Chapter 3

Solving Equations and Problems

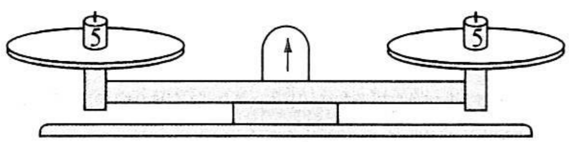
Transforming Equations

OBJECTIVES for Sections 3-1 through 3-4:

1. *To use the addition, subtraction, multiplication, and division properties of equality as reasons for statements in proving theorems.*
2. *To solve equations using an addition, subtraction, multiplication, or division transformation.*
3. *To solve equations using several transformations.*

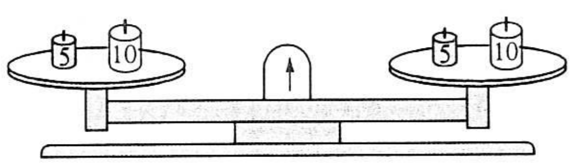
3–1 Properties of Equality

Pictured below is a device that is called a beam balance. When a 5 g mass is placed in each pan of the balance, the pans balance each other as shown.



$5 = 5$

If a 10 g mass is now added to each pan, the pans remain in balance.



$5 + 10 = 5 + 10$

The example on the preceding page illustrates the **addition property of equality:** *If the same number is added to equal numbers, the sums are equal.*

> ***Addition Property of Equality***
> For all real numbers a, b, and c, if $a = b$, then
> $$a + c = b + c \quad \text{and} \quad c + a = c + b.$$

The addition property of equality is a theorem that can be proved as follows.

PROOF

Statements	*Reasons*
1. a, b, and c are real numbers.	Hypothesis
2. $a + c = a + c$	Reflexive axiom of equality
3. $a = b$	Hypothesis
4. $\therefore a + c = b + c$	Substitution principle
5. $a + c = c + a$ $b + c = c + b$	Commutative axiom for addition
6. $\therefore c + a = c + b$	Substitution principle

You can use similar reasoning to prove the **multiplication property of equality:** *If equal numbers are multiplied by the same number, the products are equal.* (See Exercise 9 on page 116.)

> ***Multiplication Property of Equality***
> For all real numbers a, b, and c, if $a = b$, then
> $$ac = bc \quad \text{and} \quad ca = cb.$$

Since $a - c = a + (-c)$ and $b - c = b + (-c)$, you can write the **subtraction property of equality** as a special case of the addition property of equality: *If the same number is subtracted from equal numbers, the differences are equal.* (See Exercise 7 on page 117.)

> ***Subtraction Property of Equality***
> For all real numbers a, b, and c, if $a = b$, then
> $$a - c = b - c.$$

Similarly, since $\frac{a}{c} = a \cdot \frac{1}{c}$ and $\frac{b}{c} = b \cdot \frac{1}{c}$, you can write the **division property of equality** as a special case of the multiplication property of equality: *If equal numbers are divided by the same nonzero number, the quotients are equal.* (See Exercise 8 on page 117.)

Division Property of Equality

For all real numbers a and b and all nonzero real numbers c, if $a = b$, then

$$\frac{a}{c} = \frac{b}{c}.$$

EXAMPLE Name the property of equality that is illustrated.

a. If $m - 4 = -7$, then $(m - 4) + 4 = -7 + 4$.

b. If $-6n = 78$, then $\frac{-6n}{-6} = \frac{78}{-6}$.

c. If $\frac{x}{a} = b$, and $a \neq 0$, then $a\left(\frac{x}{a}\right) = ab$.

d. If $\frac{1}{2}y + d = c$, then $\left(\frac{1}{2}y + d\right) - d = c - d$.

SOLUTION **a.** addition property of equality

b. division property of equality

c. multiplication property of equality

d. subtraction property of equality

Oral Exercises

Replace each _?_ with the number that makes a true statement.

1. If $a = b$, then $a + 8 = b + \underline{?}$.
2. If $x - 8 = -10$, then $(x - 8) + 8 = \underline{?} + 8$.
3. If $a = b$, then $a - \underline{?} = b - 12$.
4. If $5y + 12 = 7$, then $(5y + 12) - 12 = \underline{?} - 12$.
5. If $a = b$, then $-5a = (\underline{?})b$.
6. If $\frac{z}{-5} = \underline{?}$, then $-5\left(\frac{z}{-5}\right) = -5(10)$.
7. If $a = b$, then $\frac{a}{\underline{?}} = \frac{b}{4}$.
8. If $4w = -68$, then $\frac{4w}{4} = \frac{?}{4}$.

Replace each _?_ with the reason that justifies the statement to its left.

9. Prove: For all real numbers a, b, and c, if $a = b$, then $ac = bc$ and $ca = cb$.

PROOF

Statements	*Reasons*
1. a, b, and c are real numbers.	_?_
2. $ac = ac$	_?_
3. $a = b$	_?_
4. $\therefore ac = bc$	Substitution principle
5. $ac = ca$ $bc = cb$	Commutative axiom for multiplication
6. $\therefore ca = cb$	_?_

10. Prove: For all real numbers a and b, if $a = b$, then $-a = -b$.

PROOF

Statements	*Reasons*
1. a and b are real numbers.	_?_
2. $a + (-a) = 0$ $b + (-b) = 0$	_?_
3. $a + (-a) = b + (-b)$	Transitive axiom of equality
4. $a = b$	_?_
5. $a + (-a) = a + (-b)$	Substitution principle
6. $-a$ is a real number.	_?_
7. $-a + [a + (-a)] = -a + [a + (-b)]$	Addition property of equality
8. $-a + [a + (-a)] = (-a + a) + (-b)$	_?_
9. $-a + 0 = 0 + (-b)$	_?_
10. $\therefore -a = -b$	_?_

Written Exercises

Name the axiom, theorem, or definition that justifies each lettered step.

A

1.

$$m + 3 = -5$$
$$(m + 3) + (-3) = -5 + (-3) \quad \textbf{(a)}$$
$$m + [3 + (-3)] = -5 + (-3) \quad \textbf{(b)}$$
$$m + 0 = -5 + (-3) \quad \textbf{(c)}$$
$$m = -5 + (-3) \quad \textbf{(d)}$$
$$m = -8$$

2.

$$-3 = -7 + n$$
$$7 + (-3) = 7 + (-7 + n) \quad \textbf{(a)}$$
$$7 + (-3) = [7 + (-7)] + n \quad \textbf{(b)}$$
$$7 + (-3) = 0 + n \quad \textbf{(c)}$$
$$7 + (-3) = n \quad \textbf{(d)}$$
$$4 = n$$

3. $3x = -21$

$\frac{1}{3}(3x) = \frac{1}{3}(-21)$ **(a)**

$\left(\frac{1}{3} \cdot 3\right)x = \frac{1}{3}(-21)$ **(b)**

$(1)x = \frac{1}{3}(-21)$ **(c)**

$x = \frac{1}{3}(-21)$ **(d)**

$x = -7$

4. $9 = -\frac{3}{4}y$

$-\frac{4}{3}(9) = -\frac{4}{3}\left(-\frac{3}{4}y\right)$ **(a)**

$-\frac{4}{3}(9) = \left[-\frac{4}{3}\left(-\frac{3}{4}\right)\right]y$ **(b)**

$-\frac{4}{3}(9) = (1)y$ **(c)**

$-\frac{4}{3}(9) = y$ **(d)**

$-12 = y$

5. $a - 9 = -2$

$a + (-9) = -2$ **(a)**

$[a + (-9)] + 9 = -2 + 9$ **(b)**

$a + (-9 + 9) = -2 + 9$ **(c)**

$a + 0 = -2 + 9$ **(d)**

$a = -2 + 9$ **(e)**

$a = 7$

6. $\frac{b}{4} = -8$

$b \cdot \frac{1}{4} = -8$ **(a)**

$\left(b \cdot \frac{1}{4}\right)4 = (-8)4$ **(b)**

$b\left(\frac{1}{4} \cdot 4\right) = (-8)4$ **(c)**

$b(1) = (-8)4$ **(d)**

$b = (-8)4$ **(e)**

$b = -32$

Give the reason that justifies each statement in the given proof.

7. Prove: For all real numbers a, b, and c, if $a = b$, then $a - c = b - c$.

PROOF

1. a, b, and c are real numbers.
2. $-c$ is a real number.
3. $a = b$
4. $a + (-c) = b + (-c)$
5. $\therefore a - c = b - c$

8. Prove: For all real numbers a, b, and c such that $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

PROOF

1. a, b, and c are real numbers such that $c \neq 0$.
2. $\frac{1}{c}$ is a real number.
3. $a = b$
4. $a \cdot \frac{1}{c} = b \cdot \frac{1}{c}$
5. $\therefore \frac{a}{c} = \frac{b}{c}$

Write a direct proof of each theorem.

B 9. For all real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

10. For all real numbers a, b, and c such that $c \neq 0$, if $ac = bc$, then $a = b$.

11. For all real numbers a, b, and c, if $a - c = b - c$, then $a = b$.

12. For all real numbers a, b, and c such that $c \neq 0$, if $\frac{a}{c} = \frac{b}{c}$, then $a = b$.

Write a direct proof of each theorem. Assume that each variable represents a real number.

13. If $a + b = 0$, then $a = -b$.

14. If $ab = 1$ and $b \neq 0$, then $a = \frac{1}{b}$.

15. If $a - b = x$, then $b + x = a$.

16. If $a \div b = x$ and $b \neq 0$, then $bx = a$.

C 17. If $a = b$ and $c = d$, then $a + c = b + d$.

18. If $a + c = b + d$ and $b = c$, then $a = d$.

19. If $a = b$ and $c = d$, then $ac = bd$.

20. If $ac = bd$, $a = d$, and $a \neq 0$, then $b = c$.

21. If $a = b$ and $c = d$, then $a - c = b - d$.

22. If $a = b$, $c = d$, $c \neq 0$, and $d \neq 0$, then $\frac{a}{c} = \frac{b}{d}$.

3–2 Transforming Equations: Addition and Subtraction

You can use the addition and subtraction properties of equality when solving certain equations. For example, study the following set of equations:

$$\begin{aligned} &(1) & y - 7 &= -3 \\ &(2) & y - 7 + 7 &= -3 + 7 \\ &(3) & y &= 4 \end{aligned}$$

When you add 7 to each side of equation (1) and simplify the result, you obtain equation (3). On the other hand, when you subtract 7 from (or add -7 to) each side of equation (3), you obtain equation (1):

$$\begin{aligned} &(3) & y &= 4 \\ &(2) & y - 7 &= 4 - 7 \\ &(1) & y - 7 &= -3 \end{aligned}$$

The addition and subtraction properties of equality guarantee that any root of equation (1) is also a root of equation (3) and that any root of equation (3) is also a root of equation (1). As a result, the equations $y - 7 = -3$ and $y = 4$ have the same solution set, namely $\{4\}$.

Equations that have the same solution set over a given domain are called **equivalent equations** over that domain. To solve an equation, you usually try to change, or **transform,** it into a simple equivalent equation whose solution or solutions can be seen at a glance, or by *inspection*. The properties of real numbers guarantee that each of the following *transformations* of a given equation will produce an equivalent equation.

> ### *Transformations that Produce an Equivalent Equation*
> **Transformation by Substitution:** Substituting for any expression in a given equation an equivalent expression.
>
> **Transformation by Addition:** Adding the same real number to each side of a given equation.
>
> **Transformation by Subtraction:** Subtracting the same real number from each side of a given equation.

EXAMPLE 1 Solve $-8 + n = 11$.

SOLUTION

$$-8 + n = 11$$
$$-8 + n + 8 = 11 + 8 \longleftarrow$$
$$n = 19$$

To obtain n alone as the left side, add the opposite of -8, or 8, to each side.

Because errors may occur in transforming equations, always check your work by showing that each root of the transformed equation satisfies the *original equation.*

Check: $-8 + n = 11 \longleftarrow$ original equation

$$-8 + 19 \stackrel{?}{=} 11$$
$$11 = 11 \quad \checkmark$$

$\therefore$ the solution set is $\{19\}$.

When solving an equation, sometimes you first need to simplify one or both sides using the properties of real numbers.

EXAMPLE 2 Solve $15 - (9 + z) = 20$.

SOLUTION

$$15 - (9 + z) = 20$$
$$15 - 9 - z = 20 \longleftarrow$$
$$6 - z = 20$$
$$6 - z - 6 = 20 - 6$$
$$-z = 14$$
$$z = -14$$

Use the property of the opposite of a sum to help simplify the left side of the equation.

Check:

$$15 - (9 + z) = 20$$
$$15 - [9 + (-14)] \stackrel{?}{=} 20$$
$$15 - (-5) \stackrel{?}{=} 20$$
$$20 = 20 \quad \checkmark$$

$\therefore$ the solution set is $\{-14\}$.

Oral Exercises

Tell what transformation can be used to obtain an equivalent equation with the variable alone as one side.

1. $m + 5 = 2$
2. $9 + g = 3$
3. $x - 3 = 6$
4. $r - 8 = 7$
5. $a + (-2) = -5$
6. $b + (-4) = -3$
7. $-4 + n = 4$
8. $-10 + s = 10$
9. $b - (-3) = -12$
10. $p - (-2) = 0$
11. $-7 = 7 + x$
12. $-5 = n + 5$
13. $0 = r - 6$
14. $-11 = m - 4$
15. $-0.6 + y = 1.5$
16. $-1\frac{1}{2} + b = -\frac{3}{4}$
17. $5 + x = -\frac{2}{3}$
18. $-4.6 = -7 + a$

Written Exercises

Solve.

A

1. $t + 7 = 5$
2. $p + 18 = 15$
3. $a - 2 = 10$
4. $r - 7 = 6$
5. $-4 + k = -3$
6. $-9 + a = -2$
7. $b + (-11) = 32$
8. $(-3) + t = 5$
9. $s - (-4) = 9$
10. $w - (-1) = 8$
11. $-15 = 6 + j$
12. $-13 = 12 + q$
13. $10 = v - 10$
14. $x - 9 = 9$
15. $-4 + f = -4$
16. $-23 = -23 + b$
17. $-13 + 2 = x - 8$
18. $f - 12 = -17 + 3$
19. $h - \frac{1}{2} = -\frac{2}{3}$
20. $-\frac{4}{5} + e = -\frac{1}{4}$
21. $w + (-0.61) = -0.39$
22. $a - 2.7 = -4.32$
23. $-t + 2 = -5$
24. $3 - p = 8$
25. $3 = -8 - k$
26. $-5 = -v + 7$
27. $-a + |-3| = 4$
28. $k - |5| = -6$
29. $-1.08 = -1.8 - t$
30. $0.1 = 0.01 - j$
31. $|g| + 4 = 18$
32. $|p| + 9 = 24$
33. $-3 + |h| = -2$
34. $|t| - 15 = -7$
35. $|k| - 4 = -6$
36. $-5 + |r| = -12$

B

37. $11 + (-1 + n) = -1$
38. $8 + (-2 + c) = -10$
39. $5 = (p - |-2|) - 16$
40. $-8 = 12 + (|-5| + r)$
41. $2 - (3 - a) = -4$
42. $7 - (12 - p) = -9$
43. $12 = 3 - (5 + b)$
44. $-2 = -5 + (-x + 7)$
45. $1\frac{1}{2} - \left(\frac{2}{3} + x\right) = \frac{7}{2}$
46. $\left(\frac{1}{4} - y\right) - \frac{3}{8} = -\frac{5}{6}$
47. $-3 = \frac{1}{6} - \left(\frac{4}{3} - a\right)$
48. $-3\frac{1}{2} = \frac{3}{2} - \left(\frac{3}{4} - k\right)$
49. $0.3 - (p - 0.5) = 0.7$
50. $2.5 - (7.8 + r) = 1.9$
51. $3 - (5 - |t|) = 6$
52. $4 - (-2 - |w|) = 10$
53. $-(|a| - 7) = 10$
54. $-(|s| - 7) = 5$

Solve each equation for the value of r when $s = -2$ and $t = 6$.

55. $-3s + 2t = 6 - (st + r)$
56. $s - (3 - t) = 2s - (t - r)$
57. $r - s = -t - (1 + s^2)$
58. $r - (s + t) = -3(t^2 - s^2)$

Solve each equation for the value of x when $a = \frac{1}{2}$ and $b = -\frac{3}{4}$.

C 59. $-a(a^2 - 2b + ab) = -\frac{2}{3}a - x + \frac{5}{16}$

60. $x - 2 = \frac{-6a^3 + (2b)^2}{a - (2b - 3a)}$

For which real values of t is the solution set of each equation empty?

EXAMPLE $|x| = t$

SOLUTION Since $|x| \geq 0$ for all real values of x, the solution set is $\emptyset$ if $t < 0$.

61. $|x| = -t$

62. $|-x| = -t$

63. $|x| = t + 1$

64. $|x| = t - 1$

65. $|x| + 2 = t$

66. $|x| - 2 = t$

3–3 Transforming Equations: Multiplication and Division

As you have just seen, certain equations can be solved using the addition and subtraction properties of equality. You will now see how certain other equations can be solved using the multiplication and division properties of equality. For example, consider the following set of equations:

$$(1) \qquad \frac{z}{-9} = -4$$

$$(2) \qquad \frac{z}{-9}(-9) = -4(-9)$$

$$(3) \qquad z = 36$$

When you multiply each side of equation (1) by -9 and simplify the result, you obtain equation (3). On the other hand, when you divide each side of equation (3) by -9, you obtain equation (1):

$$(3) \qquad z = 36$$

$$(2) \qquad \frac{z}{-9} = \frac{36}{-9}$$

$$(1) \qquad \frac{z}{-9} = -4$$

The multiplication and division properties of equality guarantee that any root of equation (1) is also a root of equation (3) and that any root of equation (3) is also a root of equation (1). Therefore the equations $\frac{z}{-9} = -4$ and $z = 36$ have the same solution set, $\{36\}$, and are equivalent equations. From this example, you see that the multiplication and division properties of equality provide two more ways to transform an equation into an equivalent equation.

Transformation by Multiplication: Multiplying each side of a given equation by the same *nonzero* real number.

Transformation by Division: Dividing each side of a given equation by the same *nonzero* real number.

EXAMPLE 1 Solve $3a = -87$.

SOLUTION

$$3a = -87$$
$$\frac{3a}{3} = \frac{-87}{3} \longleftarrow \left\{\text{To obtain } a \text{ alone as the left side, divide each side by 3 (or multiply by } \tfrac{1}{3}\text{, the reciprocal of 3).}\right.$$
$$a = -29$$

Check:
$$3a = -87$$
$$3(-29) \stackrel{?}{=} -87$$
$$-87 = -87 \quad \checkmark$$

$\therefore$ the solution set is $\{-29\}$.

EXAMPLE 2 Solve $12 = -\frac{2}{3}m$.

SOLUTION

$$12 = -\frac{2}{3}m$$
$$-\frac{3}{2}(12) = -\frac{3}{2}\left(-\frac{2}{3}m\right) \longleftarrow \left\{\text{To obtain } m \text{ alone as the right side, multiply each side by } -\tfrac{3}{2}\text{, the reciprocal of } -\tfrac{2}{3}.\right.$$
$$-18 = m$$

Check:
$$12 = -\frac{2}{3}m$$
$$12 \stackrel{?}{=} -\frac{2}{3}(-18)$$
$$12 = 12 \quad \checkmark$$

$\therefore$ the solution set is $\{-18\}$.

You know that you can never divide by zero, and so zero is certainly not allowed as a divisor in a division transformation. Do you know why zero is not allowed as a *multiplier* in transforming an equation? Look at the following set of equations:

$$(1) \qquad 7x = 28$$
$$(2) \qquad 0 \cdot 7x = 0 \cdot 28$$
$$(3) \qquad (0 \cdot 7)x = 0$$
$$(4) \qquad 0 \cdot x = 0$$

Equation (1) has just one root, namely 4, but equation (4) is satisfied by *any* real number. Therefore, equations (1) and (4) are *not* equivalent. *In transforming an equation, never multiply by zero.*

Oral Exercises

Tell what transformation can be used to obtain an equivalent equation with the variable alone as one side.

1. $7a = 35$
2. $9x = -207$
3. $-4b = -24$
4. $-3p = -54$
5. $\frac{1}{3}d = 15$
6. $\frac{1}{5}t = 5$
7. $-\frac{3}{10}f = 12$
8. $-\frac{3}{4}a = 6$
9. $-5 = -5c$
10. $0 = -6r$
11. $1.5j = 9$
12. $-5 = 0.1v$
13. $\frac{p}{3} = -4$
14. $\frac{g}{-5} = -3$
15. $x \div 5 = -3$
16. $6 = y \div (-3)$
17. $\frac{c}{-9} = 1$
18. $-1 = \frac{a}{21}$
19. $-\frac{3}{8} = \frac{3}{8}k$
20. $-\frac{4}{5} = -\frac{5}{4}h$

Written Exercises

Solve.

A

1. $13r = -143$
2. $15t = -105$
3. $\frac{1}{7}p = -3$
4. $\frac{1}{6}x = -4$
5. $-8t = -112$
6. $-144 = -9e$
7. $5 = \frac{v}{12}$
8. $7 = \frac{p}{-6}$
9. $\frac{3}{20}n = 0$
10. $0 = -\frac{5}{3}t$
11. $-60 = -\frac{2}{3}w$
12. $48 = -\frac{3}{5}t$
13. $\frac{3}{5}x = -\frac{7}{10}$
14. $-\frac{8}{9} = -\frac{4}{15}x$
15. $\frac{1}{3}t = 3\frac{2}{3}$
16. $\frac{1}{2}k = 2\frac{5}{6}$
17. $\frac{7}{2}t = -\frac{2}{7}$
18. $-\frac{3}{10}y = \frac{3}{10}$
19. $-\frac{3}{11}r = -\frac{3}{11}$
20. $\frac{5}{8}c = -\frac{8}{5}$
21. $0.01 = -0.1p$
22. $-0.15 = -0.003t$
23. $-\frac{1}{5}t = -0.3$
24. $-0.4 = \frac{1}{7}k$
25. $\frac{m}{0.2} = -4$
26. $\frac{h}{0.5} = -0.5$
27. $\frac{-4}{3} = \frac{t}{-2}$
28. $\frac{-b}{7} = \frac{-2}{3}$
29. $5|x| = 10$
30. $-3|r| = -12$

B

31. $\frac{3}{5}|d| = 9$
32. $\frac{7}{12}|x| = 14$
33. $-\frac{2}{3}|e| = -\frac{3}{4}$
34. $-\frac{5}{4}|s| = -\frac{3}{2}$
35. $-\frac{1}{3}|t| = 15$
36. $\frac{1}{4}|t| = -6$
37. $0 = -\frac{2}{5}|y|$
38. $-\frac{3}{5}|z| = 0$
39. $-0.012|p| = -4.8$
40. $2.5|x| = -4.5$
41. $7 = -1.001|t|$
42. $-5 = -1.2|k|$

Solve each equation for the value of x when $a = -3$ and $b = 4$.

43. $-\frac{2b}{a}x = |a - 2b| + 5$

44. $|a|x = -1 - |b^2 - 3a^2|$

45. $\frac{a^2b}{a + b}x = -a^2 + (-a)^2b$

46. $\frac{2a + 3b}{a^2 - b^2}x = a^2 + 2ab + b^2$

Solve each equation for the value of r when $s = -\frac{1}{4}$ and $t = \frac{2}{5}$.

C 47. $\frac{1}{10}(s + t)|r| = s^2 + 2st + t^2$

48. $s^2 - t^2 = 13s(s + t)|r|$

49. $4r + 9st = 4\left(\frac{t}{s} + \frac{s}{t}\right)$

50. $\left(\frac{7}{s^2 - 4t^2}\right)r = \frac{5}{s + 2t}$

3–4 Using Several Transformations

From your work in Section 2-7, recall the theorem which states that, for all real numbers a and b, $(a + b) + (-b) = a$. Using this theorem together with the definition of subtraction, the following related theorem has already been proved. (See Exercises 47 and 49 on page 85.)

> For all real numbers a and b,
>
> $$(a + b) - b = a \quad \text{and} \quad (a - b) + b = a.$$

This theorem can be stated informally as follows:

To "undo" addition of a number, you subtract that number.
To "undo" subtraction of a number, you add that number.

Operations that "undo" each other, such as addition and subtraction, are called **inverse operations.**

In Section 2-10, the theorem was proved which states that, for all real numbers a and all nonzero real numbers b, $(ab)\frac{1}{b} = a$. Using this theorem together with the definition of division, this related theorem has already been proved. (See Exercises 41 and 43 on page 102.)

> For all real numbers a and all nonzero real numbers b,
>
> $$(ab) \div b = a \quad \text{and} \quad (a \div b)b = a.$$

This theorem can be stated informally as follows:

To "undo" multiplication by a nonzero number, you divide by that number.
To "undo" division by a nonzero number, you multiply by that number.

That is, multiplication and division are also inverse operations.

When deciding which transformations to use in solving an equation, you may find it helpful to think of inverse operations.

EXAMPLE 1 Solve $7x + 29 = 15$.

SOLUTION

$$7x + 29 = 15$$

$$7x + 29 - 29 = 15 - 29 \longleftarrow$$ {To undo the addition of 29 to $7x$, subtract 29 from each side.

$$7x = -14$$

$$\frac{7x}{7} = \frac{-14}{7} \longleftarrow$$ {To undo the multiplication of x by 7, divide each side by 7.

$$x = -2$$

Check:

$$7x + 29 = 15$$

$$7(-2) + 29 \stackrel{?}{=} 15$$

$$-14 + 29 \stackrel{?}{=} 15$$

$$15 = 15 \quad \checkmark$$

$\therefore$ the solution set is $\{-2\}$.

The following plan summarizes the steps that are usually helpful in solving an equation in which all the variables are on the same side.

1. Simplify each side of the equation.
2. If there are still indicated additions or subtractions, use the inverse operations to undo them.
3. If there are indicated multiplications or divisions involving the variable, use the inverse operations to undo them.

EXAMPLE 2 Solve $5 - 3(2n - 3) = 44$.

SOLUTION

$$5 - 3(2n - 3) = 44$$

$$5 - 6n + 9 = 44 \longleftarrow$$ {Use the distributive axiom to help simplify the left side.

$$14 - 6n = 44$$

$$14 - 6n - 14 = 44 - 14 \longleftarrow$$ {Subtract 14 from each side.

$$-6n = 30$$

$$\frac{-6n}{-6} = \frac{30}{-6} \longleftarrow$$ {Divide each side by -6.

$$n = -5$$

Solution continued on next page.

CONDENSED SOLUTION

$$5 - 3(2n - 3) = 44$$
$$5 - 6n + 9 = 44$$
$$14 - 6n = 44$$
$$-6n = 30$$
$$n = -5$$

$\therefore$ the solution set is $\{-5\}$.

Check:
$$5 - 3(2n - 3) = 44$$
$$5 - 3[2(-5) - 3] \stackrel{?}{=} 44$$
$$5 - 3(-13) \stackrel{?}{=} 44$$
$$5 + 39 \stackrel{?}{=} 44$$
$$44 = 44 \quad \checkmark$$

Often there is more than one way to solve a given equation.

EXAMPLE 3 Solve $\frac{1}{2}x + \frac{3}{2} = 4$.

SOLUTION 1

$$\frac{1}{2}x + \frac{3}{2} = 4 \longleftarrow \begin{cases}\text{First subtract } \frac{3}{2} \text{ from each side,} \\ \text{then multiply each side by 2.}\end{cases}$$
$$\frac{1}{2}x = \frac{5}{2}$$
$$x = 5$$

SOLUTION 2

$$\frac{1}{2}x + \frac{3}{2} = 4 \longleftarrow \begin{cases}\text{First multiply each side by 2,} \\ \text{then subtract 3 from each side.}\end{cases}$$
$$2\left(\frac{1}{2}x + \frac{3}{2}\right) = 2(4)$$
$$x + 3 = 8$$
$$x = 5$$

$\therefore$ the solution set is $\{5\}$.

Check:
$$\frac{1}{2}x + \frac{3}{2} = 4$$
$$\frac{1}{2}(5) + \frac{3}{2} \stackrel{?}{=} 4$$
$$\frac{5}{2} + \frac{3}{2} \stackrel{?}{=} 4$$
$$\frac{8}{2} \stackrel{?}{=} 4$$
$$4 = 4 \quad \checkmark$$

Oral Exercises

Tell what transformation was used to transform the first equation into the second and the second equation into the third.

1. $3x - 1 = -5;\quad 3x = -4;\quad x = \frac{-4}{3}$
2. $-2x + 3 = 1;\quad -2x = -2;\quad x = 1$
3. $\frac{1}{2}b - 3 = -14;\quad \frac{1}{2}b = -11;\quad b = -22$
4. $-4 + \frac{1}{3}c = 5;\quad \frac{1}{3}c = 9;\quad c = 27$
5. $-8 = -\frac{2}{3}p + 6;\quad -14 = -\frac{2}{3}p;\quad 21 = p$

6. $5 - \frac{3}{4}x = -7; \quad -\frac{3}{4}x = -12; \quad x = 16$

7. $\frac{1}{3}t + 5 = -3; \quad t + 15 = -9; \quad t = -24$

8. $2 = 3 + \frac{k}{5}; \quad 10 = 15 + k; \quad -5 = k$

9. $\frac{c - 1}{2} = -7; \quad c - 1 = -14; \quad c = -13$

10. $\frac{-3 + b}{4} = 8; \quad -3 + b = 32; \quad b = 35$

11. $\frac{1}{2}(p - 1) = 10; \quad p - 1 = 20; \quad p = 21$

12. $-3(x - 5) = 12; \quad x - 5 = -4; \quad x = 1$

Tell which of the equations (a), (b), or (c) are equivalent to the given equation. For each equivalent equation, tell what transformation was used to obtain it from the given equation.

13. Given $\frac{1}{2}x - 3 = 7$

a. $\frac{1}{2}x = 4$ **b.** $x - 3 = 14$ **c.** $x - 6 = 14$

14. Given $-3(x - 4) = 15$

a. $x - 4 = -5$ **b.** $-3x + 12 = 15$ **c.** $-3x + 12 = -45$

15. Given $-\frac{1}{3}x + \frac{2}{3} = 3$

a. $-x + 2 = 9$ **b.** $-x + 2 = 3$ **c.** $-\frac{1}{3}x = \frac{7}{3}$

16. Given $\frac{3}{4}x + 6 = 9$

a. $3x + 24 = 36$ **b.** $x + 8 = 12$ **c.** $\frac{3}{4}x = 3$

Written Exercises

Solve.

A

1. $4c + 3 = 7$
2. $-4 + 2t = -1$
3. $0.25 = -0.5t + 1.45$
4. $-\frac{3}{8} = \frac{5}{4} - 2x$
5. $-2 = \frac{b}{3} - 4$
6. $2 - \frac{1}{4}t = 16$
7. $5 - \frac{3}{4}x = 11$
8. $5 = \frac{3}{5}p + 14$
9. $\frac{2s - 3}{4} = -5$

Solve.

10. $\dfrac{5 - 3g}{2} = 0$
11. $9t + 4 - 3t = 36$
12. $-7s + 6 - 8s = 36$
13. $315 - 22y - 13y = 0$
14. $-13 = -16b + (-10b)$
15. $3(a - 4) = -5$
16. $6(4 - 3r) = 24$
17. $\frac{1}{3}(2 - 5p) = -13$
18. $-\frac{4}{5}(x - 4) = -8$
19. $\frac{1}{2}x + \frac{1}{3} = \frac{3}{4}$
20. $-\frac{1}{4}g + \frac{1}{6} = \frac{5}{12}$
21. $2(3 - c) - 7 = -1$
22. $-3(2 - y) = 0$
23. $-8 = -3(y - 3) - 8$
24. $5(2 - 3b) - 13 = 12$
25. $15g - (13g + 14) - 5g = 0$
26. $10k - 8k - (3 - k) = -6$
27. $(3r - 7) - (4r - 15) = 6$
28. $(21 - 2p) - (15 + 6p) = 0$
29. $2v - 8 - \frac{4}{3}v = 0$
30. $5 = u + 3 - \frac{8}{9}u$
31. $3(d - 3) - 4d = 5$
32. $7y + 4(y + 1) = 15$
33. $6e - (2e - 5) = 17$
34. $7z - (2z - 8) = -2$
35. $-2(3 - g) + 2g - 9 = 1$
36. $-2b - 5(b - 1) - 3 = 16$

Solve each equation using two different methods. The first steps of the methods should be different from each other.

B

37. $\frac{1}{3}a + 4 = 2$
38. $\frac{2}{5}x + 3 = \frac{1}{5}$
39. $3(p - 2) = 12$
40. $4(k + 1) = 7$
41. $\frac{1}{4}x + \frac{3}{4} = 5$
42. $\frac{3}{2}t - \frac{1}{2} = 2$
43. $\frac{1}{3}(t - 3) = 4$
44. $\frac{1}{2}(6x - 1) = 4$
45. $\frac{4}{5}x - 3 = \frac{2}{5}$
46. $\frac{2}{3}y - 6 = 8$
47. $\frac{1}{5}a - \frac{1}{4} = \frac{1}{2}$
48. $\frac{1}{4}t + \frac{1}{6} = \frac{2}{3}$

Solve.

49. $5 = \frac{2}{3}(10 - i) + 1$
50. $8 - \frac{3}{4}(9 - d) = 5$
51. $3(k - 2) - 2(2 - 3k) = -10$
52. $5(6f - 1) - 4(3 - 4f) = 6$
53. $\frac{2}{5}(j - 4) + 2(j + 3) = -10$
54. $\frac{1}{2}(e + 5) + \frac{3}{4}(3 - e) = 5$

C

55. $6(1 - c) - 5(c + 1) + 2[(5 - 3c) + 3] = -34$
56. $3(x - 1) - 2(1 - 3x) - 3[(6 + x) - 1] = -15$
57. $3[5(4j - 3) - (5 - 8j)] + 2[2(5 - j) + 7] = -6$
58. $4[2(q - 10) + 3q - 12] - 3[3(q - 5) - (4 - q)] = 17$
59. $\frac{1}{3}(1 + p) - \frac{2}{3}[(2p + 7) - 3(p - 1)] = -7$
60. $\frac{1}{2}[5 - 2(x - 3)] + \frac{1}{4}[(2 - x) - (3 - 2x)] = 5$

Computer Exercises For students with computer experience

Write a program that will solve an equation of the form $ax + b = c$ when you input values for a, b, and c. What value of a must be excluded? RUN the program to solve each of the following equations.

1. $4x + 9 = 1$
2. $7y - 8 = 20$
3. $\frac{3}{4}z + 2 = -7$
4. $\frac{w}{2} - 6 = 15$
5. $m - 19 = 7$
6. $28 - n = -6$
7. $-3 = -5p + 2$
8. $-6q = 18$

Modify the program that you wrote for Exercises 1–8 to solve an equation of the form $a|x| + b = c$. RUN the program to solve each of the following.

9. $|x| - 5 = 9$
10. $|y| + 8 = 6$
11. $3|s| - 5 = 16$
12. $\frac{|t|}{2} = -7$

Self-Test 1

VOCABULARY

addition property of equality (p. 114)
multiplication property of equality (p. 114)
subtraction property of equality (p. 114)
division property of equality (p. 115)
equivalent equations (p. 118)
transform an equation (p. 118)
transformation by substitution (p. 119)
transformation by addition (p. 119)
transformation by subtraction (p. 119)
transformation by multiplication (p. 122)
transformation by division (p. 122)
inverse operations (p. 124)

Give the reason that justifies each statement, if x is a real number.

Obj. 1, p. 113

1. If $3x + 5 = 7$, then $(3x + 5) + (-5) = 7 + (-5)$.
2. If $\frac{x}{-2} = -8$, then $-2\left(\frac{x}{-2}\right) = -2(-8)$.

Solve.

Obj. 2, p. 113

3. $-17 = 9 + a$
4. $b - |-4| = 12$
5. $-\frac{2}{5}c = 30$
6. $-54 = -6|d|$

Obj. 3, p. 113

7. $2j - 6 = 4$
8. $-\frac{1}{3}k + 9 = 5$
9. $3(s - 4) = -15$
10. $1 = 9t - (2t - 8)$

Check your answers with those at the back of the book.

Equations and Problem Solving

OBJECTIVES for Sections 3-5 through 3-8:

1. *To use equations to solve word problems.*
2. *To solve equations having the variable in both sides.*
3. *To arrange the facts of a word problem in a chart.*
4. *To transform formulas so that one variable is expressed in terms of the other variables.*

3–5 Using Equations to Solve Problems

Often the information that you are given in a word problem can be translated into relationships among numbers. If you can represent these relationships by an equation, then you can solve the word problem by solving the equation.

EXAMPLE 1 In planning his campaign for class president, Sean has ordered three types of prints of his photograph: regular prints, wallet-size prints, and enlargements. The number of regular prints he has ordered is ten more than twice the number of enlargements. The number of wallet-size prints equals the number of the other two types combined. Sean has ordered 200 prints altogether. How many of each type of print has he ordered?

SOLUTION

Step 1 Read the problem carefully a few times. What numbers are asked for? What information is given?

The problem asks for the number of each type of print.

There are ten more regular prints than twice the number of enlargements.

There are as many wallet-size prints as there are regular prints and enlargements together.

There are 200 prints altogether.

Step 2 Choose a variable. Use it with the given facts to represent the numbers described in the problem.

Let e = the number of enlargements. Then:

Number of regular prints $= 2e + 10$

Number of wallet-size prints $= e + (2e + 10) = 3e + 10$

Step 3 Write an equation that represents the relationships among the numbers in the problem.

$$\underbrace{\text{The total number of prints}}\ \ \text{is}\ \ 200.$$
$$\downarrow \qquad \downarrow$$
$$e + (2e + 10) + (3e + 10) = 200$$

Step 4 Solve the equation. Then find the required numbers.

$$e + (2e + 10) + (3e + 10) = 200$$
$$6e + 20 = 200$$
$$6e = 180$$
$$e = 30$$

Number of enlargements $= e = 30$

Number of regular prints $= 2e + 10 = 2(30) + 10 = 70$

Number of wallet-size prints $= 3e + 10 = 3(30) + 10 = 100$

Step 5 Check your results with the words of the problem. Give the answer.

Are there 200 prints altogether?

$$30 + 70 + 100 \stackrel{?}{=} 200$$
$$200 = 200 \quad \checkmark$$

$\therefore$ Sean has ordered 30 enlargements, 70 regular prints, and 100 wallet-size prints.

In solving some problems, drawing a sketch may help you to identify the relationships among the numbers.

EXAMPLE 2 The length of a rectangular field is 10 m greater than three times its width. A roll of fencing that is 180 m long and 1 m high will enclose the field with no fencing left over. What are the dimensions of the field?

SOLUTION

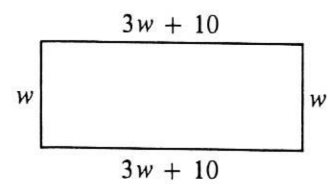
Step 1 The problem asks for the number of meters in the length and width of a rectangle.

The length of the rectangle is 10 m greater than three times the width.

The perimeter of the rectangle is 180 m.

Step 2 Let w = the width in meters.

Then $3w + 10$ = the length in meters.



Step 3 $w + (3w + 10) + w + (3w + 10) = 180$

Step 4

$$8w + 20 = 180$$
$$8w = 160$$
$$w = 20$$

Width $= 20$ m

Length $= 3(20) + 10 = 70$ m

Step 5 Is the perimeter 180 m?

$$20 + 70 + 20 + 70 \stackrel{?}{=} 180$$
$$180 = 180 \quad \checkmark$$

$\therefore$ the field is 20 m wide and 70 m long.

Notice that in Example 2 you did not use the fact that the height of the fencing is 1 m. Sometimes a word problem contains unnecessary information, and you must select only the facts needed for the solution.

The five steps used to solve the problems in Examples 1 and 2 form a plan that often helps in solving word problems.

> ***Plan for Solving a Word Problem***
>
> Step 1 Read the problem carefully a few times. Decide what numbers are asked for and what information is given. Making a sketch may be helpful.
>
> Step 2 Choose a variable and use it with the given facts to represent the number(s) described in the problem.
>
> Step 3 Reread the problem. Then write an open sentence that represents the relationships among the numbers in the problem.
>
> Step 4 Solve the open sentence and find the required numbers.
>
> Step 5 Check your results with the words of the problem. Give the answer.

Problems

Solve.

A 1. One number is six greater than another. If the sum of the two numbers is 44, find the numbers.

2. Angela has twelve fewer records than Craig. Together they have 86 records. How many records does Angela have?

3. It took Lo Kwan 10 min more to walk home from school than it took him to walk to school from home. If his total walking time from home to school and back again was 54 min, how long did it take him to walk to school?

4. A shirt cost $2.56 less than a pair of shorts. Together the two items cost $22.38. How much did the shirt cost?

5. Darcy's walking stick is three fourths as long as Damon's. Placed end-to-end, the two sticks measure 2.8 m. How long is each stick?

6. The number of boys in the Sharon Sports Club is four fifths the number of girls. If there are 360 boys and girls altogether in the club, how many girls are there?

7. The length of a certain rectangle is 4 m greater than five times its width. Find the dimensions of the rectangle if its perimeter is 20 m.

8. Tanya's age is ten years greater than half Aaron's age. If the sum of their ages is 55, how old is Tanya?

9. A certain molecule contains twice as many atoms of hydrogen as oxygen and one more atom of carbon than hydrogen. If there are 21 atoms altogether in the molecule, how many atoms of carbon are there?

10. Last month Amy sold three times as many solar calculators as printing calculators and five fewer scientific calculators than solar calculators. If Amy sold 58 calculators altogether, how many of each type did she sell?

11. One number is 35 greater than a second number. If the lesser number is subtracted from twice the greater number, the difference is 87. Find the numbers.

12. The difference of two numbers is 33. If four times the lesser number is subtracted from three times the greater number, the difference is 62. Find the numbers.

B 13. Wade has three more quarters than dimes. The total value of the dimes and quarters together is $13. How many quarters does he have?

14. Lee withdrew $200 from his account. He asked the bank clerk to give the money to him only in $5 and $10 bills. If he received two more $10 bills than $5 bills, how many bills did he receive in all?

15. The cost of an Aardvark personal computer is $40 less than twice the cost of an Afghan computer, while an Armadillo computer costs $70 more than an Afghan. If an order of two Afghans, one Aardvark, and three Armadillos totals $6820, what is the cost of each type of computer?

16. In a mail-order catalogue, a cassette recorder costs $10 less than a clock radio, and a portable stereo costs twice as much as a clock radio. Three cassette recorders and two portable stereos cost $250 in all. What is the cost of a clock radio?

17. There are eight fewer students in the Debating Club than on the Math Team, and there are only half as many students on the Ecology Force as in the Debating Club. How many students are in each group if there are 68 students in all and each student participates in only one of the activities?

18. The length of a certain rectangle is 3 cm less than twice its width. If each dimension were reduced by 5 cm, the resulting figure would have a perimeter of 16 cm. What are the dimensions of the original rectangle?

19. A collection of nickels and quarters has a total value of $6.40. If there are forty coins in the collection, how many are there of each kind?

20. Jason babysat for sixty hours during July. He charged $1.50 for each hour that he worked before midnight and $3.00 for each hour after midnight. If he earned $112.50 in all, how many hours did he work after midnight?

21. The length of a certain rectangle is 5 m less than twice its width. If the perimeter of the rectangle is 68 m, find its area.

22. In 1985 Maria was three times as old as Sue. Five years earlier, the sum of their ages was eighteen. In what year was Maria born?

C 23. The length of a rectangular field is 10 m less than four times its width. It cost \$1584 to put a fence around the field. If the fencing cost \$4.80 per meter, what is the area of the field?

24. At the time when Jay's son was born, Jay's age was twenty years less than twice his wife's age, and the sum of their ages was 64. If Jay was born in 1948, how old will his son be in 2001?

25. In Pat's Pet Palace there are two boxer pups, three Dalmatians, one Airedale, and one miniature schnauzer. The boxers are each priced \$20 less than a Dalmatian, the price of the Airedale is \$150 less than two times the price of a boxer, and the price of the miniature schnauzer is four fifths the price of the Airedale. What is the price of each type of dog if the total cost of all the dogs in the store is \$1510?

26. The three sides of a triangle are labeled r, s, and t. The length of side s is 8 m longer than two thirds the length of side r. The length of side t is 2 m shorter than one fourth the sum of the lengths of the other two sides. If the perimeter of the triangle is 33 m, determine the length of each of the three sides.

3–6 Equations Having the Variable in Both Sides

Note that the variable n appears in both sides of the equation

$$6n = 2n - 56.$$

Are you permitted to transform this equation by subtracting $2n$ from each side? The answer is yes. Because a variable represents a real number, a variable expression such as $2n$ also represents a real number. Thus the addition and subtraction properties of equality permit you to transform an equation by adding a variable expression to each side or subtracting a variable expression from each side.

EXAMPLE 1 Solve $6n = 2n - 56$.

SOLUTION

$$\begin{aligned} 6n &= 2n - 56 \\ 6n - 2n &= 2n - 56 - 2n \\ 4n &= -56 \\ \frac{4n}{4} &= \frac{-56}{4} \\ n &= -14 \end{aligned}$$

CONDENSED SOLUTION

$$6n = 2n - 56$$
$$4n = -56$$
$$n = -14$$

Check:

$$6n = 2n - 56$$
$$6(-14) \stackrel{?}{=} 2(-14) - 56$$
$$-84 \stackrel{?}{=} -28 - 56$$
$$-84 = -84 \quad \checkmark$$

$\therefore$ the solution set is $\{-14\}$.

When solving equations, be aware of the fact that not all equations have exactly one root. It is possible that an equation is true for *no* values of the variable. It is also possible that an equation is true for *all* values of the variable. Consider the following two examples.

EXAMPLE 2 Solve $3(c + 1) - c = 21 + 2(1 + c)$.

SOLUTION

$$3(c + 1) - c = 21 + 2(1 + c)$$
$$3c + 3 - c = 21 + 2 + 2c$$
$$2c + 3 = 23 + 2c$$
$$3 = 23$$

Since the given equation is equivalent to the false statement $3 = 23$, the equation has no root. $\therefore$ the solution set is $\emptyset$.

EXAMPLE 3 Solve $3(d + 1) + 8 = 2(d + 3) + d + 5$.

SOLUTION

$$3(d + 1) + 8 = 2(d + 3) + d + 5$$
$$3d + 3 + 8 = 2d + 6 + d + 5$$
$$3d + 11 = 3d + 11$$
$$11 = 11$$

Since the given equation is equivalent to the true statement $11 = 11$, the equation is satisfied by every real number. $\therefore$ the solution set is $\mathcal{R}$.

An equation that is true for all values of the variable(s) is called an **identity**. Thus, in Example 3, $3(d + 1) + 8 = 2(d + 3) + d + 5$ is an identity.

Oral Exercises

Match each equation at the left with its solution set at the right.

1. $6 - x = x$ **a.** $\mathcal{R}$

2. $4x + 3 = 4x - 2$ **b.** $\emptyset$

3. $-10x + 2 = 7 - 9x$ **c.** $\{0\}$

4. $3 - 3x = 4 - 4x$ **d.** $\{1\}$

5. $3 - 8x = 8x + 3$ **e.** $\{3\}$

6. $2x - 5 = -x - (5 - 3x)$ **f.** $\{-5\}$

Tell which of the equations (a), (b), (c), or (d) are equivalent to the given equation. For each equivalent equation, tell what transformation was used to obtain it from the given equation.

7. Given: $4(3a - 8) = 8(a - 3) - 1$
   a. $12a - 32 = 8a - 24 - 1$
   b. $3a - 8 = 2(a - 3) - 1$
   c. $3a - 8 = 2(a - 3) - \frac{1}{4}$
   d. $2(3a - 8) = 4(a - 3) - \frac{1}{2}$

8. Given: $8(x + 1) = 4(x - 2) + 2$
   a. $8x + 8 = 4x - 8 + 2$
   b. $4(x + 1) = 2(x - 2) + 1$
   c. $2(x + 1) = (x - 2) + 2$
   d. $x + 1 = \frac{1}{2}(x - 2) + \frac{1}{4}$

9. Given: $\frac{1}{2}(6a - 4) = \frac{3}{2}(2a - 5) + 6$
   a. $6a - 4 = 3(2a - 5) + 6$
   b. $3a - 2 = 3a - \frac{15}{2} + 6$
   c. $6a - 4 = 3(2a - 5) + 12$
   d. $\frac{1}{3}(6a - 4) = (2a - 5) + 4$

10. Given: $-\frac{2}{3}(6x - 1) = 3(x - 2) + 1$
   a. $6x - 1 = -\frac{9}{2}(x - 2) - \frac{3}{2}$
   b. $-4x + \frac{2}{3} = 3x - 6 + 1$
   c. $2(6x - 1) = -9(x - 2) - 3$
   d. $-2(6x - 1) = 9(x - 2) + 3$

Written Exercises

Solve.

A

1. $10a = 56 + 2a$
2. $12m = 9m - 6$
3. $b = 55 - 4b$
4. $11n - 8 = 3n$
5. $-13q = 4 - 13q$
6. $2 - 3p = 7p$
7. $10g - 7 = 3g - 70$
8. $5e - 24 = 8e - 24$
9. $23t - 10 = 8 - 2t$
10. $63x - 9 = 16 + 63x$
11. $6 + 7i = 7i + 6$
12. $70 - 2y = 6 - 10y$
13. $w + 10 = 10 - 3w$
14. $2x + (-1) = -1 + 2x$
15. $19z - 10 + 4z = 3z - 4$
16. $5n - 4 + 2n = 2 - 8n$
17. $4(2b - 1) = 5(b + 1)$
18. $-2(x - 5) = 3(3 - 5x)$
19. $\frac{2}{3}(t - 6) = \frac{3}{4}(2 - 3t)$
20. $\frac{1}{5}(a - 4) = \frac{2}{5}(3 - 2a)$
21. $10(0.1d - 0.3) = 0.2(d + 21)$
22. $0.8(1.25t - 5) = 0.1(4 - t)$
23. $\frac{1}{5}(3s - 5) = 2s + 3$
24. $2 - 3t = -\frac{1}{3}(5 - t) + 1$
25. $7(2p - 1) = 10 - 3p$
26. $-2a - 4 = 6(1 - a) - 1$
27. $5(x + 3) = -\frac{1}{2}(x - 1)$
28. $-\frac{2}{3}(3h - 1) = -3(1 - h)$
29. $4(5h - 9) - 4(1 - h) = 9(h - 11) - 1$
30. $7(v + 1) - 4(5 - 2v) = -5(v - 9)$

Solve each equation using two different methods. The first steps of the methods should be different from each other.

B 31. $\frac{1}{2}(x + 3) = \frac{1}{2}(5x - 4)$

32. $\frac{1}{4}(3x - 2) = \frac{1}{4}(8x + 1)$

33. $\frac{1}{3}(6t + 5) = \frac{1}{4}(2t + 10)$

34. $\frac{1}{5}(3b - 4) = \frac{1}{4}(b - 2)$

35. $6(c - 4) = -6(3c + 2)$

36. $-4(2 + t) = 8(3 - t)$

37. $9(2d - 5) = -6(5d - 4)$

38. $-10(4 - 7h) = -15(2 - 3h)$

39. $0.1(25t - 6) = 0.4(6t + 7)$

40. $-0.04(5b - 8) = 0.08(10b - 1)$

41. $\frac{1}{3}(2x - 4) + 5 = -\frac{2}{3}(x + 1)$

42. $\frac{3}{5}(3a - 10) = -\frac{1}{5}(2a - 9)$

Solve.

43. $10(y - 1) - 5 = -20(3 + y)$

44. $-6(2f + 1) - 3 = 7(f - 4)$

45. $12\left(\frac{1}{3}x - \frac{1}{2}\right) = 8\left(\frac{1}{2}x - 1\right)$

46. $\frac{1}{5}\left(\frac{1}{2}t + 1\right) = \frac{1}{3}\left(\frac{3}{5}t - 3\right)$

47. $\frac{1}{5}(5 - h) - \frac{1}{2}(3 - 2h) = \frac{1}{2}h + 1$

48. $\frac{3}{4}(x - 1) - \frac{1}{2} = 2(1 - 3x)$

49. $0.3(2.5t - 0.5) = 0.6(1.5t + 0.3)$

50. $\frac{1}{2}(3.6p - 2) = 3(4.1 - 3p) - 2.5$

C 51. $5 + 6[5(3i + 20) + 2(5i + 11)] = 5 - 2[4(i + 2) + 3(7i + 42)]$

52. $3[2(j - 2) - 7(2j - 3)] - 5[4(2j - 5) - (3j - 2)] = 6(j + 1) + 1$

Replace each _?_ with a numerical or a variable expression so that the following conditions are satisfied.

a. The solution set is {2}.
b. The solution set is {0}.
c. The solution set is $\mathcal{R}$.
d. The solution set is Ø.
e. The equation is an identity.

53. $2p + 3 =$ _?_

54. $4 - (t - 2) =$ _?_

55. $2k +$ _?_ $= 5k + 1$

56. $-7x +$ _?_ $= 3x - 7$

Computer Exercises For students with computer experience

Write a program to solve an equation of the form $ax + b = cx + d$ when you input values for a, b, c, and d. Be sure that the program works correctly for an equation whose solution is the empty set and for an identity. RUN the program to solve each of the following equations.

1. $3x + 5 = 7x + 1$
2. $2y - 9 = 6 - 3y$
3. $10 - z = z + 4$
4. $\frac{v}{2} + 12 = 8 + \frac{v}{4}$
5. $\frac{m}{5} - 6 = 6 + \frac{1}{5}m$
6. $-2n + 1 = 1 - 2n$

PROGRAMMING IN BASIC

The programs in this section use the asterisk, *, to produce geometric designs with a computer. For example, when you RUN the following program, you obtain the rectangular design shown at the right.

```
10  PRINT "TO CREATE GEOMETRIC"
20  PRINT "DESIGNS WITH ASTERISKS:"
30  PRINT
40  FOR I = 1 TO 10
50  FOR J = 1 TO 10
60  PRINT "*";
70  NEXT J
80  PRINT
90  NEXT I
100  END
```

```
RUN
TO CREATE GEOMETRIC
DESIGNS WITH ASTERISKS:

**********
**********
**********
**********
**********
**********
**********
**********
**********
**********
```

Notice that this program contains *nested loops.* That is, the J-loop is said to be *nested* entirely within the I-loop.

Exercises

1. Type in and RUN the program as given.
2. What is the purpose of line 80?
3. Change line 50 as follows: 50 FOR J = 1 TO I
   LIST and RUN the revised program.
4. Change line 50 as follows: 50 FOR J = 1 TO 11 − I
   LIST and RUN the revised program.
5. Change line 50 as follows: 50 FOR J = 1 TO I
   Then insert these lines:
   ```
   42  FOR J = 1 TO 11 - I
   44  PRINT " ";
   46  NEXT J
   ```
   LIST and RUN the revised program. If possible, PRINT a LISTing of the program and mark the loops.
6. Change line 50 as follows: 50 FOR J = 1 TO 2 * I − 1
   LIST and RUN the revised program.
7. Change lines 40, 42, and 50 as follows:
   ```
   40  FOR I = -9 TO 9
   42  FOR J = 1 TO ABS(I) + 1
   50  FOR J = 1 TO 18 - 2 * ABS(I) + 1
   ```
   LIST and RUN the revised program.

If possible, save this program for later use.

3–7 Using Charts in Solving Problems

Using a chart to organize the facts of a word problem can be a helpful strategy in solving the problem.

EXAMPLE Kathy is six years older than her cat. Next year she will be twice as old as her cat will be. How old is Kathy now?

SOLUTION

Step 1 The problem asks for Kathy's age now.

Step 2 Let x = the cat's age now. Make a chart of the given facts.

	Age now	Age next year
Cat	x	$x + 1$
Kathy	$x + 6$	$(x + 6) + 1$

Step 3 The only fact not recorded on the chart is that next year Kathy will be *twice as old as* her cat will be. Use this fact to write an equation:

$$(x + 6) + 1 = 2(x + 1)$$

Step 4

$$\begin{aligned}(x + 6) + 1 &= 2(x + 1)\\ x + 7 &= 2x + 2\\ 7 &= x + 2\\ 5 &= x\end{aligned}$$

$x = 5$ (cat's age)

$x + 6 = 5 + 6 = 11$ (Kathy's age)

Step 5 Checking the results is left to you.

$\therefore$ Kathy is 11 years old.

Problems

Solve. You may find a chart helpful.

A

1. When twelve is added to a certain number, the result is the number's additive inverse. What is the number?
2. Find the number that is sixteen less than its additive inverse.
3. Five more than a certain number is six less than twice the number. Find the number.
4. Three times a certain number is 32 less than five times the number. Find the number.

Solve.

5. The sum of two numbers is 16. The greater of the two numbers is one more than four times the lesser number. What are the numbers?
6. When two numbers are added together, the result is 45. Twice the greater number is six more than five times the lesser number. What are the numbers?
7. Michelle is eight years older than Adam. Three years ago Michelle was twice as old as Adam was then. How old is Michelle?
8. Maurice is now twice as old as Roberto. Maurice's present age is ten years greater than Roberto's age one year ago. How old is Maurice?
9. In 1985, Barry was 13 years old and his father was 43. In what year will Barry's age be two fifths his father's age?
10. Sheila's age is two years more than twice Nicole's age. The sum of their ages is the same as Steve's age. If Steve's age is ten years less than five times Nicole's age, find the age of each.
11. The Rhine River is 160 km shorter than the St. Lawrence River. The Amazon River is 800 km longer than five times the length of the Rhine River. The Zambezi River is twice as long as the St. Lawrence River. How long is the Amazon if it is 1440 km longer than the sum of the lengths of the other three rivers?
12. The population of Concord, New Hampshire in 1980 was 400 greater than half the population of Euclid, Ohio. The sum of the two populations was 29,600 less than twice the population of Euclid. What was the population of each city in 1980?
13. The Panama Canal is 2 km shorter than twice the length of the Suez Canal. The sum of the lengths of the two canals is 121 km greater than three fourths the length of the Panama Canal. Find the length of each canal.
14. The width of a certain rectangle is 2 m greater than half its length. Four times its length is 26 m greater than its perimeter. What are the dimensions of the rectangle?

B 15. Last year Kristen read one fourth as many biographies as mysteries and eighteen more science fiction books than biographies. The number of nature books that she read was one less than the difference between the number of mysteries and the number of biographies. If Kristen read four times as many science fiction books as nature books, how many books did she read in all?

16. Taurus is the zodiac constellation with the greatest number of stars visible to the naked eye, while Aries has the least number of visible stars. Twenty more stars are visible in Cancer than in Aries. Sagittarius has twice as many visible stars as Cancer, and Taurus has twelve more visible stars than Sagittarius. If the number of visible stars in Aries is ten greater than one third the number of visible stars in Taurus, how many stars are visible in Cancer?

17. The length of a certain rectangle is 20 m greater than its width. If the width were reduced by 20 m and the length increased by 100 m, the perimeter of the new rectangle would be twice the perimeter of the original rectangle. What are the dimensions of the original rectangle?

18. The length of a certain rectangle is 10 m greater than twice its width. If the length were doubled and the width were halved, the perimeter would be increased by 80 m. Find the original dimensions of the rectangle.

19. Penny collected some change in preparation for a garage sale. She collected two more nickels than twice the number of dimes and eight fewer quarters than twice the number of nickels. If the value of the quarters was $1.60 greater than four times the value of the nickels and dimes together, what was the total value of the change that Penny collected?

20. Jani earns $15 per hour as a music teacher and $11 per hour as a swimming instructor. During the summer she worked part-time at each job. In one week she worked a total of thirty-seven hours and earned $459. How many hours did she work at each job that week?

C 21. The East Street School opened fifteen years after the Brown School opened. In 1978, the Brown School had been open twice as long as the East Street School. In what year will the East Street School celebrate its fiftieth anniversary?

22. Dennis earns $7.25 per hour and Tony earns $8.50 per hour. In the first week of July, they worked a combined total of seventy-five hours and earned a total of $585. Who earned more money that week? How much more did he earn?

23. In 1982 Tim was twice as old as Sue. In 1977 the sum of their ages was 32.

a. In what year was Tim born?

b. In what year will Sue's age be three fourths Tim's age?

24. Ken and Roberta each sold fifty tickets for the school play. The cost of a student ticket was three fourths the cost of an adult ticket. Roberta sold ten more student tickets than adult tickets and collected a total of $212.50.

a. How many student tickets did Roberta sell?

b. What was the cost of a student ticket?

c. If Ken collected $200 in all, how many student tickets did he sell?

Careers

Accounting

Accountants prepare and analyze the financial records of individuals, businesses, and government. Because financial transactions can be so complex and diverse, many accountants choose to specialize in a particular area such as budget, tax, cost, or auditing.

For example, some accountants prepare detailed cost studies or report on the profits, losses, and inventory of a company. Other accountants work for the government and audit financial records in order to verify compliance with government regulations, while still others make credit approval decisions for banks.

EXAMPLE One of the major expenses of a business is the cost of purchasing and maintaining equipment. Each year, equipment decreases in value, or *depreciates*, as it becomes older and technically inferior to more recently produced models.

The Richland Supply Company purchases a cash register for \$920. The company also pays a 5% sales tax and a \$25 delivery fee. The estimated useful life of the cash register is four years. At the end of the four years, the cash register is valued at \$200 and is considered to be fully (100%) depreciated. Find the amount of the *annual* depreciation of the cash register.

SOLUTION One formula that can be used to find the annual depreciation, d, is

$$d = (c - s)r,$$

where c is the original cost of the item, s is the value of the item at the end of its useful life, and r is the percent of depreciation that occurs *in one year*.

Use the given information to determine the values of c, s, and r.

$$c = \text{price} + \text{tax} + \text{delivery fee} = \$920 + 0.05(\$920) + \$25 = \$991$$

$$s = \$200$$

$$r = \frac{\text{total depreciation}}{\text{number of years}} = \frac{100\%}{4} = 25\% = 0.25$$

Thus, $d = (c - s)r = (\$991 - \$200)0.25 = \$197.75$

$\therefore$ the annual depreciation of the cash register is \$197.75.

READING ALGEBRA Problem Solving

Reading word problems requires a different type of strategy than other kinds of reading. When you read a word problem, read it once all the way through to find out what it is about. Then read it a second time, more slowly and carefully, concentrating on the major facts that are given and on the question or questions that you are asked. If any words are not familiar to you, look them up in a dictionary or, if they are mathematical terms, look them up in the glossary at the back of the book.

After you have read the problem carefully, try to answer any questions that are asked. If you write an equation, check to be sure that it represents *all* the conditions of the problem. Then check your answers with those printed at the back of the book, or check with your teacher. If your answer is wrong, try reading the problem again to see if you can discover the source of your error. Remember that good problem solvers learn from their mistakes as well as from their successes.

Exercises

The exercises below refer to the following word problem.

A collection of nickels, dimes, and quarters contains three times as many nickels as dimes and four more quarters than nickels. The total value of the collection is $9. How many quarters are there?

Let *d* represent the number of dimes in the collection. Write an expression for each of the following in terms of *d*.

1. the number of nickels
2. the number of quarters
3. the total number of coins in the collection
4. the total value in cents of the coins in the collection

Let *d* represent the number of dimes in the collection. Tell whether or not each equation represents the conditions of the problem.

5. $d + 3d + (3d + 4) = 900$
6. $10d + 5(3d) + 25(3d + 4) = 9$
7. $10d + 5(3d) + 25(3d + 4) = 900$
8. $0.1d + 0.5(3d) + 0.25(3d + 4) = 9$
9. Tell why each of the following is *not* an answer to the question.
   a. 8 **b.** $7.00
10. What is the correct answer to the question?

3–8 Transforming Formulas

A **formula** is an equation that states a relationship among quantities represented by variables. For example, the following are some formulas that you may recognize.

$A = s^2$	Area of a square = square of the length of a side
$D = rt$	Distance traveled = rate × time traveled
$I = prt$	Simple interest = principal × rate of interest × time

When you are working with a formula, it is often helpful to *solve for* one of the variables. To solve for a given variable, you transform the formula until you obtain that variable alone as one side of the equation. This variable is then said to be *expressed in terms of* the other variables.

EXAMPLE The formula

$$x = \tfrac{1}{2}at^2$$

gives the distance x in meters that is traveled in t seconds by a freely falling object near the surface of a planet where the acceleration due to gravity is a m/s^2 (meters per second squared).

a. Express a in terms of x and t.

b. If it takes 10 s for a heat shield to fall 445 m to the surface of Venus, what is the acceleration due to gravity on Venus?

SOLUTION **a.** $x = \frac{1}{2}at^2$

$$2x = at^2$$

$$\frac{2x}{t^2} = a$$

$$\therefore a = \frac{2x}{t^2}, \text{ provided } t \neq 0$$

b. Substitute the values $x = 445$ and $t = 10$ into the formula as it was transformed in part (a).

$$a = \frac{2x}{t^2}$$

$$a = \frac{2(445)}{10^2} = \frac{890}{100} = 8.9$$

∴ the acceleration due to gravity on Venus is 8.9 m/s^2.

Notice that the formula obtained for the acceleration a in part (a) of the Example is not valid when $t = 0$ because division by zero has no meaning.

Oral Exercises

In these exercises, assume that the variables represent real numbers that do not result in division by zero.

Tell what steps are needed to transform the first formula into the second.

1. a. $2rs = t$
   $r = \frac{t}{2s}$

   b. $2rs = t$
   $s = \frac{t}{2r}$

   c. $2rs = t$
   $t = 2rs$

2. a. $4xyz = -t$
   $x = \frac{-t}{4yz}$

   b. $4xyz = -t$
   $y = \frac{-t}{4xz}$

   c. $4xyz = -t$
   $t = -4xyz$

3. a. $A = 2(B + C)$
   $A = 2B + 2C$

   b. $A = 2(B + C)$
   $B = \frac{A - 2C}{2}$

   c. $A = 2(B + C)$
   $C = \frac{A - 2B}{2}$

4. a. $P = 3(x - y)$
   $P = 3x - 3y$

   b. $P = 3(x - y)$
   $x = \frac{P + 3y}{3}$

   c. $P = 3(x - y)$
   $y = \frac{P - 3x}{-3}$

5. a. $h = w(x - y)$
   $w = \frac{h}{x - y}$

   b. $h = w(x - y)$
   $h = wx - wy$

   c. $h = w(x - y)$
   $y = \frac{h - wx}{-w}$

6. a. $d = -t(r - s)$
   $t = \frac{-d}{r - s}$

   b. $d = -t(r - s)$
   $d = -tr + ts$

   c. $d = -t(r - s)$
   $r = \frac{d - ts}{-t}$

Each of the following equations is solved for the variable in color. Name the property of the real numbers that justifies each lettered step.

7. $3P + 2Prt = 6$
   $P(3 + 2rt) = 6$ **(a)**
   $P = \frac{6}{3 + 2rt}$ **(b)**

8. $6w - 5wx = -3$
   $w(6 - 5x) = -3$ **(a)**
   $w = \frac{-3}{6 - 5x}$ **(b)**

9. $2t^2 = \frac{r}{s}$
   $2st^2 = r$ **(a)**
   $s = \frac{r}{2t^2}$ **(b)**

10. $5a = \frac{3h^2}{-k}$
    $-5ak = 3h^2$ **(a)**
    $k = \frac{3h^2}{-5a}$ **(b)**

11. $I = \frac{E}{A + B}$
    $I(A + B) = E$ **(a)**
    $IA + IB = E$ **(b)**
    $IB = E - IA$ **(c)**
    $B = \frac{E - IA}{I}$ **(d)**

12. $T = \frac{4 + b}{2 - a}$
    $(2 - a)T = 4 + b$ **(a)**
    $2T - aT = 4 + b$ **(b)**
    $-aT = 4 + b - 2T$ **(c)**
    $a = \frac{4 + b - 2T}{-T}$ **(d)**

Written Exercises

In these exercises assume that the variables represent real numbers that do not result in division by zero.

Solve each equation for the variable in color.

A

1. $V = IR$
2. $-3xy = z$
3. $-4xy = 3vw$
4. $-5pq = -2st$
5. $a^2b = 3cx$
6. $-2tm = 4t^2s$
7. $-2xy = 6g^2h$
8. $E = I^2R$
9. $b - 2c = -2ax$
10. $2pq = 4r - 3t$
11. $b - 2c = -2ax$
12. $2pq = 4r - 3t$
13. $a(b - c) = -4d$
14. $s(t - v) = 2w$
15. $a(b - c) = -4d$
16. $s(t - v) = 2w$
17. $a(b - c) = -4d$
18. $s(t - v) = 2w$
19. $-3a^2(x - 3) = -4c$
20. $st^2(p - 2q) = 5$
21. $A = \frac{1}{2}bh$
22. $T = -\frac{3}{4}ws$
23. $E = \frac{1}{2}mc^2$
24. $V = \frac{22}{7}r^2h$
25. $\frac{t + c}{3} = n$
26. $\frac{t - a + b}{2} = 3c$
27. $-1.5st^2 = 0.09k$
28. $0.6x^2y = -0.02z$
29. $A = \frac{1}{2}h(a + b)$
30. $V = \frac{1}{3}a(x + y)$
31. $A = wt + \frac{1}{2}at^2$
32. $T = -3p^2r + \frac{1}{2}p$
33. $C = \frac{2}{3}(a + b)$
34. $t = \frac{3}{2}(x - y)$
35. $A = \frac{5}{2}h(3x - y)$
36. $V = \frac{3}{4}t(2a - 3b)$

B

37. $A = P + Prt$
38. $T = 2A - 5AB$
39. $C = \frac{kA}{4\pi d}$
40. $F = \frac{km_1m_2}{d_1d_2}$
41. $C = \frac{c_1 - c_2}{2t}$
42. $a = \frac{w - v}{t}$
43. $C = \frac{c_1 - c_2}{2t}$
44. $a = \frac{w - v}{t}$
45. $I = \frac{E}{R + r}$
46. $\frac{y_1 - y_2}{x_1 - x_2} = m$
47. $I = \frac{E}{R + r}$
48. $\frac{y_1 - y_2}{x_1 - x_2} = m$
49. $at + 3 = bt - 4b$
50. $2rs - 4r = 4s - 5$
51. $at + 3 = bt - 4b$
52. $2rs - 4r = 4s - 5$
53. $F = \frac{f_1f_2}{f_1 + f_2}$
54. $T = \frac{d_1d_2}{d_1 - d_2}$

Solve each equation for the value of t when $r = -\frac{2}{3}$ and $s = \frac{3}{4}$.

C

55. $2rs = \frac{t + r}{t - r}$
56. $\frac{-2t + 3r}{t - r} = -3s$
57. $r(t - s) - 2(3t - s) = -5(t + 2s) + rs$
58. $-\frac{1}{2}r(s - t) + \frac{1}{3}s(r - t) = \frac{1}{2}rs$

Problems

Solve.

A

1. The formula $V = lwh$ gives the volume V of a rectangular box in terms of the length l, width w, and height h.
   a. Express h in terms of V, l, and w.
   b. If a rectangular box with length 11 cm and width 5 cm has a volume of 770 cm^3, what is the height of the box?

2. The formula $C = 2\pi r$ gives the circumference C of a circle in terms of the radius r.
   a. Express r in terms of C.
   b. If the circumference of a circle is 157 m, find the radius. Use $\pi = 3.14$.

3. The formula $SA = 2\pi r^2 + 2\pi rh$ gives the surface area SA of a cylinder in terms of the height h and the radius of the base r.
   a. Express h in terms of the other variables.
   b. Find the height of a cylinder whose base has a radius of 14 cm if its surface area is 2200 cm^2. Use $\pi = \frac{22}{7}$.

4. The formula $C = \frac{5}{9}(F - 32)$ gives the temperature C in degrees Celsius in terms of the temperature F in degrees Fahrenheit.
   a. Express F in terms of C.
   b. What Fahrenheit temperature would be equivalent to a temperature of 30 degrees Celsius?

B

5. The formula $V = \frac{1}{3}\,lwh$ gives the volume V of a pyramid having a rectangular base in terms of the length of the base l, the width of the base w, and the height of the pyramid h. Find the height of a pyramid with a base of length 21 m and width 17.5 m if the volume is 2940 m^3.

6. The formula $A = \frac{1}{2}h(a + b)$ gives the area A of a trapezoid in terms of the height h and the bases a and b. Find the length of base a if the length of base b is 14 cm, the height is 26 cm, and the area is 377 cm^2.

7. The formula $V = \pi r^2 h$ gives the volume V of a cylinder in terms of the radius of the base r and the height h. Find the height of a cylindrical tank with a base of radius $3\frac{1}{2}$ ft if it takes 1848 ft^3 of water to fill it completely. Use $\pi = \frac{22}{7}$.

8. Two cylinders have the same volume. One of the cylinders has a base of radius 12 cm and a height of 25 cm, while the second cylinder has a base of radius 10 cm. What is the height of the second cylinder? (The formula for the volume of a cylinder is given in Exercise 7.)

Self-Test 2

VOCABULARY identity (p. 135) formula (p. 144)

Write an equation that represents relationships among the numbers in the problem. Then solve the equation and answer the question.

1. One number is six less than another. If the sum of the two numbers is 42, what are the numbers? *Obj. 1, p. 130*
2. Ella has five more dimes than nickels. If the total value of her dimes and nickels is $1.10, how many dimes does she have?

Solve.

3. $5x - 4 = 12 - 3x$

4. $\frac{1}{3}(y - 2) = -4$ *Obj. 2, p. 130*

Make a chart to organize the facts of the problem. Then solve the problem.

5. Ann is six years older than Mark. Five years ago she was twice as old as Mark was then. How old is Ann? *Obj. 3, p. 130*
6. The length of a certain rectangle is 3 cm greater than its width. If the width were doubled and the length were decreased by 1 cm, the new rectangle would have the same perimeter as the original. Find the dimensions of the original rectangle.

Solve for *b*.

7. $A = \frac{1}{2}ab$

8. $A = \frac{1}{2}h(a + b)$ *Obj. 4, p. 130*

Check your answers with those at the back of the book.

EXTRA

Indirect Proof

The proofs that you have studied so far in this book are each made up of a sequence of true statements that lead *directly* from the hypothesis of a theorem to its conclusion. Such proofs are called *direct proofs*. However, it is also possible to prove a theorem by a process of *indirect* reasoning. As an example of this type of reasoning, consider the following.

Suppose that a detective who is investigating a theft narrows the list of suspects to three people: Abel, Bolton, and Carter. Further investigation reveals that, at the time of the theft, Abel was out of town and Bolton was in the hospital. On the basis of these facts, the detective knows that Abel and Bolton are innocent, since neither could have been in two places at once. The detective has eliminated all possible suspects except one, and that one must be the prime suspect. Carter is arrested.

In mathematics, this method of reasoning appears in an **indirect proof.** To write an indirect proof of a theorem, you begin by assuming that the conclusion is *false*, even though the hypothesis is accepted as true. You then show that a logical chain of statements leads you to contradict an accepted fact. That fact might be the hypothesis of the theorem, or it might be an axiom, a definition, or a previously proved theorem. As a result of this contradiction, you know your assumption must be incorrect and the conclusion of the theorem must be *true.*

The following example illustrates how indirect reasoning might be used in proving a theorem for the real numbers.

EXAMPLE 1 Prove: For all real numbers a and b such that $a \neq 0$, if $ab = 0$, then $b = 0$.

SOLUTION *Plan*: Assume $b \neq 0$.

PROOF

Statements	*Reasons*
1. $a \neq 0$ $ab = 0$	Hypothesis
2. $\frac{1}{b}$ is a real number.	Axiom of multiplicative inverses
3. $(ab) \cdot \frac{1}{b} = 0 \cdot \frac{1}{b}$	Multiplication property of equality
4. $a = 0 \cdot \frac{1}{b}$	Theorem on page 94
5. $a = 0$	Multiplicative property of zero

However, statement (5) contradicts that part of the hypothesis which states $a \neq 0$. Therefore, the assumption $b \neq 0$ must be incorrect and the conclusion $b = 0$ is true.

EXAMPLE 2 Prove $0 \neq -1$.

SOLUTION *Plan*: Assume $0 = -1$.

PROOF

Statements	*Reasons*
1. $0 + 1 = -1 + 1$	Addition property of equality
2. $0 + 1 = 0$	Axiom of additive inverses
3. $1 = 0$	Identity axiom for addition

However, statement (3) contradicts that part of the identity axiom for multiplication which states $1 \neq 0$. Therefore, the assumption $0 = -1$ must be incorrect and the conclusion $0 \neq -1$ is true.

The following is a summary of the steps used in proving a theorem by the process of indirect reasoning.

To Write an Indirect Proof of a Theorem

1. Assume that the conclusion of the theorem is false.
2. Reason from this assumption until you obtain a statement that contradicts an accepted fact.
3. Point out that the assumption must be incorrect and that the conclusion of the theorem must therefore be true.

Exercises

Write an indirect proof of each theorem.

1. For all real numbers a, b, and c, if $a + c \neq b + c$, then $a \neq b$.
2. For all real numbers a, b, and c, if $a - c \neq b - c$, then $a \neq b$.
3. For all real numbers a, b, and c, if $ac \neq bc$, then $a \neq b$.
4. For all real numbers a, b, and c such that $c \neq 0$, if $a \div c \neq b \div c$, then $a \neq b$.
5. For all real numbers a and b, if $a \neq b$, then $-a \neq -b$.
6. For all real numbers a and b such that $a \neq 0$ and $b \neq 0$, if $a \neq b$, then $\frac{1}{a} \neq \frac{1}{b}$.
7. $1 \neq 2$
8. $-1 \neq -2$

Chapter Summary

1. The following *properties of equality* are theorems that can be proved to be true for all real values of each variable except as noted.

   Addition: If $a = b$, then $a + c = b + c$ and $c + a = c + b$.

   Multiplication: If $a = b$, then $ac = bc$ and $ca = cb$.

   Subtraction: If $a = b$, then $a - c = b - c$.

   Division: If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

2. Equations that have the same solution set over a given domain are called *equivalent equations* over that domain. Each of the following

transformations of a given equation will produce an equivalent equation.

1. Substituting for any expression in the given equation an equivalent expression.
2. Adding the same real number to, or subtracting the same real number from, each side of the given equation.
3. Multiplying or dividing each side of the given equation by the same *nonzero* real number.

3. Operations that "undo" each other are called *inverse operations.* Addition and subtraction are a pair of inverse operations, as are multiplication and division. Inverse operations can be used in solving equations.

4. Using the plan outlined on page 132, a word problem can often be solved by first writing an open sentence that represents relationships among the numbers in the problem and then solving the open sentence. Organizing the facts of a word problem in a chart is often helpful.

5. A *formula* is an equation that states a relationship among quantities represented by variables. Transformations are often used to *solve for* one variable in the formula. This variable is then said to be *expressed in terms of* the other variables.

Chapter Review

Write the letter of the correct answer.

1. Give the reason that justifies the following statement: 3-1
"If $r + 5 = 8$, then $(r + 5) - 5 = 8 - 5$."

a. axiom of additive inverses

b. subtraction property of equality

c. symmetric axiom of equality

d. substitution principle

Solve.

2. $c - 14 = -5$ 3-2

a. $\{9\}$ **b.** $\{-19\}$ **c.** $\{-9\}$ **d.** $\{19\}$

3. $1 = 7 + r$

a. $\{8\}$ **b.** $\{-8\}$ **c.** $\{-6\}$ **d.** $\{6\}$

4. $-x - 3 = -4$

a. $\{1\}$ **b.** $\{-1\}$ **c.** $\{-1, 1\}$ **d.** $\emptyset$

Solve.

5. $-2y = -10$ 3-3

a. $\{-5\}$ b. $\{5\}$ c. $\{-8\}$ d. $\{20\}$

6. $-72 = \frac{2}{3}u$

a. $\{-48\}$ b. $\{48\}$ c. $\{-108\}$ d. $\{108\}$

7. $-5|d| = 30$

a. $\{-6, 6\}$ b. $\{-6\}$ c. $\{6\}$ d. $\emptyset$

8. $8a + 5 = 21$ 3-4

a. $\left\{\frac{13}{4}\right\}$ b. $\{2\}$ c. $\left\{\frac{21}{40}\right\}$ d. $\{16\}$

9. $2(5 - p) - 3 = -5$

a. $\{-6\}$ b. $\{12\}$ c. $\{-1\}$ d. $\{6\}$

10. One number is five greater than another. If the sum of the lesser number and twice the greater is 61, what is the greater number? 3-5

a. 28 b. 33 c. 17 d. 22

11. Jerry has eight more nickels than quarters. The total value of his nickels and quarters together is $10. How many nickels does he have?

a. 32 b. 40 c. 20 d. 28

12. $8k + 6 = 5k$ 3-6

a. $\left\{-\frac{6}{13}\right\}$ b. $\left\{-\frac{13}{6}\right\}$ c. $\{-2\}$ d. $\{2\}$

13. $-3(t - 7) = 4(9 - 2t)$

a. $\{3\}$ b. $\{-43\}$ c. $\left\{\frac{57}{6}\right\}$ d. $\{-15\}$

14. Kevin is five years older than Lisa. Four years ago Kevin was twice as old as Lisa was then. How old is Kevin now? 3-7

a. 9 years b. 12 years c. 14 years d. 15 years

15. The length of a certain rectangle is 4 cm less than twice its width. Five times its width is 6 cm less than its perimeter. What is the length of the rectangle?

a. 14 cm b. 24 cm c. 48 cm d. 76 cm

16. Solve for h if $V = \frac{1}{3}bh$ and $V, b, h \neq 0$. 3-8

a. $h = \frac{V}{3b}$ b. $h = \frac{3V}{b}$ c. $h = \frac{b}{3V}$ d. $h = V - \frac{b}{3}$

17. Solve for h if $SA = 2\pi r(r + h)$.

a. $h = SA - 2\pi r^2$

b. $h = \frac{SA}{2\pi r^2}, r \neq 0$

c. $h = \frac{SA}{2\pi r} - r, r \neq 0$

d. $h = \frac{SA}{2\pi r^2} + r, r \neq 0$

18. Use the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature equivalent to a temperature of 25 degrees Celsius.

a. 77°F b. 13°F c. 45°F d. −13°F

Chapter Test

1. Name the property that is illustrated by the following: "If $8x = 96$, then $\frac{1}{8} \cdot 8x = \frac{1}{8} \cdot 96$." 3-1

Solve.

2. $k + 9 = -11$ 3. $5 - t = 3$ 4. $|-9| = b + 2$ 3-2

5. $-17d = 306$ 6. $-108 = -\frac{1}{9}z$ 7. $-4|j| = 48$ 3-3

8. $3n - 7 = 8$ 9. $\frac{3h + 2}{5} = -2$ 10. $7c - 3(c - 4) = 8$ 3-4

11. One number is two thirds another number. If the sum of the numbers is forty, what is the lesser number? 3-5

12. The length of a certain rectangle is 2 cm less than three times its width. If the perimeter of the rectangle is 44 cm, what is its length?

13. $6 - 4w = 13 - 3w$ 14. $8(c + 11) = 5(c + 1)$ 3-6

15. Find the number that is four greater than three times its additive inverse. 3-7

16. Jackie is five years older than her brother. In four years, Jackie will be twice as old as her brother will be. How old is Jackie now?

17. Solve for x if $S = \frac{n(a - x)}{2}$. 3-8

18. Use the formula $V = lwh$ to find the width of a rectangular box having length 12 cm, height 4 cm, and volume 384 cm^3.

Cumulative Review

Chapter 1

Tell whether each statement is true or false.

1. $\emptyset \subset$ {the natural numbers}
2. $0 \notin$ {the whole numbers}
3. {the integers} $\subset$ {the real numbers}
4. {the irrational numbers} $\subset$ {the rational numbers}

Simplify.

5. $5\frac{7}{8} + 1\frac{2}{3}$
6. $72 \div \frac{1}{3}$
7. $44 - 6(10 - 7)$
8. $\frac{36 - 16}{12 - 2}$
9. $18 + 2^3$
10. $28 - 6 \div 2 + (9 - 7)^2$

Evaluate each expression when $a = 2$, $b = 3$, and $c = \frac{1}{2}$.

11. $\frac{ab}{c}$
12. $4b - 3ac$
13. $3b - a(a + 2c)$
14. $\frac{4abc}{b + 4c}$
15. $(a^2 + b^2)c$
16. $\frac{a^2 + 2b - 6c}{a}$

Chapter 2

Simplify.

17. $-56 + (-27)$
18. $-49 - (-31)$
19. $(-23)(-11)$
20. $-36 \div (-2)$
21. $-[-(42 - 70)]$
22. $-|-24 + 5|$
23. $-5a^2 + 7 + a^2$
24. $6 - 9b^3 - 23$
25. $(-3c)(5c)(-7c)$
26. $-76xy \div 4$
27. $-3(-7u + 4v)$
28. $9m - 2(6n - 5m)$

Give the reason that justifies each statement, assuming that *m* and *n* are real numbers.

29. $n + (-n) = 0$
30. $n + 0 = n$
31. $-(m + n) = -m + -n$
32. $-(-n) = n$
33. $(-m)(-n) = mn$
34. $n \cdot \frac{1}{n} = 1, n \neq 0$

Chapter 3

Solve.

35. $-4 + p = 16$
36. $26 = q - 13$
37. $-12s = 6$
38. $\frac{t}{-2} = -14$
39. $17 - x = 28$
40. $-2 = -\frac{1}{8}|y|$

41. $9m - 2 = -20$

42. $-9 = \frac{n}{4} + 11$

43. $-15 = -3(g - 2)$

44. $\frac{h - 8}{3} = -10$

45. $-2(j + 6) = j$

46. $6k - (1 - k) = 7k - 1$

47. The length of a certain rectangle is 4 cm longer than its width. If the width were increased by 6 cm and the length were doubled, the new perimeter would be twice the perimeter of the original rectangle. Find the dimensions of the original rectangle.

48. The formula $C = \pi d$ gives the circumference of a circle C in terms of the diameter d. If the circumference of a certain circle is 176 cm, find its diameter. Use $\pi = \frac{22}{7}$.

Contest Problems

1. If $\frac{8c - 7d}{d} = 2$, find the value of $\frac{5c - 6d}{2c}$.

2. Given that $t \in \{1, 2, \ldots, 9\}$ and $u \in \{0, 1, \ldots, 9\}$, when a two-digit number of the form $10t + u$ is divided by the sum of its digits, the quotient is 5.5. What must be the relationship between t and u?

3. If $4 = \frac{3}{y}$, $5 = \frac{2}{x}$, and $6 = \frac{5}{z}$, find the value of $\frac{15x - 10y}{9z}$.

4. Given that r, s, t, and u are positive integers and that $9t = 2u$, $5r = 3s$, and $10u = 9s$, arrange r, s, t, and u in order from least to greatest.

5. An operation $*$ is defined for real numbers u and v as

$$u * v = \tfrac{1}{2}(u + v - |u - v|).$$

   a. Is $*$ a commutative operation?

   b. If u is an integer and v is the next greater integer, what is the value of $u * v$?