is related to the graph of $y = x$.

In each of Exercises 35–38, graph the given equations on the same coordinate plane. Find the coordinates of the point at which the two graphs intersect, then determine if this ordered pair satisfies *both* equations.

C 35. $y = x + 1;\ y = -x + 1$

36. $y = x - 2;\ y = \frac{1}{3}x$

37. $2y - x = 4;\ 2y + 3x = 12$

38. $4y + 5x = 8;\ 4y + x = -8$

Graph each equation on a coordinate plane. Then tell whether or not the equation is a linear equation.

39. $y = |x|$

40. $y = -|x|$

41. $y = |x| + 2$

42. $y = |x| - 2$

43. $y = |x + 1|$

44. $y = |x - 1|$

45. $y = -|x| + 3$

46. $y = -|x + 3|$

47. For any real number k, describe how the graph of $y = |x| + k$ is related to the graph of $y = |x|$.

48. For any real number k, describe how the graph of $y = |x + k|$ is related to the graph of $y = |x|$.

5–6 The Graph of a Linear Inequality in Two Variables

The graph of the linear equation

$$y = x + 1$$

separates a coordinate plane into two regions that are called **open half-planes.** One of these regions is *above* the line, as shown by the colored shading in Figure 13, and the other region is *below* the line, as shown by the gray shading. The line itself is called the **boundary** of each half-plane.

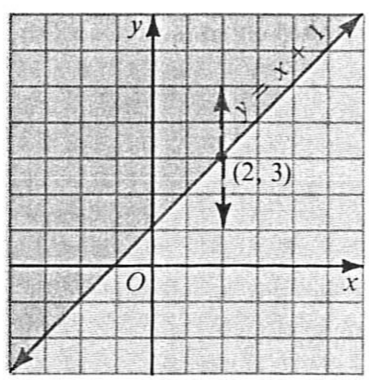


Figure 13

If you start at any point of this line, say (2, 3), and move vertically upward, the y-coordinates of the points on the plane increase. Thus the open half-plane above the line is the graph of

$$y > x + 1,$$

as indicated in Figure 13. If you move vertically *downward* from (2, 3), the y-coordinates of the points *decrease*, and so the open half-plane below the line is the graph of

$$y < x + 1.$$

Figure 14 shows the graph of four inequalities.

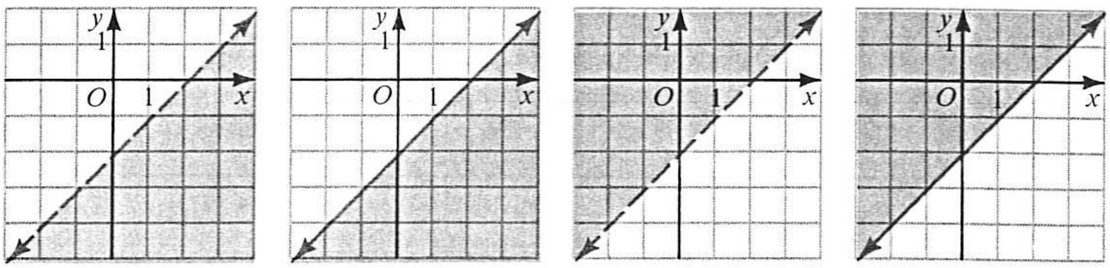


Figure 14

Each of these graphs has as its boundary the line with the equation

$$y = x - 2.$$

Notice that the boundary is drawn as a *dashed* line in parts (a) and (c) to show that the boundary is not part of the graph. That is, the dashed line indicates that the graph is an *open half-plane*. In parts (b) and (d), however, the boundary must be included as part of the graph, and so it is drawn as a *solid* line. In such cases, the graph is the *union* of the open half-plane and its boundary and is called a **closed half-plane.**

The equation of the boundary of the graph of an inequality is called the **associated equation** of the inequality. Thus, $y = x - 2$ is the associated equation of each inequality graphed in Figure 14. An inequality whose associated equation is a linear equation in two variables is called a **linear inequality in two variables.**

In general, any linear equation in two variables,

$$ax + by = c,$$

is the associated equation of four linear inequalities in two variables:

$$ax + by < c \qquad ax + by > c$$
$$ax + by \le c \qquad ax + by \ge c$$

On a coordinate plane, the *graph of a linear inequality in two variables* is either an open or closed half-plane.

EXAMPLE 1 Graph $3x - y > 1$ on a coordinate plane.

SOLUTION

1. Solve the given inequality for y.

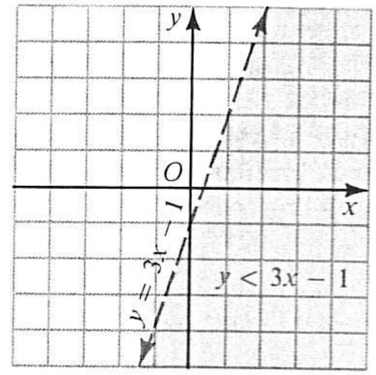
$$3x - y > 1$$
$$-y > -3x + 1$$
$$y < 3x - 1$$

2. Graph the associated equation

$$y = 3x - 1$$

and show it as a *dashed* line.

3. Shade the open half-plane *below* the line.



Check: Select any point in the shaded region and determine whether its coordinates satisfy the original inequality.

$$(2, 2): \quad 3x - y > 1$$
$$3(2) - 2 \stackrel{?}{>} 1$$
$$4 > 1 \quad \checkmark$$

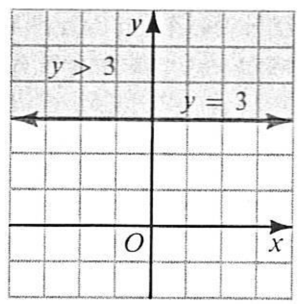
Thus, (2, 2) is in the solution set, and the correct region has been shaded.

EXAMPLE 2 Graph each inequality on a coordinate plane.

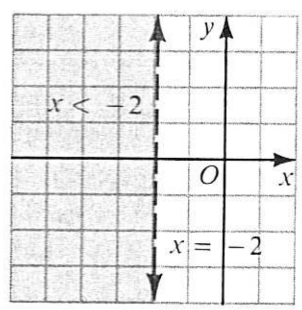
a. $y \geq 3$ **b.** $x < -2$

SOLUTION

a. Graph the associated equation $y = 3$ as a solid line. Then shade the region *above* the line to graph all the points with ordinate *greater than* 3.



b. Graph the associated equation $x = -2$ as a dashed line. Then shade the region to the *left* of the line to graph all the points with abscissa *less than* -2.



Oral Exercises

Tell whether the point (−1, 0) lies above, on, or below the graph of the given equation on a coordinate plane.

1. $y = x + 1$
2. $y = x - 3$
3. $y = x + 5$
4. $y = -2x$
5. $y = 3x + 1$
6. $y = 2 + 2x$

Tell which of the points with given coordinates belong to the graph of the inequality.

7. $(1, -2), (-1, 3); \quad x + y < 0$
8. $(-2, -3), (-3, 5); \quad x - y \geq 0$
9. $(1, 2), (3, -1); \quad 3x + y \leq 5$
10. $(5, 4), (3, 1); \quad x - 2y > 0$

Solve each inequality for *y*.

11. $x + y > 7$
12. $y + x \leq 0$
13. $3x + y < 2$
14. $-3x + y \geq 4$
15. $x - y > 2$
16. $2x - y \leq 0$
17. $4x + 2y > 6$
18. $12x + 4y < 4$
19. $3y < x$
20. $-4y \geq x$
21. $x + 3y < 6$
22. $x - 2y \geq 3$

Written Exercises

Graph each inequality on a coordinate plane.

A

1. $y < 3$
2. $y \geq -3$
3. $x \leq -1$
4. $x > 4$
5. $x > 0$
6. $y \leq 0$
7. $y > x$
8. $y \leq x$
9. $y > -x$
10. $y < -2x$
11. $y \leq 2x - 1$
12. $y > 2 - 3x$
13. $x + y \geq 4$
14. $x + y < 5$
15. $x - y < 3$
16. $x - y \geq 1$
17. $3x + y \leq 2$
18. $4x - y > 1$

B

19. $4x + 2y > 4$
20. $6x + 2y < 8$
21. $3x - 2y \leq 6$
22. $2x - 3y > 9$
23. $2x - 5y > 5$
24. $3x + 4y \leq 2$

In each of Exercises 25–30, graph the given inequalities on the same coordinate plane.

25. $y > -1; \; y \leq 2$
26. $y > 1; \; y < x$
27. $y > -3; \; x < 1$
28. $y \leq 5; \; x \geq -2$
29. $x + y > -1; \; 2x + y > 3$
30. $4x + y < -1; \; 2x - y < 5$

Graph each inequality on a coordinate plane.

C

31. $|x| > 2$
32. $|x| < 5$
33. $|y| > 3$
34. $|y| < 1$
35. $y > |x|$
36. $y < -|x|$

Self-Test 2

VOCABULARY

solution of an open sentence in two variables (p. 228)
graph of an open sentence in two variables (p. 229)
graph of an equation (p. 232)
equation of a line (p. 232)
linear equation in two variables (p. 232)
linear function (p. 234)
open half-plane (p. 236)
boundary of a half-plane (p. 236)
closed half-plane (p. 237)
associated equation (p. 237)
linear inequality in two variables (p. 237)

Let $x \in \{-2, 0, 2\}$ and $y \in \{-4, -2, 4\}$. Solve each open sentence.

1. $y = -x$
2. $3x + 2y = -2$
3. $y < x + 2$

Obj. 1, p. 228

Graph each open sentence on a coordinate plane.

4. $y = x$
5. $2x - 5y = 10$
6. $x = -3$
7. $y \geq -2$
8. $x + y < 6$
9. $2x - y > 1$

Obj. 2, p. 228

Check your answers with those at the back of the book.

Ada Lovelace
1815–1852

Ada Lovelace was born Augusta Ada Byron, daughter of Lord Byron, the famous English poet, and Annabella Milbanke. Encouraged by her mother and by friends, she studied mathematics enthusiastically. In 1835 she married William, Lord King, who soon became the Earl of Lovelace.

During the 1830's, the English mathematician Charles Babbage was designing his "Analytical Engine," a machine that could make calculations, store data, and print out results. Lady Lovelace first wrote to him in her search for a mathematics tutor, but she soon became involved in the project of translating and annotating a paper describing his Analytical Engine. Her detailed notes describing the process of communicating with the machine are considered to be the first description of computer programming.

In 1979 a newly developed programming language was named *Ada* in recognition of Lady Lovelace's contribution to computer science.

READING ALGEBRA Independent Study

When you begin a new chapter or section of your textbook, you will want to have some goals in mind. The title of each main chapter division and the objectives that follow it will tell you what you will be reading about. Do not skip over them. Like the headlines in a newspaper story, they will highlight what is important.

After you know what you will be reading about, read through each section at a moderate speed. Look at each diagram and read through the examples and their solutions. If you encounter an unfamiliar word, look it up in the Glossary at the back of the book or in a dictionary. When you are finished, try to say aloud in your own words what you have read.

Usually you will need to go back and read a section a second time. When you do this, have a pencil, your notebook, and a piece of scrap paper at hand. Read slowly and carefully this time, making sure that you understand each word. Pay particular attention to mathematical symbols, translating each one into words. For example, on page 69, the mathematical sentence "$a + (-a) = 0$" translates into "a plus the opposite of a equals zero."

When you come to an example in the text, cover the solution with your scrap paper and try to work it out yourself. Then compare your solution with the one given in the book. If you did not work the example correctly, copy the given solution, one step at a time, understanding each step before going on to the next. Then cover the solution and try again to do the example by yourself.

Exercises

Name the objective for the given section.

1. Section 5–4
2. Section 5–5
3. Section 5–6

Tell which section you would review if you had trouble with the following exercises in Self-Test 2 on page 240.

4. Exercise 3
5. Exercise 5
6. Exercise 9

Translate into words.

7. $-(a + b) = -a + (-b)$
8. $f: x \longrightarrow 7 - \frac{3}{2}x,\ x \in \{0, 2, 4\}$

9. Write out a solution to the following example.

EXAMPLE Given $f: x \longrightarrow 5 - |x|$, find all values of x such that $f(x) = 0$.

Lines on a Coordinate Plane

OBJECTIVES for Sections 5-7 through 5-9:

1. *To find the slope of a line on a coordinate plane.*
2. *To use the slope-intercept form of a linear equation.*
3. *To determine an equation of a line given its slope and y-intercept.*
4. *To determine an equation of a line given the slope of the line and the coordinates of one point through which the line passes, or given the coordinates of two points on the line.*

5–7 The Slope of a Line

There are many everyday terms that have a special meaning in mathematics. Sometimes the everyday use of such terms helps in understanding their use in mathematics. As an example, consider the word "slope."

To describe the steepness, or slope, of a hill, you may estimate the amount of vertical *rise* of the hill that corresponds to a certain amount of horizontal *run*, then calculate the ratio of the rise to the run. For example, Figure 15 represents a hill that rises 10 m for every 50 m of horizontal run. Its slope is the ratio

$$\frac{\text{rise}}{\text{run}} = \frac{10}{50} = 0.2.$$

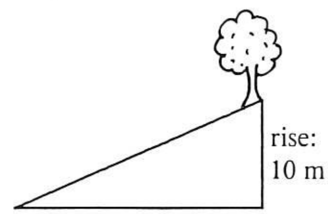
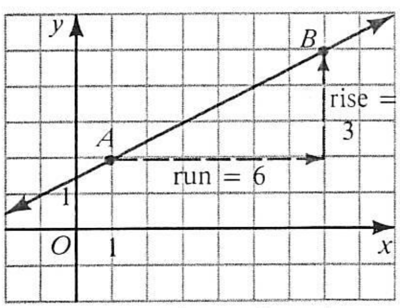


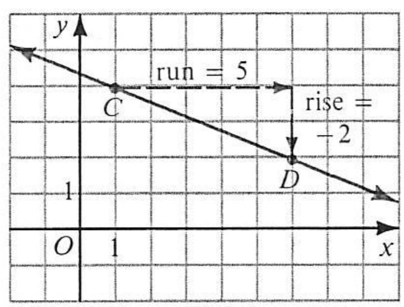
Figure 15

Similarly, to describe the steepness, or slope, of a line on a coordinate plane, you may choose any two points on the line, compute the units in the rise and the run from one point to the other, and calculate the ratio of the rise to the run. For example, consider the lines that are graphed in Figures 16 and 17.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3}{6} = \frac{1}{2}$$

Figure 16



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-2}{5} = -\frac{2}{5}$$

Figure 17

In Figure 16, the line is shown passing through the points $A(1, 2)$ and $B(7, 5)$. Notice that the rise, or *vertical change*, in moving from A to B is equal to the difference between the ordinates of these points: $5 - 2 = 3$.

The run, or *horizontal change*, in moving from A to B is equal to the difference between the abscissas: $7 - 1 = 6$. Similarly, in Figure 17, the rise in moving from $C(1, 4)$ to $D(6, 2)$ is equal to the difference between the ordinates, $2 - 4 = -2$, and the run is equal to the difference between the abscissas, $6 - 1 = 5$. Thus, the slope of a line on a coordinate plane may be defined as follows.

$$\textbf{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{difference between ordinates}}{\text{difference between abscissas}}$$

More formally, if (x_1, y_1), read "x sub one, y sub one," and (x_2, y_2), read "x sub two, y sub two," are any two different points on a line,

$$\textbf{slope} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2).$$

Notice that the differences between the ordinates and the abscissas must be taken *in the same order*.

EXAMPLE 1 Find the slope of the line that passes through $(-5, 1)$ and $(7, -3)$.

SOLUTION Let $x_1 = -5$, $y_1 = 1$, $x_2 = 7$, and $y_2 = -3$.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 1}{7 - (-5)} = \frac{-4}{12} = -\frac{1}{3}$$

$\therefore$ the slope of the line is $-\frac{1}{3}$.

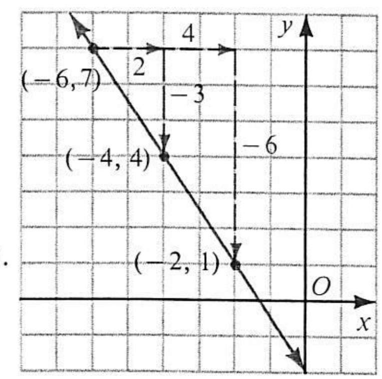
EXAMPLE 2 Graph the line that passes through the point $(-6, 7)$ and has slope $-\frac{3}{2}$.

SOLUTION

1. Plot $(-6, 7)$ on a coordinate plane.
2. Since $-\frac{3}{2} = \frac{-3}{2}$, count 3 units *down* and 2 units to the *right* to obtain a second point on the line.

$$(-6 + 2,\quad 7 - 3) = (-4, 4)$$

3. Draw a line through $(-6, 7)$ and $(-4, 4)$.



Check: Since $-\frac{3}{2} = -\frac{6}{4}$ a third point on the line should be

$$(-6 + 4,\quad 7 - 6) = (-2, 1).$$

The line is graphed on the coordinate plane above.

Notice in Example 2 that $-\frac{3}{2} = \frac{3}{-2}$ is also true. Thus, in Step 2 you may obtain a second point on the line by counting 3 units *up* and 2 units to the *left* from the point $(-6, 7)$.

A basic property of a line is that its slope is constant.

EXAMPLE 3 Determine whether or not the points $(0, -2)$, $(2, 1)$, $(4, 4)$, and $(8, 10)$ lie on the same line.

SOLUTION Arrange the ordered pairs in a table in increasing order of abscissas. Then compute the differences between the abscissas and ordinates from point to point.

Differences between abscissas: 2 2 4

x	0	2	4	8
y	−2	1	4	10

Differences between ordinates: 3 3 6

The ratio $\dfrac{\text{difference between ordinates}}{\text{difference between abscissas}}$ is constant: $\dfrac{3}{2} = \dfrac{3}{2} = \dfrac{6}{4}$

$\therefore$ the points lie on the same line.

Points that lie on the same line are called **collinear points**.

When a line rises from left to right on a coordinate plane, as in Figure 16 (page 242), the slope of the line is *positive*. When a line "falls" from left to right, as in Figure 17, its slope is *negative*.

Now consider the following figures.

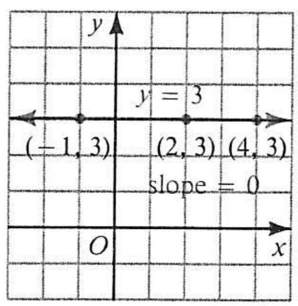


Figure 18

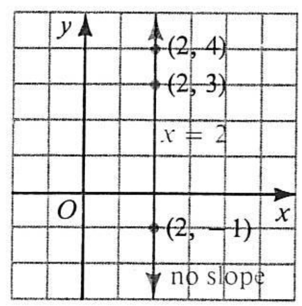


Figure 19

What is the slope of the horizontal line $y = 3$ that is shown in Figure 18? Because the ordinate of every point on this line is 3, the *difference* between the ordinates of any two points will always be 0. Thus, the numerator of the ratio

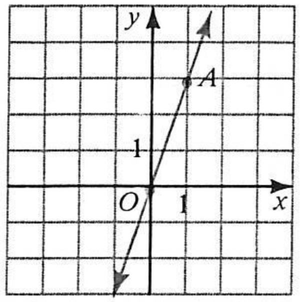
$$\frac{y_2 - y_1}{x_2 - x_1}$$

will always be 0, and so the slope of the line is 0. In fact, *the slope of any horizontal line is 0.*

On the other hand, every point on the vertical line $x = 2$, shown in Figure 19, has the same abscissa. Thus, the *denominator* of the above ratio will always be 0. Since division by 0 has no meaning, the line has no slope. In fact, *vertical lines have no slope.*

Oral Exercises

Determine the slope of each line.

1.


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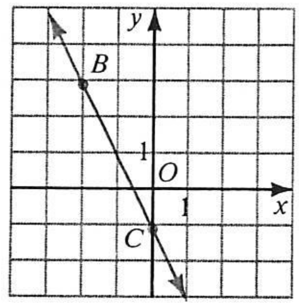

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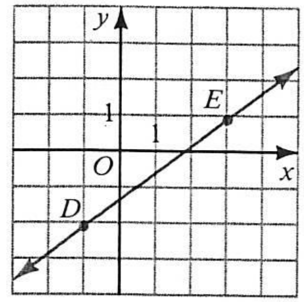

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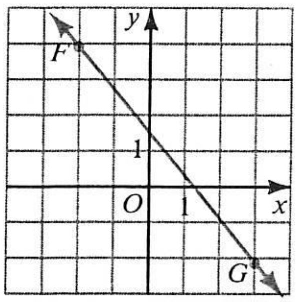
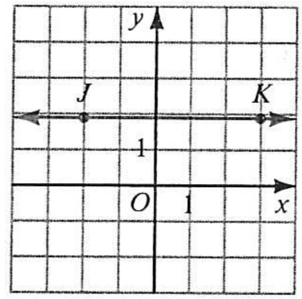
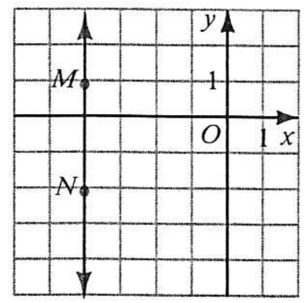


5.


6.


Tell whether or not the points that have the coordinates listed in each table are collinear. If the points are collinear, determine the slope of the line that passes through them.

7.

x	0	1	2	3
y	5	7	9	11

8.

x	2	3	4	5
y	9	6	3	0

9.

x	−1	0	1	2
y	4	4	4	4

10.

x	−2	0	2	4
y	−2	1	4	10

11.

x	0	2	6	8
y	11	6	−4	−9

12.

x	1	3	7	11
y	−5	−1	7	15

13.

x	−4	−1	5	14
y	−1	−4	−10	−19

14.

x	−1	−3	−11	−17
y	−10	−10	−10	−10

Written Exercises

Determine the slope of the line that passes through the given points.

A

1. $(2, 3), (1, 1)$
2. $(3, 5), (4, 2)$
3. $(5, -4), (-3, -4)$
4. $(-2, 1), (-2, 5)$
5. $(2, -5), (4, -7)$
6. $(-3, 2), (0, 8)$
7. $(4, -6), (-1, 4)$
8. $(3, 2), (-2, -1)$
9. $(-1, -7), (1, 0)$
10. $(8, 1), (-1, 1)$
11. $(-7, -2), (-7, -3)$
12. $(-4, -1), (0, 2)$

Graph the line that passes through the given point and has the given slope.

13. $(1, 4)$; slope $= 2$
14. $(-2, 1)$; slope $= -3$
15. $(-3, -2)$; slope $= -\frac{1}{2}$
16. $(4, -2)$; slope $= \frac{2}{3}$
17. $(2, -5)$; slope $= 0$
18. $(1, 3)$; no slope

Determine whether or not the points with the given coordinates are collinear. If the points are collinear, determine the slope f the line that passes through them.

B

19. $(0, 3), (1, 5), (2, 7), (3, 9)$
20. $(4, 5), (5, 2), (6, -1), (7, -4)$
21. $(-4, 0), (-1, -1), (5, -3), (8, -4)$
22. $(1, -5), (3, -2), (7, 1), (9, 4)$
23. $(-6, -1), (-4, -1), (-3, -1), (-1, -1)$
24. $(-3, 5), (-3, 3), (-3, 1), (-3, -1)$

Determine the value of t so that the slope of the line through each pair of points has the given value.

25. $(6, 3), (-4, t)$; slope $= 2$
26. $(t, 5), (5, 1)$; slope $= -2$
27. $(5, 2t), (1, 3t)$; slope $= -\frac{1}{2}$
28. $(-4, 5t), (5, 2t)$; slope $= \frac{1}{3}$
29. $(7, -4t), (-8, 6t)$; slope $= -\frac{1}{3}$
30. $(3t, 5), (-9t, -5)$; slope $= \frac{5}{4}$
31. The vertices of a triangle are $A(-3, 2)$, $B(-1, -2)$, and $C(3, 0)$. Determine the slope of each side of the triangle.
32. The vertices of a parallelogram are $A(-3, -3)$, $B(2, -5)$, $C(5, -1)$, and $D(0, 1)$. Determine the slope of each side of the parallelogram and the slope of the two diagonals that connect the opposite vertices.

C

33. The vertices of a right triangle are $A(-3, 2)$, $B(-3, -1)$, and $C(3, 2)$. Show algebraically that the point $P(1, 1)$ lies on one of the sides of the triangle.
34. The vertices of a square are $A(-1, -3)$, $B(7, 3)$, $C(1, 11)$, and $D(-7, 5)$. Show algebraically that the point $Q(0, 4)$ lies both on the diagonal that connects A and C and on the diagonal that connects B and D.

PROGRAMMING IN BASIC

The following program will print a graph of an open sentence in two variables, X and Y, when you input the same replacement set for both X and Y. Note that, in line 110, the program uses a negative STEP in the loop for values of Y. Thus, if you input M = 9, the values of Y will run from 9 down to −9, and in line 120, the values of X will run from −9 to 9.

```
10   PRINT "TO GRAPH AN OPEN SENTENCE"
20   PRINT "IN TWO VARIABLES"
30   PRINT "(SENTENCE IS IN LINE 130):"
40   PRINT
50   PRINT "INPUT EXTENT OF GRAPH, -M TO M"
60   PRINT "(INPUT M> 0):";
70   INPUT M
80   REM *PRINT Y
90   PRINT TAB(M);"Y"
100  PRINT TAB(M);"!"
110  FOR Y = M TO -M STEP -1
120  FOR X = -M TO M
130  IF Y = X + 2 THEN 250
140  REM *PRINT Y-AXIS
150  IF X <> 0 THEN 190
160  PRINT "!";
170  GOTO 260
180  REM *PRINT X-AXIS
190  IF Y <> 0 THEN 220
200  PRINT "-";
210  GOTO 260
220  PRINT " ";
230  GOTO 260
240  REM *PRINT GRAPH
250  PRINT "*";
260  NEXT X
270  REM *PRINT X
280  IF Y = 0 THEN 310
290  PRINT
300  GOTO 320
310  PRINT "-X"
320  NEXT Y
330  PRINT TAB(M);"!"
340  END
```

Be aware that, in lines 90, 100, and 330, you may need to adjust the PRINT TAB statements in order to accommodate the specifics of the TAB function on the computer that you are using. (See page 196.)

Exercises

1. Type in and RUN the program as given. INPUT 9 for the extent of the graph.

Change line 130 as necessary to RUN the program for each of the following open sentences. INPUT 9 for the extent of the graph.

2. $y > x + 2$	3. $y \le x + 2$	4. $x + y = 8$	5. $x - y = 6$
6. $x + y > 5$	7. $x - y > 3$	8. $2x + y \le -6$	9. $x - 2y \ge 6$

If possible, save this program for later use.

5–8 The Slope-Intercept Form of a Linear Equation

The graph of the equation

$$y = 2x$$

is shown on the coordinate plane in Figure 20. As you can see, whenever the ordinates of two points on this line differ by 2, their abscissas differ by 1. Therefore, the slope of this line is $\frac{2}{1}$, or 2. Notice that the line passes through the origin.

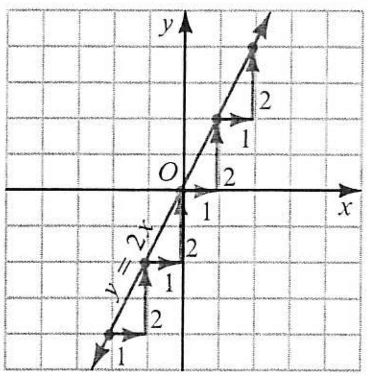


Figure 20

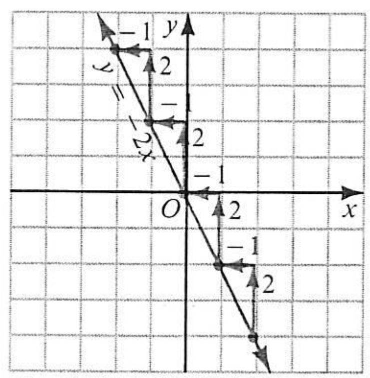


Figure 21

In Figure 21 you see the graph of the equation

$$y = -2x.$$

In this case, whenever the ordinates of two points on the line differ by 2, the abscissas differ by −1. Thus the slope of this line is $\frac{2}{-1}$, or −2. Notice that this line, too, passes through the origin.

The graphs in Figures 20 and 21 illustrate the following fact.

> For every real number m, the graph on a coordinate plane of the equation
>
> $$y = mx$$
>
> is the line that has slope m and passes through the origin.

Now consider the following equations, which are graphed on the coordinate plane in Figure 22.

$$y = 2x \qquad \text{and} \qquad y = 2x + 3$$

Notice that the lines have equal slopes, but they intersect the y-axis at different points. The ordinate of the point at which a line intersects the y-axis is called the **y-intercept** of the line. Since the abscissa of any point on the y-axis is 0, you can determine the y-intercept of a line by replacing the variable x with 0 in the equation of the line. Thus, for the lines graphed in Figure 22:

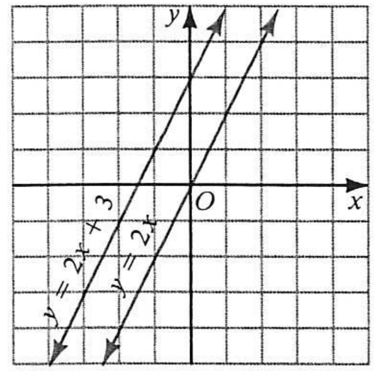


Figure 22

$y = 2x$	$y = 2x + 3$
$y = 2(0)$	$y = 2(0) + 3$
$y = 0$	$y = 3$
The y-intercept is 0.	The y-intercept is 3.

Notice that the lines graphed in Figure 22 are *parallel*. **Parallel lines** are lines in a plane that do not intersect. On a coordinate plane, nonvertical lines that have the same slope, but different y-intercepts, are parallel. Also, all vertical lines are parallel to each other.

Since the equations

$$y = 2x \qquad \text{and} \qquad y = 2x + 0,$$

are equivalent, the lines graphed in Figure 22 suggest the following fact about linear equations.

> For all real numbers m and b, the graph on a coordinate plane of
>
> $$y = mx + b$$
>
> is a line whose slope is m and whose y-intercept is b.

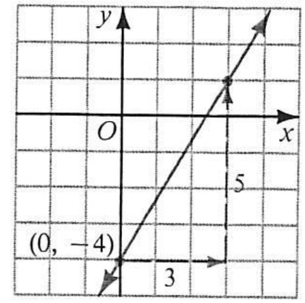
When a linear equation is written in the form $y = mx + b$, it is said to be written in **slope-intercept form**.

EXAMPLE 1 Graph $y = \frac{5}{3}x - 4$ on a coordinate plane.

SOLUTION

1. Since the y-intercept is -4, plot $(0, -4)$.
2. The slope of the line is $\frac{5}{3}$. Count 5 units up from $(0, -4)$ and 3 units to the right to obtain a second point on the line.
3. Draw a line through the two points.

The equation is graphed on the coordinate plane at the right.



Often you are given a linear equation in a different form, and you need to determine the slope-intercept form.

EXAMPLE 2 Find the slope and the y-intercept of the line whose equation is $5x - 2y = 6$.

SOLUTION Solve the given equation for y to obtain the slope-intercept form.

$$5x - 2y = 6$$
$$-2y = -5x + 6$$
$$y = \frac{5}{2}x - 3$$

$\therefore$ the slope is $\frac{5}{2}$ and the y-intercept is -3.

Sometimes you will need to find an equation of a line in some form other than slope-intercept form.

EXAMPLE 3 Find an equation of the line with slope $-\frac{2}{3}$ and y-intercept 5. The equation should be in the form $ax + by = c$, where a, b, and c are integers.

SOLUTION Substitute the given slope and y-intercept into the slope-intercept form for a linear equation. Then transform the slope-intercept form into an equation of the required form.

$$y = mx + b$$
$$y = -\frac{2}{3}x + 5$$
$$3y = -2x + 15$$
$$2x + 3y = 15$$

$\therefore$ an equation of the line is $2x + 3y = 15$.

Oral Exercises

State the slope-intercept form of the equation of the line that has the given slope m and y-intercept b.

1. $m = 2; b = 3$ **2.** $m = -5; b = 4$ **3.** $m = -3; b = -1$
4. $m = -1; b = 6$ **5.** $m = 4; b = 0$ **6.** $m = 0; b = -2$

State the slope and the y-intercept, if any, of the line whose equation is given.

7. $y = 4x + 3$ **8.** $y = 5x - 1$ **9.** $2x + y = 5$ **10.** $y - 8x = 5$
11. $x + y = 7$ **12.** $x - y = 2$ **13.** $2y = 4x - 12$ **14.** $6x + 3y = 3$
15. $2x + 3y = 9$ **16.** $4x - 2y = 7$ **17.** $y = 3$ **18.** $x = -1$

Tell whether or not the lines whose equations are given are parallel.

19. $y = 3x + 1$, $y = 3x - 2$ **20.** $y = 2x + 9$, $y = -7x + 9$ **21.** $x + y = 4$, $x - y = 4$ **22.** $x + y = 3$, $x + y = -5$

Written Exercises

Determine the slope and the y-intercept of the line whose equation is given. Then use the slope and y-intercept to graph the equation.

A **1.** $y = 2x + 1$ **2.** $y = 3x - 4$ **3.** $y = -3x + 2$ **4.** $y = -2x - 3$
5. $x - y = 4$ **6.** $x + y = -1$ **7.** $3x - y = -5$ **8.** $y - 2x = 2$
9. $2y - x = 6$ **10.** $x - 4y = 4$ **11.** $6x + 2y = 1$ **12.** $5x - 3y = 3$

Determine an equation of the line with the given slope m and y-intercept b. Use the form $ax + by = c$, where a, b, and c are integers.

13. $m = 5; b = -1$ **14.** $m = 6; b = 0$ **15.** $m = -2; b = 0$
16. $m = -3; b = -5$ **17.** $m = 0; b = 2$ **18.** $m = 0; b = -6$
19. $m = \frac{1}{2}; b = -4$ **20.** $m = -\frac{1}{3}; b = -2$ **21.** $m = \frac{3}{2}; b = 2$
22. $m = \frac{4}{3}; b = 4$ **23.** $m = -\frac{3}{5}; b = \frac{1}{2}$ **24.** $m = -\frac{5}{2}; b = -\frac{1}{3}$

Determine an equation of the line that satisfies the given requirements. Use the form $ax + by = c$, where a, b, and c are integers.

B **25.** y-intercept 3; parallel to the graph of $y = -5x + 7$
26. y-intercept -1; parallel to the graph of $3x + y = 8$
27. slope $= 3$; y-intercept the same as the graph of $y = 4x - 11$
28. slope $= -1$; y-intercept the same as the graph of $y - 2x = 9$
29. y-intercept -7; parallel to the graph of $y = x$
30. y-intercept 2; parallel to the graph of $y = -3$
31. no slope; passes through the point $(-3, 5)$
32. no slope; intersects the x-axis at the same point as the graph of $y = x$

Determine the value of r so that the graph of the equation has the given slope.

33. $y = 2rx - 4$; slope $= 3$

34. $y = \frac{2x}{r}$; slope $= 4$

35. $ry = 8x - 1$; slope $= -2$

36. $2ry = 5x + 9$; slope $= 1$

C 37. Rewrite the linear equation $ax + by = c$ in slope-intercept form. What restrictions, if any, must be placed on the values of a, b, and c?

38. If two points on the nonvertical line whose equation is $ax + by = c$ have coordinates (p, r) and (q, s), show that

$$\frac{a}{b} = \frac{s - r}{p - q}.$$

39. Show that, if k is the y-intercept of the graph of $ax + by = c$, k is a root of the equation $by - c = 0$.

Computer Exercises For students with computer experience

1. Write a program that will allow you to input the coordinates of any two points on a coordinate plane and will determine the slope of the line that passes through the two points. Be careful to account for lines that have no slope.
2. Write a program that will allow you to input the coordinates of any *three* points on a coordinate plane and will determine whether the three points lie on the same line.
3. Write a program that will determine the coordinates of four other points on the line when you input the coordinates of a point and the slope of a line that passes through the point.
4. Write a program that will allow you to input values for a, b, and c and will compute the slope, the y-intercept, and the x-intercept of the line whose equation is $ax + by = c$. (The x-intercept is the abscissa of the point at which the line intersects the x-axis.) Be careful to account for situations in which the line has no slope, no y-intercept, or no x-intercept.

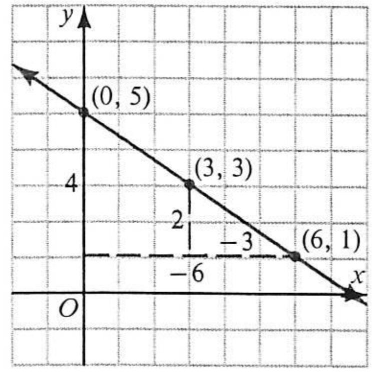
5–9 Determining an Equation of a Line

One way to describe the position of a nonvertical line on a coordinate plane is to state its slope and the coordinates of a point through which it passes. A second way is to give the coordinates of any two points through which the line passes. When a nonvertical line is described in either of these two ways, it is possible to determine an equation of the line by using the slope-intercept form of a linear equation.

EXAMPLE 1 Determine an equation of the line that passes through the point (6, 1) and has slope $-\frac{2}{3}$.

SOLUTION

1. The slope-intercept form of the equation of the line with slope $-\frac{2}{3}$ is
$$y = -\tfrac{2}{3}x + b.$$
2. Since the point (6, 1) lies on the line, its coordinates must satisfy this equation.
$$\begin{aligned} y &= -\tfrac{2}{3}x + b \\ 1 &= -\tfrac{2}{3}(6) + b \\ 1 &= -4 + b \\ 5 &= b \end{aligned}$$

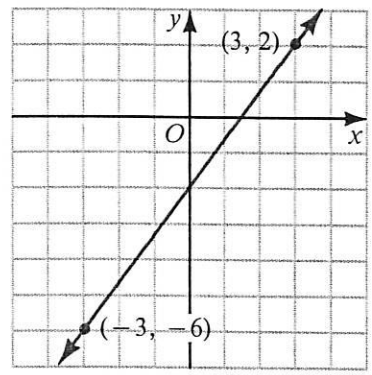


∴ an equation of the line is $y = -\frac{2}{3}x + 5$.
An equivalent equation for this line is $2x + 3y = 15$.

EXAMPLE 2 Determine an equation of the line that passes through the points (3, 2) and (−3, −6).

SOLUTION

1. Use the coordinates of the two points to compute m, the slope of the line.
$$m = \frac{2 - (-6)}{3 - (-3)} = \frac{8}{6} = \frac{4}{3}$$



2. The slope-intercept form of the equation of the line with slope $\frac{4}{3}$ is
$$y = \tfrac{4}{3}x + b.$$
3. Since the points (3, 2) and (−3, −6) both lie on the line, the coordinates of either point, say (3, 2), may be used to find the value of b.
$$\begin{aligned} y &= \tfrac{4}{3}x + b \\ 2 &= \tfrac{4}{3}(3) + b \\ 2 &= 4 + b \\ -2 &= b \end{aligned}$$
Thus $y = \frac{4}{3}x - 2$ is an equation of the line.
4. To check, show that the coordinates of the other point, (−3, −6), satisfy the equation.
$$\begin{aligned} y &= \tfrac{4}{3}x - 2 \\ -6 &\stackrel{?}{=} \tfrac{4}{3}(-3) - 2 \\ -6 &= -6 \quad \checkmark \end{aligned}$$

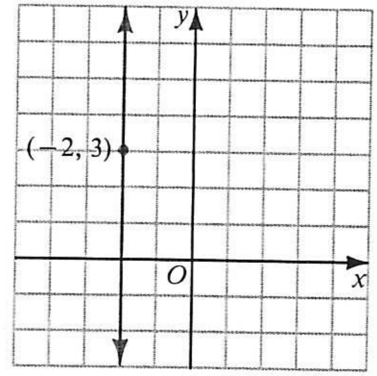
∴ an equation of the line is $y = \frac{4}{3}x - 2$.
An equivalent equation for this line is $-4x + 3y = -6$.

Since vertical lines have no slope, you cannot use the slope-intercept form of a linear equation to determine the equation of a vertical line.

EXAMPLE 3 Determine an equation of the vertical line through the point $(-2, 3)$.

SOLUTION Every point on the vertical line through the point $(-2, 3)$ has -2 as its abscissa.

$\therefore$ an equation of this line is $x = -2$.



Written Exercises

Determine an equation of the line that has slope m (if any) and passes through the given point. The equation should be in the form $ax + by = c$, where a, b, and c are integers.

A

1. $m = 2$; $(1, 1)$
2. $m = 5$; $(-1, 3)$
3. $m = -2$; $(3, -2)$
4. $m = -1$; $(-1, -2)$
5. $m = \frac{1}{2}$; $(4, 5)$
6. $m = -\frac{1}{3}$; $(3, -2)$
7. $m = -\frac{1}{5}$; $(2, -3)$
8. $m = \frac{3}{4}$; $(-3, 2)$
9. $m = 0$; $(2, 5)$
10. $m = 0$; $(-1, -4)$
11. no slope; $(-2, 7)$
12. no slope; $(-3, 3)$

Determine an equation of the line that passes through the given points. The equation should be in the form $ax + by = c$, where a, b, and c are integers.

13. $(1, 5)$, $(2, 4)$
14. $(3, 2)$, $(2, 1)$
15. $(2, -6)$, $(0, 0)$
16. $(0, 0)$, $(-1, -4)$
17. $(4, 4)$, $(5, -1)$
18. $(3, 1)$, $(5, 5)$
19. $(5, -1)$, $(-1, -3)$
20. $(-1, -4)$, $(5, -2)$
21. $(-1, 7)$, $(6, 7)$
22. $(7, 2)$, $(7, -1)$
23. $(-3, 3)$, $(-3, 2)$
24. $(5, -4)$, $(-1, -4)$

Determine the value of k so that the graph of the equation passes through the given point.

B

25. $3x + ky = 15$; $(2, 3)$
26. $kx + 4y = 2$; $(-1, 2)$
27. $kx - 5y = 14$; $(2, -4)$
28. $3x - ky = 1$; $(3, 4)$
29. $3x + ky = 9$; $(3, 5)$
30. $kx - 2y = -10$; $(-2, 5)$

Find the linear function f with domain $\mathcal{R}$ that satisfies the given requirements.

31. $f(4) = -1; f(-2) = 5$
32. $f(0) = 3; f(-2) = 7$
33. $f(1) = 3; f(3) = 8$
34. $f(-1) = 5; f(2) = -3$
35. $f(-5) = 0; f(1) = 8$
36. $f(4) = -6; f(-4) = 0$

Find the value of r and the value of s for which the graphs of the given equations are the same line.

C 37. $sx - y = -3; 12x - 3y = r$
38. $sx = 5(y - 2); ry = 2(3x + 10)$

39. Show that the y-intercept of the line with slope m that passes through the point (x_1, y_1) is $y_1 - mx_1$.

40. Use the result of Exercise 39 to show that an equation of the line with slope m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

Self-Test 3

VOCABULARY
slope (p. 243)
collinear points (p. 244)
y-intercept (p. 249)
parallel lines (p. 249)
slope-intercept form of a linear equation (p. 249)

Determine the slope of the line that passes through the given points.

1. (2, 5), (3, 8)
2. (−5, 1), (−3, −2)

Obj. 1, p. 242

Determine the slope and the y-intercept of the line with the given equation.

3. $y = 3x - 5$
4. $3x + 2y = 7$

Obj. 2, p. 242

Determine an equation of the line that satisfies the given requirements. The equation should be in the form $ax + by = c$, where a, b, and c are integers.

5. slope $= 4$; y-intercept $= -1$ *Obj. 3, p. 242*
6. slope $= -\frac{2}{5}$; y-intercept $= 3$
7. passes through the point $(-2, -6)$ and has slope 3 *Obj. 4, p. 242*
8. passes through the points $(0, -2)$ and $(3, 2)$

Check your answers with those at the back of the book.

EXTRA

Transformations of the Plane: Translations

Figure 23 shows coordinate axes on a plane, together with a square whose vertices are at the points (1, 1), (2, 1), (2, 2), and (1, 2).

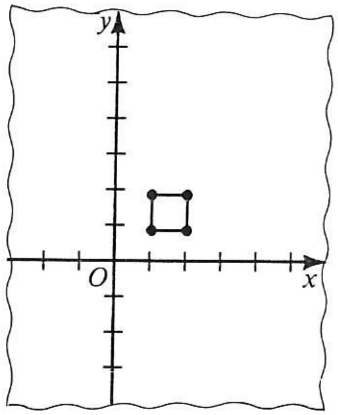


Figure 23

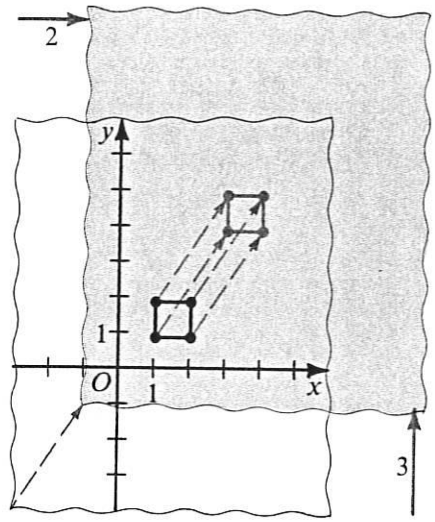


Figure 24

Now, imagine that the axes remain fixed while the plane, like a sheet of glass, slides rigidly into a new position, such as the one shown in Figure 24. The new coordinates of the vertices of the square after this "sliding" are (3, 4), (4, 4), (4, 5), and (3, 5).

In fact, *each* point on the plane now has a new pair of coordinates, (x', y'), where x' is 2 greater than the original x-coordinate and y' is 3 greater than the original y-coordinate. Thus for each point (x', y'),

$$x' = x + 2 \quad \text{and} \quad y' = y + 3.$$

Such a "sliding" of the plane is an example of a *transformation*, or *mapping*, of the plane called a **translation.** In a translation of the plane, if the point originally coinciding with the origin slides to a new location with coordinates (h, k), then every point on the plane has new coordinates (x', y') that are related to the original coordinates (x, y) by the *equations of translation*

$$x' = x + h \quad \text{and} \quad y' = y + k.$$

EXAMPLE A translation maps the point with coordinates (5, 4) onto the point with coordinates (8, −2).

a. Find the equations of translation.

b. Find the new coordinates of the point with old coordinates (−4, 0).

SOLUTION a. Let $(x, y) = (5, 4)$ and $(x', y') = (8, -2)$, and substitute in the general translation equations:

$$x' = x + h \qquad \text{and} \qquad y' = y + k$$
$$8 = 5 + h \qquad\qquad -2 = 4 + k$$
$$h = 3 \qquad\qquad k = -6$$

$\therefore$ the equations of translations are $x' = x + 3$ and $y' = y - 6$.

b. Use the equations $x' = x + 3$ and $y' = y - 6$, and substitute -4 for x and 0 for y.

$$x' = -4 + 3 \qquad\qquad y' = 0 - 6$$
$$x' = -1 \qquad\qquad y' = -6$$

$\therefore$ the new coordinates of $(-4, 0)$ are $(-1, -6)$.

The following result is proved in more advanced courses.

> Under a translation:
>
> 1. Every line is mapped (translated) onto itself or onto a line parallel to the original line.
> 2. Every line segment is mapped (translated) onto a line segment of equal length.
> 3. Every angle is mapped (translated) onto an angle of equal measure.

Exercises

In Exercises 1–8:

a. Find equations of translation that map the first point onto the second.

b. Find the new coordinates under this translation of the point whose original coordinates are $(-7, 3)$.

1. $(5, 5)$, $(6, 2)$
2. $(3, 8)$, $(5, 1)$
3. $(-7, 2)$, $(6, -1)$
4. $(3, -2)$, $(8, 1)$
5. $(-7, 0)$, $(0, 2)$
6. $(0, -3)$, $(5, 0)$
7. $(-5, -2)$, $(4, -7)$
8. $(-6, -3)$, $(-7, 1)$

9. Under what kind of translation of the plane will a vertical line be mapped onto itself? a horizontal line be mapped onto itself?

10. Use the slope formula to show that the slope of the line through the points (a, b) and (c, d), $a \neq c$, is equal to the slope of the line through the points $(a + h, b + k)$ and $(c + h, d + k)$, and thus show that under a translation a nonvertical line is mapped onto itself or onto a parallel line.

Chapter Summary

1. A *coordinate plane* can be set up using two perpendicular number lines that intersect at the origin of each. One line, called the *x-axis*, is usually horizontal and the other, called the *y-axis*, is usually vertical.
2. Each point on a coordinate plane is assigned a unique pair of *coordinates* that together form an *ordered pair* of real numbers. The first coordinate of an ordered pair is called the *x-coordinate*, or *abscissa*, and the second coordinate is called the *y-coordinate*, or *ordinate*. Conversely, each ordered pair of real numbers is assigned a unique point on a coordinate plane.
3. A *relation* is any set of ordered pairs. The set of all the first coordinates of the ordered pairs is called the *domain* of the relation, and the set of all the second coordinates is called the *range*.
4. A *function* is a relation in which no two ordered pairs have the same first coordinate.
5. Members of the range of a function are called the *values* of the function. The notation $f(x)$ is used to denote the specific value of the function f that is paired with the number x from the domain.
6. A *solution of an open sentence in two variables* is an ordered pair of values of the variables that together make the sentence a true statement. The set of all such ordered pairs is called the *solution set* of the open sentence. The *graph of an open sentence in two variables* is the graph of its solution set.
7. A *linear equation in two variables*, x and y, is any equation that can be written equivalently in the form $ax + by = c$, where a, b, and c are real numbers and a and b are not both zero. If the replacement set of each variable is $\mathcal{R}$, the graph of such an equation on a coordinate plane is a straight line.
8. A function whose ordered pairs satisfy a linear equation is called a *linear function*.
9. On a coordinate plane, the graph of a *linear inequality in two variables* is a *half-plane*. The equation of the line that forms the *boundary* of the half-plane is called the *associated equation* of the inequality.
10. Given that (x_1, y_1) and (x_2, y_2) are any two different points on a line on a coordinate plane and $x_1 \neq x_2$, the *slope* of the line is defined as the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

If $x_1 = x_2$, the line has no slope.

11. The point at which a line intersects the y-axis is called the *y-intercept* of the line.

12. A line with slope m and y-intercept b is the graph of the equation $y = mx + b$, which is called the *slope-intercept* form of the equation.

13. On a coordinate plane, nonvertical lines with the same slope but different y-intercepts are *parallel*. Also, all vertical lines are parallel.

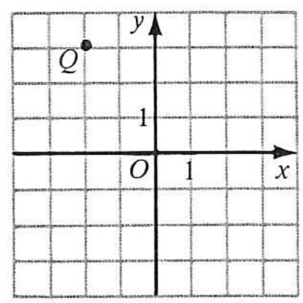
14. An equation of a line can be found given: (a) the slope and the y-intercept of the line; (b) the slope and any point on the line; (c) any two points on the line.

Chapter Review

Write the letter of the correct answer.

1. Which of the following statements is true? *5-1*
   a. The y-coordinate of every point on the y-axis is 0.
   b. The x-coordinate of every point in the second quadrant is positive.
   c. The x-coordinate of every point in the fourth quadrant is greater than the y-coordinate of the point.
   d. None of the above statements is true.

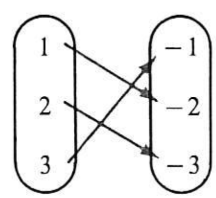
2. Given the point $P(-4, 1)$, which of the following terms describes the number -4?
   a. slope   b. ordinate
   c. abscissa   d. y-intercept



3. Give the coordinates of the point plotted on the coordinate plane at the right.
   a. $(3, -2)$ b. $(-3, 2)$
   c. $(2, -3)$ d. $(-2, 3)$

4. Give the domain D and the range R of the relation $\{(-3, 4), (-2, 3), (4, 3), (7, 4)\}$. *5-2*
   a. $D = \{-3, -2, 4, 7\}$; $R = \{-4, -3, 3, 4\}$
   b. $D = \{-3, -2, 4, 7\}$; $R = \{3, 4\}$
   c. $D = \{3, 4\}$; $R = \{-3, -2, 4, 7\}$
   d. $D = \{-4, -3, 3, 4\}$; $R = \{-3, -2, 4, 7\}$

5. Which of the following relations is pictured in the mapping diagram at the right?
   a. $\{(1, -1), (2, -2), (3, -3)\}$
   b. $\{(1, -2), (2, -3), (3, -1)\}$
   c. $\{(-1, 1), (-2, 2), (-3, 3)\}$
   d. $\{(-1, 3), (-2, 2), (-3, 1)\}$



6. Which of the following relations is *not* a function?

a. $\{(16, 2), (1, 1), (0, 0), (1, -1), (16, -2)\}$

b. $\{(2, 16), (1, 1), (0, 0), (-1, 1), (-2, 16)\}$

c. $\{(2, 16), (1, 16), (0, 16), (-1, 16), (-2, 16)\}$

d. $\{(2, 2), (1, 1), (0, 0), (-1, 1), (-2, 2)\}$

7. Determine the function that is specified by $f: x \longrightarrow x - 5$ if the domain of the function is $\{0, -2, -3\}$. 5-3

a. $\{(0, 5), (-2, 3), (-3, 2)\}$

b. $\{(0, 5), (-2, -3), (-3, 2)\}$

c. $\{(0, -5), (-2, -3), (-3, -2)\}$

d. $\{(0, -5), (-2, -7), (-3, -8)\}$

8. Given the function $g: x \longrightarrow 3 - x^2$, which of the following statements is *not* true?

a. $g(-1) = g(1)$ b. $g(-2) \leq g(-1)$

c. $g(0) \leq g(1)$ d. $g(0) + 3g(2) = 0$

9. Which of the following ordered pairs is *not* a solution of $3x - 2y \geq 5$? 5-4

a. $(-1, -3)$ b. $(2, -1)$ c. $(-2, -6)$ d. $(1, -2)$

10. Solve $x - 6y \leq 3$ for y.

a. $y \leq -\frac{1}{6}x + \frac{1}{2}$ b. $y \leq -\frac{1}{6}x + 2$

c. $y \geq \frac{1}{6}x - \frac{1}{2}$ d. $y \geq -\frac{1}{6}x + \frac{1}{2}$

11. Solve $y - x < 1$ if $x \in \{-1, 0, 1\}$ and $y \in \{-2, 0, 2\}$.

a. $\{(-2, -1), (-2, 0), (-2, 1), (0, -1), (0, 0)\}$

b. $\{(0, -1), (0, 0), (2, -1), (2, 0), (2, 1)\}$

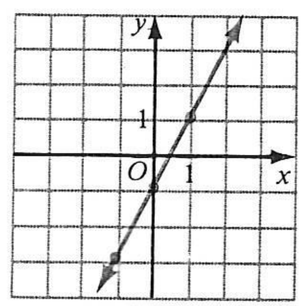
c. $\{(-1, -2), (-1, 0), (0, -2), (0, 0), (1, 0)\}$

d. $\{(-1, -2), (0, -2), (0, 0), (1, -2), (1, 0)\}$

12. Which of the following equations is graphed on the coordinate plane at the right? 5-5

a. $2x + y = 1$ b. $2x - y = 1$

c. $x + 2y = 1$ d. $x - 2y = 1$



13. Which of the following describes the graph of $y = 5$ on a coordinate plane?

a. parallel to the y-axis

b. passes through quadrants I and IV

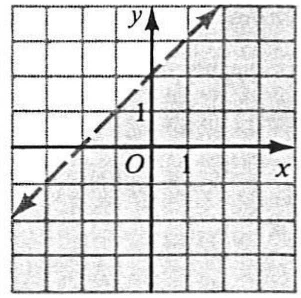
c. passes through the point $(-9, 5)$

d. none of the above

14. Which of the following inequalities is graphed on the coordinate plane at the right? 5-6

a. $x - y < -2$ b. $x - y > -2$

c. $x - y \leq -2$ d. $x - y \geq -2$



15. Determine the slope of the line that passes through the points $(-7, -1)$ and $(5, -6)$. 5-7

a. $\frac{5}{12}$ b. $-\frac{12}{5}$ c. $-\frac{5}{12}$ d. $\frac{12}{5}$

16. Through which of the following points does a line pass if it passes through $(-4, 1)$ and has slope $-\frac{2}{3}$?

a. $(0, 0)$ b. $(-6, 4)$ c. $(-1, -1)$ d. $(-6, 2)$

17. Determine the y-intercept of the line whose equation is $6x - 5y = 10$. 5-8

a. 2 b. -2 c. 10 d. -10

18. Determine an equation of the line that has slope $-\frac{1}{3}$ and y-intercept 2.

a. $x - 3y = 2$ b. $3x - y = 2$ c. $x + 2y = 6$ d. $x + 3y = 6$

19. Determine an equation of the line that passes through the point $(-2, -6)$ and has slope $\frac{4}{3}$. 5-9

a. $4x + 3y = -10$ b. $-4x + 3y = -10$

c. $3x - 4y = 10$ d. $3x + 4y = -10$

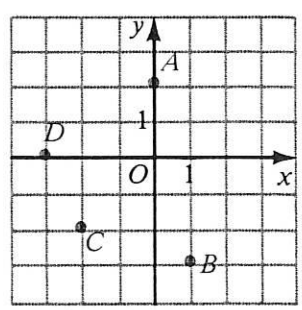
20. Determine an equation of the line that passes through the points $(-3, 2)$ and $(5, -2)$.

a. $x + 2y = 1$ b. $-x + 2y = 1$ c. $2x + y = 1$ d. $2x - y = -1$

Chapter Test

Name the coordinates of each point plotted on the coordinate plane at the right.

1. *A* 2. *B* 3. *C* 4. *D* 5-1



Plot each point on a coordinate plane.

5. $N(1, 0)$ 6. $O(0, 0)$

7. $P(3, -1)$ 8. $Q(-2, -3)$

Given the relation $\{(1, 6), (2, -6), (2, 6), (3, -6)\}$:

9. Draw a mapping diagram that represents the relation. *5-2*
10. Determine whether or not the relation is a function.

Given $f: x \longrightarrow 3 - 2x$, with domain $D = \{-2, -1, 0, 1, 2\}$:

11. Graph the function.
12. Compute $f(-2) + 3f(1)$. *5-3*

Solve when $x \in \{-2, 0, 2\}$ and $y \in \{-3, -1, 0, 3\}$.

13. $y = 1 - x^2$
14. $y < -x$ *5-4*

Graph on a coordinate plane.

15. $2x - 2y = 3$
16. $x = 4$
17. $y = x$ *5-5*
18. $x - y < 1$
19. $x \leq -3$
20. $y \geq 0$ *5-6*

21. Graph the line that passes through the point $(0, -3)$ and has slope $\frac{1}{4}$. *5-7*
22. Determine the slope of the line that passes through the points $(-1, 2)$ and $(4, 6)$.

Determine an equation of the line that satisfies the given requirements. Use the form $ax + by = c$, where a, b, and c are integers.

23. has slope $\frac{1}{2}$ and y-intercept -3 *5-8*
24. has no slope and passes through $(-3, 7)$
25. passes through $(-4, -5)$ and has slope $-\frac{1}{2}$ *5-9*
26. passes through the points $(-3, -5)$ and $(2, -1)$

Cumulative Review

Chapter 1

Graph each set of numbers on a number line.

1. $\left\{-3, -\frac{7}{3}, -2, -\frac{5}{3}, \frac{2}{3}\right\}$
2. $\{-1.75, -0.25, 0, 0.5, 2.75\}$
3. {the integers greater than -4 and less than 2}
4. {the natural numbers greater than 6}
5. {the real numbers between -4 and 3}
6. $\left\{\text{the positive real numbers less than } 4\frac{1}{4}\right\}$

Simplify.

7. $3^2 + 5(3)^2$
8. $[5(4) - 4] + [7 + 6(4)]$
9. $\dfrac{(3)7 + 5}{(8)4 - 19}$
10. $2^2(6) - 2(3^2) + 2(2) + 1$
11. $2^6 \div 2^4 \div 2^2 \div 2$
12. $\frac{1}{3}[18 \div 2(6)] \div [(12)2 \div 6]$

Chapter 3

Name the property that justifies each statement.

13. If $a + 9 = 14$, $(a + 9) - 9 = 14 - 9$.
14. If $7x = 21$, $\frac{1}{7} \cdot 7x = \frac{1}{7} \cdot 21$.

Solve.

15. $m + (-17) = 22$
16. $-39 - n = -56$
17. $\frac{1}{8}a = -12$
18. $18 = -4b + 18$
19. $3y + 4y + 7 = 49$
20. $2z + z + 5 = 3z + 5$
21. $2|s| = -3$
22. $|t| - 9 = -1$
23. $(1 - 2j) - (3 - 3j) = 0$
24. $[2(k - 4) + 6] + k = -117$
25. Larry is twice as old as Amy. In five years, Larry will be three times as old as Amy was seven years ago. How old is Larry now?
26. Seven less than a certain number is nine more than three times the number. Find the number.

Chapter 4

Solve each inequality and graph its solution set.

27. $-3v \geq 24$
28. $-4w + 9 < w - 21$
29. $6(a - 2) < 7(a - 3)$
30. $-4(b + 5) + 8b \geq 4(b - 5)$
31. $p + 2 > 6$ or $p + 2 < -6$
32. $10 - 2q > 12$ and $5q < 2q + 9$
33. $8 > 5 - 3g > -13$
34. $7 \geq -2h + 3 \geq -1$
35. $|x - 4| \geq 3$
36. $|2y + 9| - 9 < 2$

Specify the union and the intersection of the given sets.

37. $\{-2, -1, 2, 3\}$, $\{-1, 0, 1\}$
38. $\{1, 3, 5\}$, $\emptyset$
39. {the odd whole numbers}, $\{1, 2, 3, 4\}$
40. {the integers greater than -2}, {the integers less than 1}
41. {the real numbers greater than $-\frac{1}{3}$}, {the real numbers less than $2\frac{2}{3}$}
42. {the positive real numbers}, {the negative real numbers}

APPLICATION

Days of the Week

On what day of the week did astronauts first land on the moon? On what day of the week will New Year's Day fall in the year 2001? If you ever wonder about questions like these, you do not necessarily need to look for a calendar for the year in question. Using a specially devised formula, you can arithmetically calculate the day of the week for a given date, provided that the date occurs later than 1752. In that year, England and its colonies adopted the *Gregorian calendar*, which is the calendar still in use today. (The Gregorian calendar was created in order to correct errors that had been made in past calculations of leap years.)

The formula is as follows.

$$w = d + 2m + \left[\frac{3(m + 1)}{5}\right] + y + \left[\frac{y}{4}\right] - \left[\frac{y}{100}\right] + \left[\frac{y}{400}\right] + 2$$

In this formula, d represents the day of the month, y represents the number of the year, and m represents the number of the month. The months March through December are numbered 3 through 12, as you might expect. However, January and February must be considered as the 13th and 14th months of the *previous* year, and so they are numbered 13 and 14, respectively. Correspondingly, when using the formula for a date in January or February, the value of y must be the number of the previous year.

The square brackets are symbols for the *greatest integer function*, meaning that any expression within the brackets is assigned its greatest integral value. For example:

$$\left[\frac{3(4 + 1)}{5}\right] = [3] = 3$$

$$\left[\frac{3(5 + 1)}{5}\right] = [3.6] = 3$$

Once you have used the formula to find the value of w, you divide w by 7, the number of days in a week. The *remainder* of this division represents the day of the week. Sunday through Friday are represented by remainders of 1 through 6, in that order, and Saturday is represented by 0. Thus, a remainder of 3 indicates that the day of the week was Tuesday.

EXAMPLE 1 Astronauts first landed on the moon on July 16, 1969. What day of the week was this?

SOLUTION Given the date July 16, 1969, $d = 16$, $m = 7$, and $y = 1969$.

Substitute these values into the formula:

$$w = 16 + 2(7) + \left[\frac{3(8)}{5}\right] + 1969 + \left[\frac{1969}{4}\right] - \left[\frac{1969}{100}\right] + \left[\frac{1969}{400}\right] + 2$$
$$= 16 + 14 + 4 + 1969 + 492 - 19 + 4 + 2$$
$$= 2482$$

Divide the value of w by 7: $2482 \div 7 = 354$ R4

$\therefore$ the day of the week was the 4th day, or Wednesday.

EXAMPLE 2 On what day of the week will New Year's Day fall in the year 2001?

SOLUTION Given the date January 1, 2001, $d = 1$, $m = 13$, and $y = 2000$.

Substitute these values into the formula:

$$w = 1 + 2(13) + \left[\frac{3(14)}{5}\right] + 2000 + \left[\frac{2000}{4}\right] - \left[\frac{2000}{100}\right] + \left[\frac{2000}{400}\right] + 2$$
$$= 1 + 26 + 8 + 2000 + 500 - 20 + 5 + 2$$
$$= 2522$$

Divide the value of w by 7: $2522 \div 7 = 360$ R2

$\therefore$ the day of the week will be the 2nd day, or Monday.

Exercises

Determine the day of the week for each event.

1. the first telephone call: June 3, 1875
2. the first scheduled passenger service using airplanes: August 25, 1919
3. the first commercial television broadcast: July 1, 1941
4. the day you were born