# Chapter 2

# *Working with Real Numbers*

## *Basic Assumptions and Definitions*

*OBJECTIVES for Sections 2-1 through 2-4:*
1. *To determine whether a statement that contains a quantifier is true or false.*
2. *To use basic axioms of addition, multiplication, and equality to simplify expressions.*
3. *To simplify expressions and solve open sentences involving opposites and absolute values.*

## 2–1 Using Quantifiers

As you have seen, an open sentence such as

$$x + 3 = 3 + x$$

is by itself neither true nor false. If a domain is specified for the variable in an open sentence, however, you can determine whether the sentence is true for any particular value of the variable from that domain. In the open sentence just given, if the domain of $x$ is specified as $\mathcal{R}$, you obtain a true statement when you replace $x$ with *any* value from its domain.

For example, each of the following is a true statement.

$$7 + 3 = 3 + 7$$

$$\frac{1}{5} + 3 = 3 + \frac{1}{5}$$

$$\sqrt{2} + 3 = 3 + \sqrt{2}$$

Thus the following statement is also true.

For any real number $x$, $x + 3 = 3 + x$.

Other ways to state this same assertion are the following:

For each real number $x$, $x + 3 = 3 + x$.
For every real number $x$, $x + 3 = 3 + x$.
For all real numbers $x$, $x + 3 = 3 + x$.

Now consider the following statement.

For each integer $y$, $y + 2 > 3$.

This statement is certainly false, since you obtain a false statement when you replace $y$ with the integer 1:

$1 + 2 > 3$ is false.

When you replace $y$ with the integer 4, however, you obtain a *true* statement:

$4 + 2 > 3$ is true.

Because there is *some* integer $y$ for which "$y + 2 > 3$" is a true statement, the following statement is also true.

For some integer $y$, $y + 2 > 3$.

Other ways to state this same assertion are the following:

There is an integer $y$ such that $y + 2 > 3$.
There exists an integer $y$ such that $y + 2 > 3$.
For at least one integer $y$, $y + 2 > 3$.

Words and phrases such as *any, each, every, all, some, there is, there exists,* and *at least one* are used to convey the idea of *how many*, or *quantity*. For this reason, when such a word or phrase is used in combination with a variable in an open sentence, the word or phrase is called a **quantifier.**

**EXAMPLE 1** Tell whether each statement is true or false.

**a.** There exists a real number $m$ such that $3m + 7 = 16$.

**b.** For some real number $r$, $r + 1 < r + 2$.

**SOLUTION**

**a.** True, since $3(3) + 7 = 16$ is true.

**b.** True; for example, $7 + 1 < 7 + 2$ is true.

**EXAMPLE 2** Find a value of the variable that makes the statement true: For some whole number $w$, $2 + w = 2w$.

**SOLUTION** The statement $2 + 2 = 2(2)$ is true. Therefore, the statement is true when $w = 2$.

## *Oral Exercises*

**Tell whether each statement is true or false.**

1. There exists a real number $a$ such that $a < 0$.
2. There is a natural number $b$ such that $b < 0$.
3. For any whole number $w$, $w > 0$.
4. For any negative integer $z$, $0 > z$.
5. For at least one positive integer $m$, $m + 1 > 0$.
6. For every integer $n$, $n(n + 1) > 0$.
7. For some integer $c$, $c^4 = 0$.
8. For all positive integers $d$, $d^2 > d$.
9. There is a positive integer $p$ such that $2p = p^2$.
10. For some positive integer $q$, $3q = 10$.
11. For each positive integer $x$, $3x > x$.
12. For any real number $y$, $3y > y$.

## *Written Exercises*

**Find a value of the variable that makes each statement true.**

A

1. There is a real number $u$ such that $3u + 1 = 16$.
2. For at least one rational number $v$, $5v = 4$.
3. For some whole number $j$, $j + 1 = 2j$.
4. There exists a natural number $k$ such that $k(k - 1) = 0$.
5. For at least one whole number $s$, $6 > 2s + 5$.
6. There is an integer $t$ such that $14 \leq 3 + 2t \leq 15$.

**Find a value of the variable that makes each statement false.**

7. For all real numbers $p$, $p^2 > 0$.
8. For each positive integer $q$, $q < q^3$.
9. For any natural number $a$, $a^2 + a > 2$.
10. For any whole number $b$, $0 < b^2 - b$.
11. For each integer $m$, $5m \neq \frac{m}{5}$.
12. For every real number $n$, $2n \neq n^2$.

**Rewrite each statement, using a variable expression to represent the relationship between $y$ and $x$.**

EXAMPLE For every natural number $x$, there is a natural number $y$ that is one less than twice $x$.

SOLUTION For every natural number $x$, there is a natural number $y$ such that $y = 2x - 1$.

**B** 13. For any real number $x$, there is a real number $y$ such that $y$ is three greater than $x$.

14. For each real number $x$, there exists a real number $y$ such that $y$ is one fourth $x$.

15. For every whole number $x$, there exists a whole number $y$ that is five more than three times $x$.

16. For any natural number $x$, there is a natural number $y$ that exceeds $x$ by one.

17. For all real numbers $x$ and $y$, the sum when $x$ is increased by $y$ is the same as the sum when $y$ is increased by $x$.

18. There is a real number $y$ such that, for any real number $x$, the sum of $x$ and $y$ is $x$.

19. There exists a nonzero real number $y$ such that, for each real number $x$, the quotient when $x$ is divided by $y$ is $x$.

20. For each positive real number $x$, there exists a positive real number $y$ such that the quotient when $x$ is divided by $y$ is $y$.

**C** 21. Rewrite each of the statements in Written Exercises 7–12, changing the quantifier so that the statement becomes true.

22. Rewrite each of the statements in Written Exercises 1–6, changing the quantifier so that the statement becomes false.

## 2–2 Basic Assumptions: Axioms for the Real Numbers

In working with real numbers, you use two basic operations, *addition* and *multiplication*. Each of these operations is called a **binary operation** because it pairs any *two* real numbers with a third real number.

The operation of addition pairs any two real numbers $a$ and $b$ with a real number that is called their **sum.** The sum is denoted as $a + b$, and in this sum the real numbers $a$ and $b$ are called **terms.** Multiplication pairs any two real numbers $a$ and $b$ with a real number that is called their **product.** The product may be denoted as $ab$, and in this product the real numbers $a$ and $b$ are called **factors.**

The rules that you use when adding or multiplying real numbers are all based on a few basic statements, called **axioms** or **postulates,** that are *assumed* to be true. For example, in describing the sum and the product of two real numbers, it was assumed that the result of adding or multiplying two real numbers is always a real number. That is, the set of real numbers was assumed to be *closed* under the operations of addition and multiplication. Furthermore, it was assumed that any two real numbers have one and only one sum and one and only one product. In other words, the sum and the product of two real numbers were each assumed to be *unique*. These assumptions are stated formally as the following axioms.

### *Axioms of Closure*

For all real numbers $a$ and $b$:

$a + b$ is a unique real number.
$ab$ is a unique real number.

There are other axioms for the real numbers that are general statements of familiar properties of addition and multiplication. For example, you know that when you add or multiply two real numbers, you get the same sum or product no matter what order you use in performing the operation:

$$5 + 12 = 12 + 5 \quad \text{and} \quad 5 \times 12 = 12 \times 5$$

This fact can be stated formally as follows.

### *Commutative Axioms*

For all real numbers $a$ and $b$:

$$a + b = b + a$$
$$ab = ba$$

Since addition and multiplication are both binary operations, expressions such as $2 + 3 + 5$ and $2 \times 3 \times 5$, in which *more than* two numbers are to be added or multiplied, must be defined. The following pattern is used in defining sums and products of three or more real numbers:

$a + b + c = (a + b) + c,$ $\qquad abc = (ab)c,$
$a + b + c + d = (a + b + c) + d,$ $\qquad abcd = (abc)d,$
$a + b + c + d + e = (a + b + c + d) + e,$ $\qquad abcde = (abcd)e,$

and so on.

When you add or multiply three or more real numbers, you get the same sum or product no matter how you group, or *associate*, the numbers. For example:

$$(19 + 47) + 23 = 19 + (47 + 23) \quad \text{and} \quad (9 \times 2) \times 5 = 9 \times (2 \times 5)$$

The following axiom is a formal statement of this fact.

### *Associative Axioms*

For all real numbers $a$, $b$, and $c$:

$$(a + b) + c = a + (b + c)$$
$$(ab)c = a(bc)$$

The commutative and associative axioms permit you to add or multiply numbers *in any order* and *in any groups of two*. Thoughtful use of these axioms can sometimes help you in simplifying numerical expressions.

**EXAMPLE 1** Simplify.

**a.** $37 + 2\frac{1}{3} + 23 + 5\frac{2}{3}$ **b.** $\frac{2}{3} \times 4 \times 90 \times 75$

**SOLUTION**

**a.** $37 + 2\frac{1}{3} + 23 + 5\frac{2}{3} = (37 + 23) + \left(2\frac{1}{3} + 5\frac{2}{3}\right)$
$= 60 + 8$
$= 68$

**b.** $\frac{2}{3} \times 4 \times 90 \times 75 = \left(\frac{2}{3} \times 90\right)(4 \times 75)$
$= (60)(300)$
$= 18{,}000$

Since variables represent real numbers, you can also use the commutative and associative axioms in simplifying variable expressions and in solving open sentences.

**EXAMPLE 2** Simplify.

**a.** $9 + 6m^3 + 14$ **b.** $(7n)(13n)$

**SOLUTION**

**a.** $9 + 6m^3 + 14 = 6m^3 + (9 + 14) = 6m^3 + 23$

**b.** $(7n)(13n) = (7 \cdot 13)(n \cdot n) = 91n^2$

**EXAMPLE 3** Solve each open sentence over $\mathcal{R}$.

**a.** $(p + 97) + 86 = p + (97 + 86)$

**b.** $8 \cdot 15q < 15 \cdot 8q$

SOLUTION **a.** By the associative axiom for addition,
$(p + 97) + 86 = p + (97 + 86)$ for every real value of $p$.
Therefore, the solution set is $\mathcal{R}$.

**b.** By the commutative axiom for multiplication,
$8 \cdot 15q = 15 \cdot 8q$ for every real value of $q$.
Therefore, the solution set is $\emptyset$.

In Example 3, notice that the domain of the variables was specified as $\mathcal{R}$. *Throughout the rest of this book, the domain of all variables is $\mathcal{R}$ unless otherwise specified.*

The way that the symbol = is used in mathematical sentences is consistent with the following axioms.

### *Axioms of Equality*

For all real numbers $a$, $b$, and $c$:

| | |
|---|---|
| **Reflexive Axiom** | $a = a$ |
| **Symmetric Axiom** | If $a = b$, then $b = a$. |
| **Transitive Axiom** | If $a = b$ and $b = c$, then $a = c$. |

EXAMPLE 4 Name the axiom of equality that is illustrated.

**a.** If $8 = x$, then $x = 8$.
**b.** If $2 + 5 = 7$ and $7 = 4 + 3$, then $2 + 5 = 4 + 3$.
**c.** If $a = b$, $b = c$, and $c = d$, then $a = d$.

SOLUTION **a.** The symmetric axiom **b.** The transitive axiom

**c.** The transitive axiom, in two steps:
*Step 1.* If $a = b$ and $b = c$, then $a = c$.
*Step 2.* If $a = c$ and $c = d$, then $a = d$.

## *Oral Exercises*

**Name the axiom that is illustrated.**

1. $4 + (^-5)$ is a real number.
2. $^-3(^-7)$ is a real number.
3. $2 \times (50 \times 64) = (2 \times 50) \times 64$
4. $3\frac{1}{5} + \left(6\frac{4}{5} + 2\frac{3}{8}\right) = \left(3\frac{1}{5} + 6\frac{4}{5}\right) + 2\frac{3}{8}$
5. $m \cdot 8 = 8 \cdot m$
6. $(n + 12) + 29 = n + (12 + 29)$
7. $\frac{1}{2}(6x) = \left(\frac{1}{2} \cdot 6\right)x$
8. $51 + y = y + 51$
9. $(38 + p) + 46 = (p + 38) + 46$
10. $(25 + 7) + q = q + (25 + 7)$
11. If $35 = 3z$, then $3z = 35$.
12. If $a = 2$ and $2 = b$, then $a = b$.

Name the axiom that justifies each lettered step. A check (√) indicates that the step is justified by the substitution principle. Note that the transitive axiom of equality is also used in writing each step after the first.

13. $(79 + 57) + 21 = (57 + 79) + 21$ (a)
$= 57 + (79 + 21)$ (b)
$= 57 + 100$ √
$= 157$ √

14. $(v + 9) + (w + 8) = v + [9 + (w + 8)]$ (a)
$= v + [9 + (8 + w)]$ (b)
$= v + [(9 + 8) + w]$ (c)
$= v + (17 + w)$ √
$= v + (w + 17)$ (d)
$= (v + w) + 17$ (e)

15. $(125r)(8s) = [(125r)8]s$ (a)
$= [(r \cdot 125)8]s$ (b)
$= [r(125 \cdot 8)]s$ (c)
$= (r \cdot 1000)s$ √
$= (1000 \cdot r)s$ (d)
$= 1000(rs)$ (e)

## Written Exercises

**Simplify.**

A

1. $540 + 37 + 60 + 43$
2. $9 + 38 + 912 + 4391$
3. $5 \times 23 \times 3 \times 2$
4. $3 \times 4 \times 9 \times 75$
5. $\frac{1}{2} \times 45 \times 24 \times \frac{1}{5}$
6. $\frac{2}{3} \times \frac{5}{11} \times 44 \times 45$
7. $8.7 + 5.9 + 1.3 + 0.1$
8. $2.24 + 3.3 + 4.51 + 17.76$
9. $3\frac{1}{5} + 7\frac{1}{2} + 1\frac{4}{5} + 3\frac{1}{2}$
10. $98\frac{3}{8} + 4\frac{1}{9} + 11\frac{4}{9} + 1\frac{5}{8}$
11. $\frac{9}{2} \times \frac{3}{7} \times \frac{2}{9} \times 49$
12. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6}$
13. $22 + 6z + 95$
14. $(2x + 7) + (5y + 9)$
15. $13(4m)$
16. $(15n)6$
17. $(7p)(13q)$
18. $(17r)(5s)(3t)$

**Solve.**

19. $127 + (a + 96) = 47 + 96 + 127$
20. $(12b)11 = 11 \times 12 \times 13$
21. $(c + 12) + 23 = c + (12 + 23)$
22. $3(5d) < (3 \cdot 5)d$
23. $7(22f) \geq 22(7f)$
24. $41 + (g + 72) \leq 72 + (g + 41)$

**B** 25. $25 + (x + 39) < x + 64$

26. $(11y)18 > 198y$

27. $(1.4)3w > 7w(0.6)$

28. $(2.7 + z) + 7.3 \leq (z + 6.09) + 3.91$

**In each of Exercises 29–36, an operation ☆ is defined over the set of natural numbers.**

**a. Find 3 ☆ 4.**

**b. Tell whether the statement "For all natural numbers *a* and *b*, *a* ☆ *b* is a natural number" is true or false.**

**c. Tell whether or not ☆ is a commutative operation.**

**d. Tell whether or not ☆ is an associative operation.**

29. $a \star b = a + (b + 2)$

30. $a \star b = a(2b)$

31. $a \star b = 3a + b$

32. $a \star b = 3(a + b)$

33. $a \star b = (a + b)^2$

34. $a \star b = a + b^2$

35. $a \star b = \frac{a}{2} + b$

36. $a \star b = \frac{ab}{2}$

**Tell whether each relationship is reflexive, symmetric, and/or transitive.**

**C** 37. is greater than

38. is less than or equal to

39. is not equal to

# 2–3 The Distributive Axiom

In order to simplify the expression 12(10 + 3), you may first add 10 and 3, as indicated by the parentheses, and then multiply this sum by 12.

$$12(10 + 3) = 12(13) = 156$$

Another way to simplify this same expression is to first *distribute* 12 as the multiplier of both 10 and 3, then find the sum.

$$(12 \times 10) + (12 \times 3) = 120 + 36 = 156$$

Either way, the result is the same. Thus you can write:

$$12(10 + 3) = (12 \times 10) + (12 \times 3)$$

This last equation illustrates another fact that is often used in working with real numbers: multiplication is *distributive with respect to addition.*

*Distributive Axiom of Multiplication with Respect to Addition*

For all real numbers $a$, $b$, and $c$,

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca.$$

By applying the symmetric axiom of equality, you can also state the distributive axiom in the following form.

> For all real numbers $a$, $b$, and $c$,
>
> $$ab + ac = a(b + c) \quad \text{and} \quad ba + ca = (b + c)a.$$

The distributive axiom can be helpful in simplifying both numerical and variable expressions.

**EXAMPLE 1** Simplify.

**a.** $24\left(\frac{5}{6} + \frac{3}{8}\right)$ **b.** $\left(3\frac{4}{5}\right)10$ **c.** $\frac{1}{9} \times 20 + \frac{1}{9} \times 16$

**SOLUTION**

**a.** $24\left(\frac{5}{6} + \frac{3}{8}\right) = 24 \times \frac{5}{6} + 24 \times \frac{3}{8} = 20 + 9 = 29$

**b.** $\left(3\frac{4}{5}\right)10 = \left(3 + \frac{4}{5}\right)10 = 3 \times 10 + \frac{4}{5} \times 10 = 30 + 8 = 38$

**c.** $\frac{1}{9} \times 20 + \frac{1}{9} \times 16 = \frac{1}{9}(20 + 16) = \frac{1}{9} \times 36 = 4$

**EXAMPLE 2** Show that, for every real number $y$, $8y + 7y = 15y$.

**SOLUTION**

$8y + 7y = (8 + 7)y$ by the distributive axiom
$= 15y$ by the substitution principle

As shown in Example 2, basic properties of the real numbers guarantee that, for all values of the variable, the expressions

$$8y + 7y \qquad \text{and} \qquad 15y$$

represent the same number. Therefore, the two expressions are said to be **equivalent expressions.** Note that the expression $8y + 7y$ has two terms, $8y$ and $7y$, while the expression $15y$ has only one term. When you replace a variable expression with an equivalent expression that has as few terms as possible, you **simplify the expression.**

**EXAMPLE 3** Simplify.

**a.** $5a + 3 + 8a + 4$ **b.** $(7b^2 + 10b) + 9b^2$

**SOLUTION**

**a.** $5a + 3 + 8a + 4 = (5a + 8a) + (3 + 4)$
$= (5 + 8)a + (3 + 4)$
$= 13a + 7$

**b.** $(7b^2 + 10b) + 9b^2 = (7b^2 + 9b^2) + 10b$
$= (7 + 9)b^2 + 10b$
$= 16b^2 + 10b$

Frequently you need to use *both* forms of the distributive axiom in simplifying an expression.

**EXAMPLE 4** Simplify $9(2m + 3) + 2(m + 5)$.

**SOLUTION**
$$\begin{aligned} 9(2m + 3) + 2(m + 5) &= 18m + 27 + 2m + 10 \\ &= (18m + 2m) + (27 + 10) \\ &= (18 + 2)m + (27 + 10) \\ &= 20m + 37 \end{aligned}$$

## *Oral Exercises*

**Name the axiom or principle that justifies each step.**

**1.** $\left(\frac{1}{2} + \frac{1}{3}\right)24 = \frac{1}{2} \times 24 + \frac{1}{3} \times 24$
$= 12 + 8$
$= 20$

**2.** $\frac{1}{7} \times 23 + \frac{1}{7} \times 33 = \frac{1}{7}(23 + 33)$
$= \frac{1}{7} \times 56$
$= 8$

**3.** $35 \times 2\frac{2}{5} = 35\left(2 + \frac{2}{5}\right)$
$= 35 \times 2 + 35 \times \frac{2}{5}$
$= 70 + 14$
$= 84$

**4.** $216 \times 101 = 216(100 + 1)$
$= 216 \times 100 + 216 \times 1$
$= 21{,}600 + 216$
$= 21{,}816$

**5.** $3m + (8 + 6m) = 3m + (6m + 8)$
$= (3m + 6m) + 8$
$= (3 + 6)m + 8$
$= 9m + 8$

**6.** $4(n + 6) + 5 = (4 \times n + 4 \times 6) + 5$
$= (4n + 24) + 5$
$= 4n + (24 + 5)$
$= 4n + 29$

**Simplify.**

**7.** $4(50 + 7)$

**8.** $8(60 + 2)$

**9.** $20 \times 3\frac{1}{5}$

**10.** $14 \times 2\frac{3}{7}$

**11.** $6(5.5)$

**12.** $(3.25)8$

**13.** $\frac{1}{3} \times 8 + \frac{1}{3} \times 19$

**14.** $11 \times \frac{5}{8} + 13 \times \frac{5}{8}$

**15.** $4a + 12a$

**16.** $7b + 3b + 9b$

**17.** $16c + 14 + 4c$

**18.** $(23d^2 + 8) + 17d^2$

**19.** $3f + 7g + 8f$

**20.** $12k + (5j + 6k) + 2j$

**21.** $4x + 9x^2 + 6x$

**22.** $(6y^2 + 7y) + (2y^2 + 8y)$

**23.** $2(z + 6) + 9$

**24.** $8w^2 + 4(w^2 + 5)$

## *Written Exercises*

**Simplify.**

**A**

1. $42\left(\frac{1}{3} + \frac{5}{7}\right)$
2. $\left(\frac{6}{5} + \frac{3}{4}\right)40$
3. $80 \times 3\frac{9}{10}$
4. $(12.75)4$
5. $(18.25)27 + (1.75)27$
6. $\frac{5}{8} \times 21 + \frac{5}{8} \times 75$
7. $17x + 64x$
8. $49y + 82y$
9. $85z^2 + 15z^2$
10. $5v^3 + 78v^3$
11. $1.6g + 3g + 4.4g$
12. $\frac{1}{3}h^2 + 5h^2 + \frac{5}{3}h^2$
13. $12s + 17s + 43$
14. $11t^2 + 14 + 27t^2$
15. $6p + 11p + 9q$
16. $8m + 15n + 13m$
17. $5a^2 + 6a^2 + 2a$
18. $2b^3 + 5b + 4b^3$
19. $2c^2 + 9c^3 + 14c^3$
20. $16d^4 + 8d^2 + 9d^4$
21. $4m + 17 + 2m + 8$
22. $24n^2 + 8 + 14 + 9n^2$
23. $17x + 3y + 15y + 5x$
24. $21z^3 + 4z + 41z + 19z^3$
25. $5p^4 + 8 + 9p^4 + 11p^4 + 2$
26. $4q + 6 + 3 + 9q + 7q$
27. $3(u + 5) + 6u$
28. $8v^2 + 4(1 + v^2)$
29. $6(c^2 + 2) + 5(c^2 + 6)$
30. $2(d + 5) + 4(d + 9) + 3d$
31. $4(3a + 5) + 8(a + 1)$
32. $7(3b + 1) + 3(2b + 7) + 1$
33. $5(y^2 + 2y) + 7(2y^2 + 3y)$
34. $(5z + 3z^2)3 + (z + z^2)6$
35. $12(j + k) + 3(j + k) + 3j$
36. $2(m + n) + 11n + 3(m + n)$

**B**

37. $(2 + 3g + 2g^2) + (4g^2 + 9g + 7)$
38. $(4 + 10h^2 + 3h) + (2h^2 + 1 + 5h)$
39. $(6a + 8a^3 + 2a^2) + (3a + 9a^2)$
40. $(11b^4 + 2b^2) + (3b + 5b^2 + b^3 + 6b^4)$
41. $(2 + 7p^2 + 8p + 4p^3) + (9p + 8p^3 + 3p^2 + 9)$
42. $(6q^3 + 4 + 3q + 7q^2) + (6 + 2q^3 + 5q + 6q^2)$
43. $2(1 + 3y + 4y^2 + y^3) + (4y^3 + 5y^2 + 8y + 3)$
44. $(5x^2 + 3x + 4 + 2x^3) + 3(x^2 + 4x + x^3 + 2)$
45. $5(m + 2m^2) + 3(3m^2 + 4)$
46. $6(n^2 + 2) + 2(3n + 1) + 9n^2$
47. $5(3r + s) + 9(2r + 1) + 7(s + 6)$
48. $2(u + 2v) + 3(v + 3w) + 5(u + w)$

**Write a variable expression for each word phrase. Then simplify the variable expression.**

49. three times the sum of a number $u$ and a number $v$, increased by twice the sum of $u$ and twice $v$
50. twice the sum when a number $b$ is added to three times a number $a$, increased by three times the sum when twice $b$ is added to $a$

51. five times the sum of three and the square of a number $w$, increased by twice the sum when five is added to the square of $w$

52. twice the sum when three is added to the square of a number $b$, increased by triple the sum of one and four times the square of $b$

53. the product of two and the cube of a number $y$, increased by six times the sum of three times the square of $y$ and two times the cube of $y$

54. triple the sum when the square of a number $c$ is added to the fourth power of $c$, increased by four times the square of $c$

**Simplify.**

55. $4[2a + 5(2 + 3a)] + 2(a + 1)$

56. $3[4b + 2(b + 3) + 2] + 5(2 + 3b)$

57. $2[3(3x + 1) + 2(2x + 5)] + 3[1 + 4(x + 2) + 5x]$

58. $3[2(1 + 4y) + 2(y + 4) + 2y] + 2[5y + 4(2 + y)]$

59. $4[5(m + 1) + 2(m + 3n) + 1] + 5[4 + 2(2m + n)]$

60. $2[3(2u + v) + 2(5 + v)] + 3[2(u + v) + 3(v + 1)]$

61. $4[3p^2 + 2(p^2 + 2p + 1)] + 3[2 + 3(3p^2 + p + 2)] + 2(p^2 + p + 1)$

62. $6[2q^2 + 3(q^2 + q + 4)] + 3[3(q^2 + q + 7) + 4q] + 3(2 + 2q + q^2) + 5q$

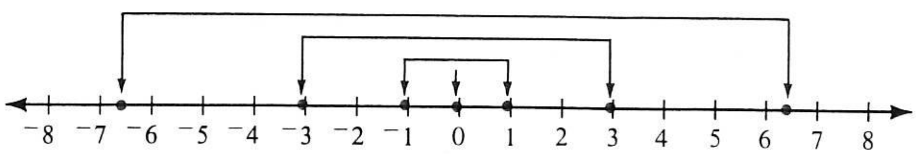
---

# Karl Friedrich Gauss

**1777–1855**

# 2–4 Opposites and Absolute Values

The figure below shows pairings of selected points on a number line. Note that the paired points are at the same distance from the origin, but on *opposite* sides of the origin. The origin is paired with itself.




The coordinates of the paired points can also be paired.

0 with 0 $\quad$ ⁻1 with 1 $\quad$ ⁻3 with 3 $\quad$ ⁻6.5 with 6.5

Each number in such a pair is called the **opposite** of the other number. The symbol for the opposite of a number $a$ is $-a$ (note the lowered position of the minus sign). For example:

$-8 = {}^{-}8$, read "The opposite of eight equals negative eight."
$-({}^{-}4) = 4$, read "The opposite of negative four equals four."
$-0 = 0$, read "The opposite of zero equals zero."

Notice that the numerals $-8$ (lowered minus sign) and ${}^{-}8$ (raised minus sign) name the same number. Thus you can read $-8$ as "negative eight" as well as "the opposite of eight." Therefore, in order to simplify notation, *lowered minus signs will be used in the numerals for negative numbers throughout the rest of this book.*

Be sure that you understand the meaning of the variable expression $-a$, the opposite of $a$. For example, if the value of $a$ is 8, then the value of $-a$ is $-8$; if the value of $a$ is $-8$, then the value of $-a$ is 8. In general, the following statements hold true.

1. If $a$ is a positive number, then $-a$ is a negative number.
2. If $a$ is a negative number, then $-a$ is a positive number.
3. If $a$ is zero, then $-a$ is zero.

The following property of opposites is often helpful in simplifying expressions and in solving equations.

***Cancellation Property of Opposites***

For all real numbers $a$,

$$-(-a) = a.$$

That is, the opposite of $-a$ is $a$.

**EXAMPLE 1** Simplify.

a. $-(-1.75)$   b. $-0$   c. $-\left(\frac{1}{4} + \frac{3}{4}\right)$   d. $-[-(-2)]$

**SOLUTION** a. 1.75   b. 0   c. $-1$   d. $-2$

**EXAMPLE 2** Solve $-n = 12$.

**SOLUTION** If $-n = 12$, then $-(-n) = -12$, and $n = -12$.
Therefore, the solution set is $\{-12\}$.

In any pair of nonzero opposites, such as $-8$ and 8, one number is negative and the other is positive. The positive number of any pair of opposite nonzero real numbers is called the **absolute value** of *each* number in the pair. Thus, 8 is the absolute value of $-8$; 8 is also the absolute value of 8. The absolute value of a number is denoted by writing the numeral for the number between a pair of vertical bars, $| \ |$. For example,

$$|-8| = 8 \qquad \text{and} \qquad |8| = 8.$$

The absolute value of 0 is defined to be 0.

$$|0| = 0$$

Formally, the absolute value of any real number is defined as follows.

> For any real number $a$,
>
> $|a| = a$, if $a$ is nonnegative;
> $|a| = -a$, if $a$ is negative.

**EXAMPLE 3** Solve $|x| = 12$.

**SOLUTION** Since $|12| = 12$ *and* $|-12| = 12$, there are two values of $x$ that make a true statement, 12 and $-12$.
Therefore, the solution set is $\{12, -12\}$.

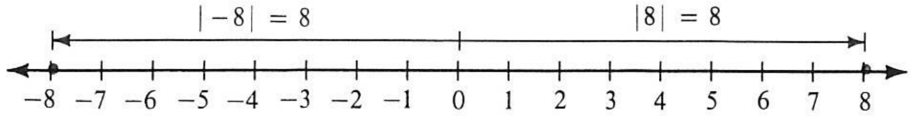
In computations involving absolute values, the vertical bars also act as grouping symbols.

**EXAMPLE 4** Simplify $7|-8| - |9 + 7|$.

**SOLUTION**

$$\begin{aligned} 7|-8| - |9 + 7| &= 7|-8| - |16| \\ &= 7(8) - 16 \\ &= 56 - 16 \\ &= 40 \end{aligned}$$

The absolute value of a number may also be thought of as the distance of the graph of the number from the origin on a number line. For example, the graphs of both $-8$ and $8$ are 8 units from the origin.

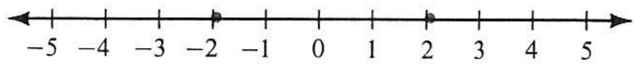



This interpretation of absolute value may be especially helpful in solving inequalities that involve absolute values.

**EXAMPLE 5** Graph each open sentence.

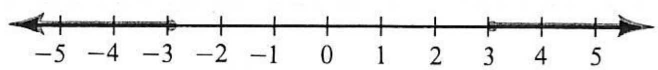
**a.** $|a| = 2$ **b.** $|x| < 1$ **c.** $|y| \geq 3$

**SOLUTION** **a.** The graph consists of the two points that are exactly two units from the origin, that is, the graphs of $-2$ and $2$.




**b.** The graph consists of all the points that are *less than* one unit from the origin in either direction.


**c.** The graph consists of all the points that are *at least* three units from the origin in either direction.




Notice in part (b) of Example 5 that the graph of the open sentence $|x| < 1$ is also the graph of the inequality $-1 < x < 1$.

## *Oral Exercises*

**Name the opposite and the absolute value of each number.**

| | | | |
|---|---|---|---|
| **1.** $5$ | **2.** $-4$ | **3.** $-\frac{3}{7}$ | **4.** $\frac{1}{3}$ |
| **5.** $-6.92$ | **6.** $6.92$ | **7.** $0$ | **8.** $1$ |

**Simplify.**

| | | | |
|---|---|---|---|
| **9.** $-(-11)$ | **10.** $-[-(-15)]$ | **11.** $-(9 + 14)$ | **12.** $-[-(37 - 12)]$ |
| **13.** $\|-17\|$ | **14.** $-\|17\|$ | **15.** $-\|-23\|$ | **16.** $-\|0\|$ |
| **17.** $5\|12\|$ | **18.** $\|-8\|6$ | **19.** $-\|11 + 7\|$ | **20.** $-\|19 - 5\|$ |

**Tell whether each statement is true or false.**

| | | | |
|---|---|---|---|
| **21.** $\|-9\| = 9$ | **22.** $\|7\| = -7$ | **23.** $\|-8\| = -\|8\|$ | **24.** $-\|-2\| = -\|2\|$ |
| **25.** $\|5\| > \|-5\|$ | **26.** $\|3\| \leq \|-3\|$ | **27.** $-1 < \|-4\|$ | **28.** $\|-10\| > 0$ |

29. The opposite of any real number is less than zero.
30. The absolute value of any real number is greater than zero.
31. The absolute value of any real number is equal to the absolute value of the opposite of the number.
32. The absolute value of the opposite of any real number is equal to the opposite of the absolute value of the number.
33. The opposite of the opposite of any real number is equal to the number itself.
34. The absolute value of the absolute value of any real number is equal to the number itself.

## Written Exercises

**Simplify.**

A

1. $-(-3.5)$
2. $-[-(-93)]$
3. $-(36 + 49)$
4. $-(8.4 - 1.2)$
5. $[-(-79)] - 22$
6. $24 + [-(-137)]$
7. $\left|-6\frac{1}{7}\right|$
8. $-|0.77|$
9. $-|10.5 - 4.3|$
10. $|-(223 + 19)|$
11. $6|-(-14)|$
12. $\left|-\frac{1}{5}\right|3$
13. $5|-0.2| + |3|$
14. $6|6| - 7|-5|$
15. $|9 + 22| + |-29|$
16. $6|-4| - |4 + 11|$
17. $3|9 + 5| - 6|-7|$
18. $7|3 + 8| - 4|8 - 3|$

**Solve.**

19. $-x = 18$
20. $2 = -y$
21. $-z = -4$
22. $-9 = -w$
23. $|a| = 5$
24. $16 = |b|$
25. $|c| = 0$
26. $|d| = -7$
27. $|-f| = 10$
28. $-|g| = 1$
29. $-|j| = -6$
30. $-|-k| = 3$

B

31. $-r + 1 = 5$
32. $5 + (-s) = 9$
33. $-u - 3 = 12$
34. $6 = -v - 14$
35. $|p| + 2 = 8$
36. $9 - |q| = 7$
37. $17 = |s| - 8$
38. $4 + |t| = 13$
39. $|-m| + 3 = 4$
40. $16 - |-n| = 5$
41. $|j| + |3| = 9$
42. $|-k| - |-7| = 15$

**Graph each inequality.**

43. $|a| \le 4$
44. $|b| > 5$
45. $|c| \ge 0$
46. $|d| < 0$
47. $|x| < -1$
48. $|y| \ge -4$
49. $|-w| > 5$
50. $|-z| \le 6$
51. $0 \le |j| \le 2$
52. $3 > |k| > 0$
53. $-1 \le |m| < 5$
54. $3 \ge |n| > -7$

**Name the set of all values of $x$ for which each inequality is a true statement.**

C

55. $|x| > x$
56. $|x| < -x$
57. $|x| \le -x$
58. $|x| \ge x$

## *Computer Exercises* For students with computer experience

**Write a program that uses the computer's absolute value function to evaluate each expression when $n \in \{25, -25\}$.**

1. $|n|$
2. $-|n|$
3. $|-n|$
4. $-|-n|$
5. $|n| + 2$
6. $|n + 2|$
7. $|2n|$
8. $\frac{|n|}{2}$
9. Write a program that will print the absolute value of a number *without* using the computer's absolute value function. RUN your program for a positive number, a negative number, and zero.

# Self-Test 1

**VOCABULARY**

quantifier (p. 50)
binary operation (p. 52)
sum (p. 52)
terms (p. 52)
product (p. 52)
factors (p. 52)
axiom or postulate (p. 53)
axioms of closure (p. 53)
commutative axioms (p. 53)
associative axioms (p. 54)
reflexive axiom (p. 55)
symmetric axiom (p. 55)
transitive axiom (p. 55)
distributive axiom (p. 57)
equivalent expressions (p. 58)
simplify a variable expression (p. 58)
opposite of a number (p. 62)
cancellation property of opposites (p. 62)
absolute value (p. 63)

**Find a value of the variable that makes each statement true.**

*Obj. 1, p. 49*

1. For some natural number $n$, $2n - 3 = 11$.
2. There exists a whole number $w$ such that $3w + 4 < 6$.

**Simplify.**

*Obj. 2, p. 49*

3. $9.5 + 6.8 + 0.5 + 3.2$
4. $\frac{1}{3} \times 44 \times 21 \times \frac{1}{4}$
5. $8 + 2x^2 + 7 + 3x^2$
6. $2(3y + 8) + 3(4 + 5y)$

*Obj. 3, p. 49*

7. $-[-(41 - 26)]$
8. $9|-7| - |12 + 16|$

**Solve.**

9. $13 + (-a) = 21$
10. $|b| - |-4| = 15$

**Check your answers with those at the back of the book.**

# Addition and Subtraction

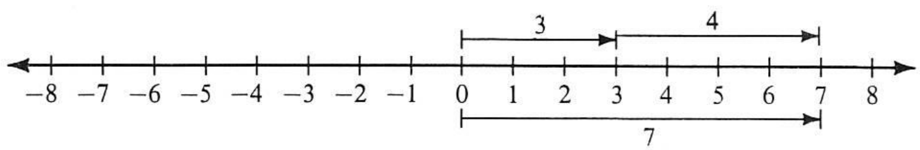
*OBJECTIVES for Sections 2-5 through 2-8:*

1. *To picture the addition of real numbers on a number line.*
2. *To add two or more real numbers and to solve word problems involving addition.*
3. *To use basic axioms of addition and equality as reasons for steps in proving theorems.*
4. *To subtract real numbers and to solve word problems involving subtraction.*

## 2–5 Addition on a Number Line

The addition of real numbers can be pictured as a series of moves, or *displacements*, along a number line. A positive number is represented by a displacement in the positive direction, and a negative number is represented by a displacement in the negative direction. Arrows are used to picture these displacements.

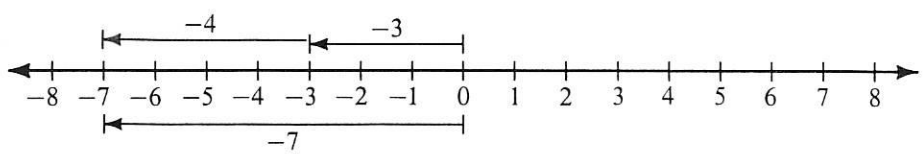
For example, to add 3 and 4 on a horizontal number line that is marked with positive numbers to the right, first start at the origin and move 3 units to the right. In the diagram that follows, the short black arrow represents this displacement, that is, the number 3. Then, starting at the graph of 3, move 4 units *farther* to the right, as represented by the red arrow. Together the two displacements amount to a displacement of 7 units to the right from the origin.




Therefore, this diagram may be used to picture the fact

$$3 + 4 = 7.$$

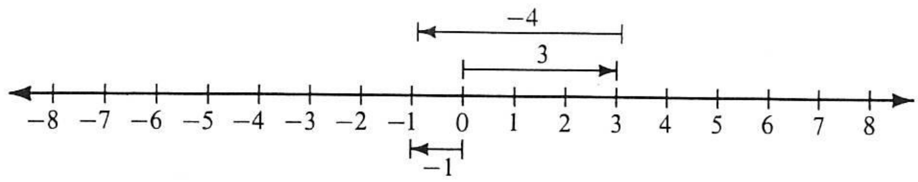
To find the sum of $-3$ and $-4$, first move 3 units to the *left* from the origin, then move 4 units farther left from this point.




$$-3 + (-4) = -7$$

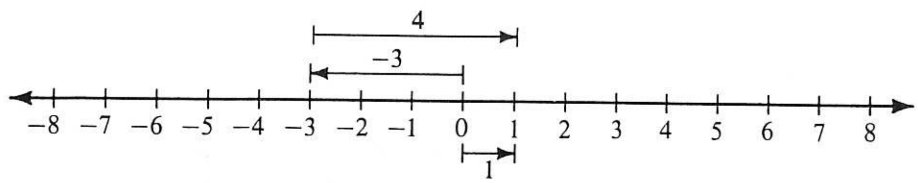
Notice the use of parentheses in the expression "$-3 + (-4)$." These parentheses separate the plus sign that means "add" from the minus sign that is part of the numeral for negative four.

To find the sum $3 + (-4)$, first move 3 units to the *right* from the origin. Then move 4 units to the *left* from this point.




$$3 + (-4) = -1$$

The following diagram shows the displacements that would be used to find the sum $-3 + 4$.




$$-3 + 4 = 1$$

Can you visualize $-3 + 0$ on a number line? If you interpret "add 0" as "no displacement," you can see that

$$-3 + 0 = -3 \qquad \text{and} \qquad 0 + (-3) = -3.$$

These equations illustrate the special property of zero for addition of real numbers: When 0 is added to any given real number, the sum is *identical* to the given number. Thus 0 is called the **identity element for addition,** and the following statement is assumed to be true.

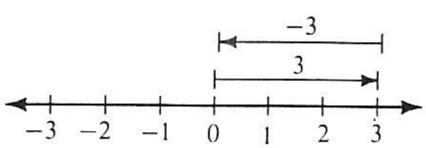
### *Identity Axiom for Addition*

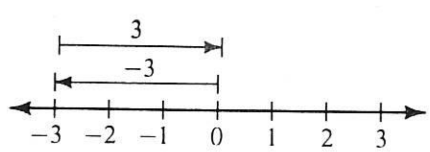
There is a unique real number 0 such that, for every real number $a$,

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

What is the result of adding a pair of opposites, such as $-3$ and 3, on a number line? As shown in the figures that follow,

$$3 + (-3) = 0 \qquad \text{and} \qquad -3 + 3 = 0.$$




Because the sum of a number and its opposite is always zero, the identity element for addition, the opposite of a number is also called the **additive inverse** of that number. The numeral $-3$ can then be read "negative three," "the opposite of three," or "the additive inverse of three."

The following axiom is a formal way of saying that the sum of a number and its opposite is always zero.

### *Axiom of Additive Inverses*

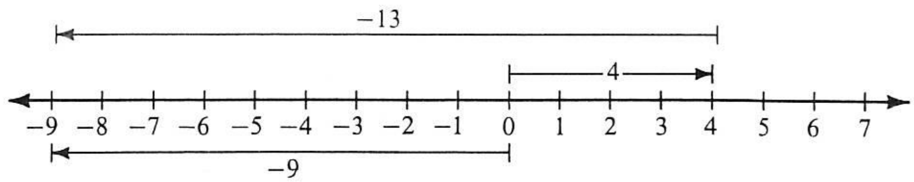
For every real number $a$, there is a unique real number $-a$ such that

$$a + (-a) = 0 \quad \text{and} \quad -a + a = 0.$$

A number line is sometimes helpful in solving equations that involve addition.

**EXAMPLE** Solve $4 + t = -9$.

**SOLUTION** The equation states that 4 plus a number $t$ is equal to $-9$. On a number line, to go from 4 to $-9$ you move 13 units to the left.



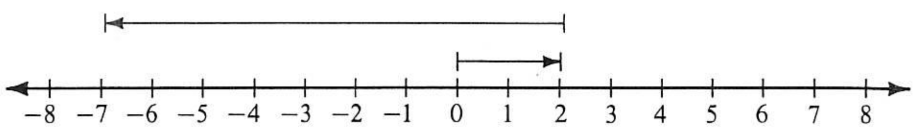

$$4 + (-13) = -9$$

Therefore, the solution set is $\{-13\}$.

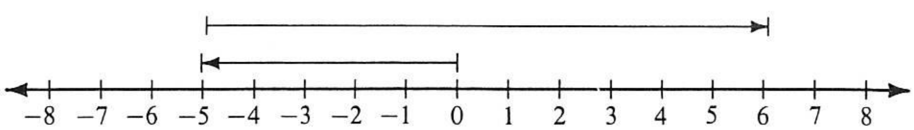
## *Oral Exercises*

**Give an addition statement pictured by each diagram.**

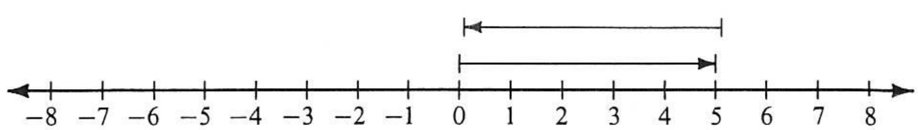
**1.**



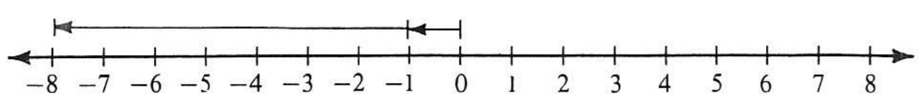

**2.**




**3.**




**4.**



Simplify. (Think of displacements along a number line.)

5. $-5 + 0$
6. $0 + (-7)$
7. $4 + (-4)$
8. $-9 + 9$
9. $-3 + (-6)$
10. $-8 + (-2)$
11. $9 + (-5)$
12. $-3 + 4$
13. $-\frac{2}{5} + \left(-\frac{1}{5}\right)$
14. $\frac{6}{7} + \left(-\frac{1}{7}\right)$
15. $\frac{1}{9} + \left(-\frac{5}{9}\right)$
16. $\frac{9}{10} + \left(-\frac{9}{10}\right)$
17. $3.5 + (-1)$
18. $-2.9 + 1$
19. $4.5 + (-3.5)$
20. $-8.9 + 1.9$

# Written Exercises

**Simplify.**

**A**

1. $(-5 + 6) + (-9)$
2. $-10 + (-4 + 7)$
3. $12 + [7 + (-9)]$
4. $[5 + (-8)] + 14$
5. $[-4 + (-7)] + 8$
6. $15 + [-6 + (-3)]$
7. $-10 + [-14 + 6]$
8. $[7 + (-16)] + (-5)$
9. $[-12 + (-8)] + (-5)$
10. $-9 + [-11 + (-13)]$
11. $\left[\frac{4}{7} + \left(-\frac{1}{7}\right)\right] + \frac{2}{7}$
12. $-\frac{1}{5} + \left[\frac{2}{5} + \left(-\frac{4}{5}\right)\right]$
13. $2\frac{1}{9} + \left[\frac{4}{9} + \left(-\frac{5}{9}\right)\right]$
14. $\left[1\frac{6}{7} + \left(-\frac{1}{7}\right)\right] + \left(-\frac{4}{7}\right)$
15. $(-16 + 7) + (-11 + 2)$
16. $[13 + (-6)] + [17 + (-9)]$
17. $[7.5 + (-4)] + [8.5 + (-6)]$
18. $(-9.5 + 7) + (-6.5 + 1)$
19. $[6.3 + (-1.3)] + (-9.3 + 1.3)$
20. $[4.7 + (-9.7)] + [3.4 + (-10.4)]$
21. $[-5 + (-14 + 19)] + (-23)$
22. $-28 + [(-6 + 13) + 21]$
23. $35 + [(-21 + 4) + (-18)]$
24. $[14 + (-27 + 13)] + (-41)$

**Solve.**

25. $-2 + a = -6$
26. $-6 + b = -11$
27. $3 + r = -5$
28. $2 + s = -7$
29. $-4 + x = 7$
30. $-5 + y = 9$
31. $4 + j = 3$
32. $9 + k = 1$
33. $-9 + c = -9$
34. $-6 + d = 0$
35. $-10 + u = 10$
36. $8 + v = -8$

**B**

37. $-9 = 8 + s$
38. $6 = -7 + t$
39. $a + (-6) = -15$
40. $b + 9 = -3$
41. $c + (-6) = 8$
42. $d + 8 = 2$
43. $-8 + j = -3 + (-5)$
44. $-5 + 5 = k + (-7)$
45. $x + (-5) = -3 + 7$
46. $1 + (-6) = -3 + d$
47. $-7 + (-3) + v = 6$
48. $-12 = w + (-6) + 4$
49. $m + m = -8$
50. $-18 = n + n$
51. $p + p = -5$
52. $q + q = 0$
53. $x + x + x = -12$
54. $y + y + y + y = -28$

**Evaluate each expression when $a = -2$, $b = 3$, $c = 4$, and $d = -8$.**

**C**

55. $(-a + b) + (-c + d)$
56. $-(a + b) + [-(c + d)]$
57. $-a + [(-b + c) + d]$
58. $a + [-b + (c + d)]$
59. $-[a + (-b + c) + (-d)]$
60. $-[-(-a + b) + (-c + d)]$

# 2–6 Rules for Addition

The expression $-(4 + 5)$ represents the opposite of the sum of 4 and 5. Since $4 + 5 = 9$, it follows that

$$-(4 + 5) = -9.$$

The expression $-4 + (-5)$ represents the sum of the opposite of 4 and the opposite of 5. Using a number line, you can show that

$$-4 + (-5) = -9.$$

Since $-(4 + 5) = -9$ and $-4 + (-5) = -9$, it follows that

$$-(4 + 5) = -4 + (-5).$$

Using similar reasoning, you can also show the following facts.

$$-[-4 + (-5)] = 4 + 5$$
$$-[4 + (-5)] = -4 + 5$$
$$-(-4 + 5) = 4 + (-5)$$

These examples suggest the following property of addition.

### *Property of the Opposite of a Sum*

The opposite of a sum of real numbers is equal to the sum of the opposites of the numbers. That is, for all real numbers $a$ and $b$,

$$-(a + b) = -a + (-b).$$

By using the property of the opposite of a sum along with axioms you have learned and the familiar addition facts for positive numbers, you can compute sums of *any* real numbers without using a number line.

**EXAMPLE 1** Simplify.

**a.** $-11 + (-5)$ **b.** $11 + (-5)$ **c.** $5 + (-11)$

**SOLUTION**

**a.** $-11 + (-5) = -(11 + 5)$
$= -16$

**b.** $11 + (-5) = (6 + 5) + (-5)$
$= 6 + [5 + (-5)]$
$= 6 + 0 = 6$

**c.** $5 + (-11) = 5 + [-(5 + 6)]$
$= 5 + [-5 + (-6)]$
$= [5 + (-5)] + (-6)$
$= 0 + (-6) = -6$

After computing many sums by using either a number line or the methods of Example 1, you would probably discover the short-cut methods that are permitted by the following rules.

### *Rules for Addition of Positive and Negative Numbers*

1. If $a$ and $b$ are both positive, then $a + b = |a| + |b|$.
   *Example*: $9 + 5 = 14$
2. If $a$ and $b$ are both negative, then $a + b = -(|a| + |b|)$.
   *Example:* $-8 + (-3) = -(8 + 3) = -11$
3. If $a$ is positive and $b$ is negative and $a$ has the greater absolute value, then $a + b = |a| - |b|$.
   *Example*: $7 + (-6) = 7 - 6 = 1$
4. If $a$ is positive and $b$ is negative and $b$ has the greater absolute value, then $a + b = -(|b| - |a|)$.
   *Example*: $2 + (-9) = -(9 - 2) = -7$
5. If $a$ and $b$ are opposites, then $a + b = 0$.
   *Example*: $6 + (-6) = 0$

The following example illustrates how to use these rules when adding more than two real numbers.

**EXAMPLE 2** Simplify $6 + (-9) + 7 + (-8)$.

**SOLUTION 1** Add the numbers in order from left to right.
$6 + (-9) = -3;\ -3 + 7 = 4;\ 4 + (-8) = -4$

**SOLUTION 2**
Add the positive numbers: $6 + 7 = 13$
Add the negative numbers: $-9 + (-8) = -17$
Add the sums: $13 + (-17) = -4$

The rules for addition of positive and negative numbers also apply when simplifying variable expressions.

**EXAMPLE 3** Simplify.
**a.** $(-10)m + 2m$ **b.** $2n^2 + (-3)n + 7n + (-8)n^2$

**SOLUTION**
**a.** $(-10)m + 2m = (-10 + 2)m$
$= (-8)m$

**b.** $2n^2 + (-3)n + 7n + (-8)n^2 = [2n^2 + (-8)n^2] + [(-3)n + 7n]$
$= [2 + (-8)]n^2 + [(-3) + 7]n$
$= (-6)n^2 + 4n$

# Oral Exercises

**Add.**

1. $\begin{array}{r} 12 \\ \underline{18} \end{array}$ 2. $\begin{array}{r} -12 \\ \underline{-18} \end{array}$ 3. $\begin{array}{r} -14 \\ \underline{20} \end{array}$ 4. $\begin{array}{r} 14 \\ \underline{-20} \end{array}$ 5. $\begin{array}{r} -25 \\ \underline{25} \end{array}$ 6. $\begin{array}{r} -25 \\ \underline{-25} \end{array}$

7. $\begin{array}{r} -1.2 \\ \underline{-2.4} \end{array}$ 8. $\begin{array}{r} 1.2 \\ \underline{-2.4} \end{array}$ 9. $\begin{array}{r} -1.2 \\ \underline{2.4} \end{array}$ 10. $\begin{array}{r} -\frac{3}{7} \\ \underline{-\frac{2}{7}} \end{array}$ 11. $\begin{array}{r} \frac{3}{7} \\ \underline{-\frac{2}{7}} \end{array}$ 12. $\begin{array}{r} -\frac{3}{7} \\ \underline{\frac{2}{7}} \end{array}$

**Replace each _?_ with one of the words *always*, *sometimes*, or *never* to make a true statement that has the widest application.**

13. The sum of two negative numbers is _?_ a negative number.
14. The sum of a positive number and a negative number is _?_ a negative number.
15. The sum of two positive numbers is _?_ zero.
16. The sum of a positive number and zero is _?_ a negative number.
17. The sum of a positive number and a negative number is _?_ zero.
18. The sum of zero and a negative number is _?_ a negative number.

**Tell whether each statement is true or false.**

19. $-(3 + 5) = -3 + 5$
20. $-[-7 + (-9)] = 7 + 9$
21. $-(-10 + 4) = -10 + (-4)$
22. $-[14 + (-8)] = -14 + 8$
23. $4 + 7 = |4| + |7|$
24. $-17 + (-8) = |-17| - |-8|$
25. $-16 + 9 = |-16| + |9|$
26. $5 + (-13) = |-13| - |5|$
27. $-12 + 3 = -(|-12| - |3|)$
28. $11 + (-3) = -(|11| - |-3|)$

# Written Exercises

**Add.**

A 1. $\begin{array}{r} 11 \\ 19 \\ -14 \\ \underline{-10} \end{array}$ 2. $\begin{array}{r} -21 \\ 7 \\ 19 \\ \underline{-12} \end{array}$ 3. $\begin{array}{r} -35 \\ -24 \\ 18 \\ \underline{-31} \end{array}$ 4. $\begin{array}{r} -47 \\ 26 \\ -19 \\ \underline{-38} \end{array}$ 5. $\begin{array}{r} 146 \\ -97 \\ -128 \\ \underline{119} \end{array}$ 6. $\begin{array}{r} -223 \\ 351 \\ -269 \\ \underline{-144} \end{array}$

**Simplify.**

7. $19 + (-11) + (-3) + 4$
8. $-17 + 5 + (-21) + 14$
9. $209 + (-401) + (-20) + (-103)$
10. $-821 + 579 + (-37) + (-163)$
11. $4.2 + (-1.3) + (-0.8) + 0.9$
12. $-8.6 + (-2.2) + 9.4 + (-3.9)$

**Simplify.**

13. $-\frac{4}{9} + \left(-\frac{1}{9}\right) + \left(-\frac{2}{9}\right) + \frac{8}{9}$

14. $-\frac{3}{5} + \frac{1}{5} + \frac{2}{5} + \left(-\frac{4}{5}\right)$

15. $-(-5 + 8) + (-4) + 3$

16. $-[14 + (-9)] + 17 + (-2)$

17. $-(-12 + 5) + [-14 + (-8)]$

18. $-(-8 + 3) + -[16 + (-7)]$

19. $6x + (-14)x$

20. $(-19)y^2 + 3y^2$

21. $(-5)m + (-9)n + (-12)m$

22. $6a + (-13)b + (-9)b$

23. $3r^2 + (-5)r + (-7)r^2 + 13r$

24. $(-6)s + 18s^2 + (-4)s^2 + 6s$

**B**

25. $(-9)a + 6b + (-16)c + 9b + (-12)a + 21c$

26. $4z + (-10)y + (-8)x + (-12)y + 17x + (-13)z$

27. $10j^2 + (-1)j^3 + 12j + (-19)j^2 + (-14)j^3 + (-8)j$

28. $(-5)k^4 + 7k^2 + (-13) + (-9)k^2 + 8 + (-10)k^4$

29. $17p^2 + (-15)p^2 + (-4)p + (-17) + 7p + (-2)p^2$

30. $(-12)q + 19q^2 + (-6)q^3 + (-1)q^2 + (-14)q + 2q$

**Evaluate each expression when $r$ and $s$ have the given values.**

a. $-(r + s)$ b. $-[r + (-s)]$ c. $-|r + s|$ d. $-(|r| + |s|)$

31. $r = -1;\ s = 2$

32. $r = 3;\ s = 0$

33. $r = -4;\ s = 4$

34. $r = -5;\ s = -5$

35. $r = -0.7;\ s = 0.3$

36. $r = \frac{1}{2};\ s = -\frac{1}{4}$

**Replace each _?_ with one of the symbols $=, <, >, \leq,$ or $\geq$ to make a true statement that has the widest application.**

**C**

37. For all real numbers $a \geq 0$ and $b \geq 0$, $|a + b|$ _?_ $|a| + |b|$.

38. For all real numbers $a \leq 0$ and $b \leq 0$, $|a + b|$ _?_ $|a| + |b|$.

39. For all real numbers $a < 0$ and $b > 0$, $|a + b|$ _?_ $|a| + |b|$.

40. For all real numbers $a$ and $b$, $|a + b|$ _?_ $|a| + |b|$.

## *Problems*

**a. Name a positive or a negative number to represent each measurement in the given problem.**

**b. Compute the sum of the numbers.**

**c. Answer the question.**

EXAMPLE An elevator started at the 38th floor of an office building. It then went down 15 floors, up 6 floors, and down 18 floors. At what floor was the elevator then located?

SOLUTION a. 38, $-15$, 6, $-18$

b. $38 + (-15) + 6 + (-18) = 11$

c. The elevator was located at the 11th floor.

**A**

1. A helicopter flew 43 km directly north from its base at Eagle Point and then flew 51 km directly south. Where was the helicopter then located relative to its base?
2. A salesperson drove 97 km directly east from the sales office to see a customer and then drove 123 km directly west to see a second customer. Where was the salesperson then located relative to the sales office?
3. A glider that was flying at an altitude of 3200 m dropped 340 m, rose 75 m, and then dropped 800 m. What was its new altitude?
4. A diving bell that was at a depth of 285 m below the surface of the ocean ascended 105 m, descended 220 m, and then ascended 175 m. What was its new depth?
5. Andrea bought four silver coins at prices of $12.05, $13.15, $13.45, and $12.05. She later sold these coins for $12.25, $12.85, $13.75, and $12.95, respectively. How much money did she make or lose in selling these coins?
6. A stock that opened on Monday morning at $45 per share gained $2.50 per share on each of Monday and Tuesday. It then dropped $1.75 per share on each of Wednesday, Thursday, and Friday. What was the closing price of this stock on Friday afternoon?
7. The temperature at midnight of one day was 8°C. The temperature fell 15°C during the next six hours, rose 9°C during the next twelve hours, and then fell 2°C during the next six hours. What was the temperature at midnight of the second day?
8. Robert's normal pulse rate is 72 beats per minute. After he jogged for 10 min, his pulse rate rose by 31 beats. It dropped by 29 beats after a 5 min rest and then rose by 34 beats after another 10 min of jogging. What was his pulse rate after this 25 min period?
9. Using her new revolving charge account, Alice Thornton purchased clothing worth $38.50 and stereo equipment worth $220.95. She then made two monthly payments of $84 each to her account. The finance charge on her account for this two-month period was $6.55. How much did Alice still owe on her account?
10. Earl Washington had a balance of $453.80 in his checking account at the start of the month. During the month, he wrote checks for $33.90, $14.62, $119.65, and $164.80, and he made a deposit of $360 from his paycheck. On the last day of the month, a $3.00 service charge and a $6.50 check-printing fee were deducted from his account. What was the starting balance of his account for the next month?

**Use what you know about addition of positive and negative numbers to answer each question.**

**B** 11. A submarine fired a rocket that rose a total of 138 m and that reached an altitude of 125 m above the surface of the ocean. What was the position of the submarine relative to the surface?

12. A passenger on a moving train walked toward the back of the train at a rate of 7 km/h. As a result, the passenger's rate of travel relative to the ground was 143 km/h. At what rate was the train traveling?

13. In one week, Diane deposited $150 in her savings account and then withdrew $65. If the amount of money in her account at the end of the week was $340, how much was in her account at the beginning?

14. The population of Bay City increased by 15,000 in one decade, then decreased by 8500 in the next decade. If the population at the end of these two decades was 23,000, what was the population at the beginning?

15. In one 5 min period, an elevator in an office building went up 9 floors, down 12 floors, up 5 floors, down 7 floors, and up 10 floors. If the elevator was then at the 11th floor, where was it located at the beginning of this 5 min period?

16. At the four bus stops on Walnut Street, the 10:00 bus discharged 5 passengers and picked up 7; discharged 10 and picked up 3; discharged 8 and picked up 2; and discharged 6 and picked up 15. If there were 26 passengers on the bus after the fourth stop, how many were on the bus before the first stop?

**C** 17. Luis walked the following route as he was sightseeing in the city: 6 blocks north and 4 blocks east; 2 blocks south and 6 blocks east; 8 blocks west and 3 blocks north; and 7 blocks south and 2 blocks west. Where was Luis then located relative to his starting point?

## PROGRAMMING IN BASIC

The computer program on page 34 can be expanded to provide graphs of additional open sentences over different domains. For example, in line 70 of the program that follows,

$$ABS(X) = |X|.$$

In line 60, the domain of the variable is changed to $\{-10, -9, \ldots, 9, 10\}$. Lines 150–220 locate and mark the zero point of the graph.

```
10  }
 :  }  From program on page 34
50  }
60  FOR X = -10 TO 10
70  IF ABS(X) = 2 THEN 120
```

```
80
 .
 .
140
150  FOR I = -10 TO 10
160  IF I <> 0 THEN 210
170  REM *ELSE
180  PRINT "0"
190  GOTO 230
200  REM *THEN
210  PRINT " ";
220  NEXT I
230  END
```

Lines 80–140: *From program on page 34*

Line 210: Type a single space between the quotation marks.

## *Exercises*

1. Type in and RUN the revised program. Compare the result with the graph shown in part (a) of Example 5 on page 64.

**Change line 70 as indicated, then RUN the revised program.**

2. 70  IF ABS(X) < 1 THEN 120
3. 70  IF ABS(X) >= 3 THEN 120
4. 70  IF 0 <= ABS(X) AND ABS(X) <= 2 THEN 120
5. 70  IF 3 > ABS(X) AND ABS(X) > 0 THEN 120
6. 70  IF 7 >= ABS(X) AND ABS(X) >= 3 THEN 120

The following change in this graphing program uses

INT(X) = the *greatest integer* less than or equal to X

to mark every fifth unit on the number line in the graphs.

```
72   REM *ELSE
74   IF X/5 = INT(X/5) THEN 104
102  REM *THEN
104  PRINT "+";
106  GOTO 130
```

7. Type in these new lines and RUN the revised program.

8–12. Change line 70 as indicated in Exercises 2–6 and RUN the program again.

13. What changes do you need to make in the program if you wish to obtain graphs of open sentences over the domain {−20, −19, . . . , 1, 2}?

14. What changes do you need to make in the program if you wish to mark *every other unit* on the number line in the graphs?

# 2–7 Proving Statements: Theorems About Addition

As you have seen, axioms for the real numbers are statements that are *assumed* to be true. Other statements about the real numbers can be *proved* to be true, and these statements are called **theorems.**

A theorem consists of two parts, a *hypothesis* and a *conclusion*. The **hypothesis** states what is assumed to be true, and the **conclusion** states something that follows logically from this assumption. When you start with the hypothesis of a theorem and arrive at its conclusion through a logical chain of statements, you give a **direct proof** of the theorem. The reason that justifies each statement in a direct proof may be the hypothesis itself, or it may be an axiom, a definition, or a previously proved theorem.

For example, consider the following equations.

$$(8 + 5) + (-5) = 8$$
$$[2 + (-9)] + 9 = 2$$
$$[-4.5 + (-7)] + 7 = -4.5$$

These equations are true sentences that suggest the following theorem.

> ***Theorem.*** For all real numbers $a$ and $b$,
> $$(a + b) + (-b) = a.$$

The hypothesis of this theorem is the statement that $a$ and $b$ are real numbers. In order to prove the theorem, you must reason from this hypothesis to the conclusion that $(a + b) + (-b) = a$. The following is a direct proof of this theorem.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ and $b$ are real numbers. | Hypothesis |
| 2. $-b$ is a real number. | Axiom of additive inverses |
| 3. $(a + b) + (-b) = a + [b + (-b)]$ | Associative axiom for addition |
| 4. $b + (-b) = 0$ | Axiom of additive inverses |
| 5. $(a + b) + (-b) = a + 0$ | Substitution principle |
| 6. $a + 0 = a$ | Identity axiom for addition |
| 7. $\therefore (a + b) + (-b) = a$ | Transitive axiom of equality |

The three dots, $\therefore$, in statement (7) of the preceding proof are read as "Therefore."

To shorten the writing of a proof, simple statements involving closure, substitution, and basic properties of equality are frequently omitted. Also, the left side of an equation generally is not written when it is the same as the right side of the equation in the previous statement. For example, the proof just given may be shortened to the following form.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ and $b$ are real numbers. | Hypothesis |
| 2. $-b$ is a real number. | Axiom of additive inverses |
| 3. $(a + b) + (-b) = a + [b + (-b)]$ | Associative axiom for addition |
| 4. $= a + 0$ | Axiom of additive inverses |
| 5. $= a$ | Identity axiom for addition |
| 6. $\therefore (a + b) + (-b) = a$ | Transitive axiom of equality |

Some familiar properties of the real numbers are actually theorems that can be proved using the fundamental axioms for the real numbers. The following example outlines a proof of the *property of the opposite of a sum*, which was discussed in Section 2-6.

**EXAMPLE** Prove: For all real numbers $a$ and $b$, $-(a + b) = -a + (-b)$.

**SOLUTION** *Plan*: By the axiom of additive inverses, the additive inverse of the real number $(a + b)$ is a *unique* real number $-(a + b)$. If it can be shown that $(a + b) + [-a + (-b)] = 0$, then by the axiom of additive inverses $[-a + (-b)]$ is the additive inverse of $(a + b)$ and is therefore equal to $-(a + b)$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ and $b$ are real numbers. | Hypothesis |
| 2. $-a$ and $-b$ are real numbers. | Axiom of additive inverses |
| 3. $(a + b) + [-a + (-b)]$ $= [a + (-a)] + [b + (-b)]$ | Commutative and associative axioms for addition |
| 4. $= 0 + 0$ | Axiom of additive inverses |
| 5. $= 0$ | Identity axiom for addition |
| 6. $(a + b) + [-a + (-b)] = 0$ | Transitive axiom of equality |
| 7. $-a + (-b) = -(a + b)$ | Axiom of additive inverses |
| 8. $\therefore -(a + b) = -a + (-b)$ | Symmetric axiom of equality |

Another theorem for the real numbers is the *cancellation property of opposites*, which was discussed in Section 2-4. Exercise 14 below illustrates how the axioms can be used to prove this theorem.

Note that any theorem which has been proved can thereafter be used as a reason in other proofs.

## Oral Exercises

**Simplify.**

**1.** $(9 + 4) + (-4)$ **2.** $(-7 + 6) + 7$ **3.** $8 + [5 + (-8)]$

**4.** $-3 + [3 + (-9)]$ **5.** $-(9 + 5) + 5$ **6.** $-(10 + 2) + 10$

**7.** $11 + [-(7 + 11)]$ **8.** $5 + [-(5 + 12)]$ **9.** $-(-3 + 8) + 8$

**10.** $-(-9 + 7) + (-9)$ **11.** $-10 + [-(-10 + 3)]$ **12.** $4 + [-(-16 + 4)]$

**Replace each __?__ with the reason that justifies the statement to its left.**

**13.** Prove: For all real numbers $a$ and $b$, $(a + b) + (-a) = b$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ and $b$ are real numbers. | Hypothesis |
| 2. $-a$ is a real number. | __?__ |
| 3. $(a + b) + (-a) = (b + a) + (-a)$ | Commutative axiom for addition |
| 4. $= b + [a + (-a)]$ | __?__ |
| 5. $= b + 0$ | __?__ |
| 6. $= b$ | __?__ |
| 7. $\therefore (a + b) + (-a) = b$ | Transitive axiom of equality |

**14.** Prove: For all real numbers $a$, $-(-a) = a$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ is a real number. | __?__ |
| 2. $-a$ and $-(-a)$ are real numbers. | __?__ |
| 3. $-(-a) = -(-a) + 0$ | Identity axiom for addition |
| 4. $= -(-a) + (-a + a)$ | Axiom of additive inverses |
| 5. $= [-(-a) + (-a)] + a$ | __?__ |
| 6. $= 0 + a$ | __?__ |
| 7. $= a$ | __?__ |
| 8. $\therefore -(-a) = a$ | __?__ |

## Written Exercises

**Simplify.**

**A**

1. $(82 + 65) + (-65)$
2. $(-56 + 90) + 56$
3. $38 + [104 + (-38)]$
4. $-22 + [22 + (-141)]$
5. $-(93 + 52) + 93$
6. $17 + [-(47 + 17)]$
7. $-(-18 + 49) + (-18)$
8. $61 + [-(-14 + 61)]$
9. $-[53 + (-16)] + (-16)$
10. $-85 + [-(-85 + 91)]$
11. $-[24 + (-81)] + 24$
12. $-[-43 + (-78)] + (-78)$

**Give the reason that justifies each statement in the given proof.**

13. Prove: For all real numbers $a$ and $b$, $-b + (a + b) = a$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-b$ is a real number.
3. $-b + (a + b) = -b + (b + a)$
4. $= (-b + b) + a$
5. $= 0 + a$
6. $= a$
7. $\therefore -b + (a + b) = a$

14. Prove: For all real numbers $a$ and $b$, $(-a + b) + a = b$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-a$ is a real number.
3. $(-a + b) + a = [b + (-a)] + a$
4. $= b + (-a + a)$
5. $= b + 0$
6. $= b$
7. $\therefore (-a + b) + a = b$

15. Prove: For all real numbers $a$ and $b$, $-(a + b) + a = -b$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-(a + b) + a = [-a + (-b)] + a$
3. $= [-b + (-a)] + a$
4. $= -b + (-a + a)$
5. $= -b + 0$
6. $= -b$
7. $\therefore -(a + b) + a = -b$

16. Prove: For all real numbers $a$ and $b$, $-a + [-(-a + b)] = -b$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-a$ is a real number.
3. $-a + [-(-a + b)]$
   $= -a + [-(-a) + (-b)]$
4. $= -a + [a + (-b)]$
5. $= (-a + a) + (-b)$
6. $= 0 + (-b)$
7. $= -b$
8. $\therefore -a + [-(-a + b)] = -b$

**Write a direct proof of each theorem.**

**B**

17. For all real numbers $a$ and $b$, $-a + (a + b) = b$.
18. For all real numbers $a$ and $b$, $[a + (-b)] + b = a$.
19. For all real numbers $a$ and $b$, $[a + (-b)] + (-a) = -b$.
20. For all real numbers $a$ and $b$, $-b + (-a + b) = -a$.

**Write a direct proof of each theorem.**

21. For all real numbers $a$ and $b$, $-(a + b) + b = -a$.
22. For all real numbers $a$ and $b$, $a + [-(a + b)] = -b$.
23. For all real numbers $a$ and $b$, $-[a + (-b)] + (-b) = -a$.
24. For all real numbers $a$ and $b$, $b + [-(-a + b)] = a$.

C 25. For all real numbers $a$ and $b$, $(a + b) + [-a + (-b)] = 0$.
26. For all real numbers $a$ and $b$, $-[a + (-b)] + [-(-a + b)] = 0$.
27. For all real numbers $a$, $b$, and $c$, $-[(a + b) + c] = [-a + (-b)] + (-c)$.
28. For all real numbers $a$, $b$, and $c$, $-[a + (b + c)] = -(a + b) + (-c)$.

# 2–8 Subtracting Real Numbers

The first column below lists a few examples of subtracting 3. The second column lists related examples of adding $-3$.

| Subtracting 3 | Adding $-3$ |
|---|---|
| $7 - 3 = 4$ | $7 + (-3) = 4$ |
| $6 - 3 = 3$ | $6 + (-3) = 3$ |
| $5 - 3 = 2$ | $5 + (-3) = 2$ |
| $4 - 3 = 1$ | $4 + (-3) = 1$ |

Comparing the entries in the two columns shows that *subtracting* 3 gives the same result as *adding the opposite* of 3. This relationship between addition and subtraction suggests the following definition.

*Definition of Subtraction*

For all real numbers $a$ and $b$, the **difference** denoted as $a - b$ is defined by:

$$a - b = a + (-b)$$

That is, to subtract $b$ from $a$, add the opposite of $b$ to $a$.

EXAMPLE 1 Simplify.

**a.** $4 - 15$ **b.** $6 - (-21)$ **c.** $-9 - 17$ **d.** $-3 - (-18)$

SOLUTION

**a.** $4 - 15 = 4 + (-15) = -11$
**b.** $6 - (-21) = 6 + 21 = 27$
**c.** $-9 - 17 = -9 + (-17) = -26$
**d.** $-3 - (-18) = -3 + 18 = 15$

**EXAMPLE 2** Simplify $14 - 29 + 16 - 43$.

**SOLUTION 1** Add or subtract in order from left to right.

$14 - 29 = -15; -15 + 16 = 1; 1 - 43 = -42$

**SOLUTION 2** Group positive and negative numbers.

$14 - 29 + 16 - 43 = (14 + 16) - (29 + 43) = 30 - 72 = -42$

Subtraction of real numbers is *not* commutative. For example:

$$7 - 2 = 5 \quad \text{but} \quad 2 - 7 = -5$$

Nor is subtraction of real numbers associative. For example:

$$(11 - 4) - 1 = 7 - 1 = 6 \quad \text{but} \quad 11 - (4 - 1) = 11 - 3 = 8$$

On the other hand, note that

$$7(9 - 4) = 7 \times 9 - 7 \times 4,$$

since $7(9 - 4) = 7(5) = 35$ and $7 \times 9 - 7 \times 4 = 63 - 28 = 35$. This example illustrates the fact that multiplication is *distributive with respect to subtraction*, which will be proved as a theorem in Exercise 62 on page 92.

> For all real numbers $a$, $b$, and $c$,
>
> $$a(b - c) = ab - ac \quad \text{and} \quad (b - c)a = ba - ca.$$

Certain sums are usually replaced by differences. For example:

$5n + (-7)$ is usually written $5n - 7$.
$10y^2 + (-3)y$ is usually written $10y^2 - 3y$.

**EXAMPLE 3** Simplify $12 - 3a - 21 + 14a$.

**SOLUTION**

$$\begin{aligned} 12 - 3a - 21 + 14a &= (-3a + 14a) + (12 - 21) \\ &= 11a + (-9) \\ &= 11a - 9 \end{aligned}$$

When you subtract a sum within grouping symbols, you use the property of the opposite of a sum.

**EXAMPLE 4** Simplify $(10b - 7) - (2b + 5)$.

**SOLUTION**

$$\begin{aligned} (10b - 7) - (2b + 5) &= (10b - 7) + [-(2b + 5)] \\ &= (10b - 7) + [-2b + (-5)] \\ &= [10b + (-2b)] + [-7 + (-5)] \\ &= 8b + (-12) \\ &= 8b - 12 \end{aligned}$$

What is the opposite of the *difference* of two real numbers $a$ and $b$? If the difference is denoted $a - b$, then

$$-(a - b) = -a + b.$$

This fact will be proved as a theorem in Exercise 48 on page 85.

## Oral Exercises

**Simplify.**

**1.** $15 - 8$ **2.** $15 - (-8)$ **3.** $-15 - 8$ **4.** $-15 - (-8)$

**5.** $8 - 15$ **6.** $8 - (-15)$ **7.** $-8 - 15$ **8.** $-8 - (-15)$

**9.** $10 - 0$ **10.** $0 - 10$ **11.** $0 - (-10)$ **12.** $-10 - 0$

**13.** $(9 - 3) - 5$ **14.** $9 - (3 - 5)$ **15.** $(3 - 5) - 9$ **16.** $3 - (5 - 9)$

**17.** $6(8 - 3)$ **18.** $6 \times 8 - 6 \times 3$ **19.** $6 \times 8 - 3 \times 8$ **20.** $(6 - 3)8$

**21.** $2a - (2a + 7)$ **22.** $2a - (7 + 2a)$ **23.** $2a - (2a - 7)$ **24.** $2a - (7 - 2a)$

**25.** $(3x + 20) - (3x + 12)$ **26.** $(3x + 20) - (3x - 12)$

**27.** $(3x - 20) - (3x + 12)$ **28.** $(3x - 20) - (3x - 12)$

**29.** $(20 - 3x) - (12 + 3x)$ **30.** $(20 - 3x) - (12 - 3x)$

**Replace each __?__ with the reason that justifies the statement to its left.**

**31.** Prove: For all real numbers $a$ and $b$, $a - (-b) = a + b$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ and $b$ are real numbers. | Hypothesis |
| 2. $-b$ is a real number. | __?__ |
| 3. $a - (-b) = a + [-(-b)]$ | Definition of subtraction |
| 4. $\quad = a + b$ | __?__ |
| 5. $\therefore a - (-b) = a + b$ | __?__ |

## Written Exercises

**Simplify.**

A

**1.** $25 - 48$ **2.** $39 - (-81)$ **3.** $-16 - 37$ **4.** $-34 - (-56)$

**5.** $61 - 17$ **6.** $-42 - (-35)$ **7.** $78 - (-22)$ **8.** $-97 - 32$

**9.** $-6.9 - 3.7$ **10.** $3.6 - 8.5$ **11.** $-1.4 - (-5.8)$ **12.** $-9.2 - (-4.9)$

**13.** $0 - \left(-\frac{5}{8}\right)$ **14.** $-\frac{4}{9} - \frac{7}{9}$ **15.** $8\frac{2}{7} - 1\frac{5}{7}$ **16.** $1\frac{2}{3} - \left(-\frac{2}{3}\right)$

17. $18 - 54 - 33$

18. $-97 - (-43) - 24$

19. $74 - 92 - 16 - (-58)$

20. $-26 - 83 + 65 - 49$

21. $-47 + 50 + 66 - 12 - 34$

22. $36 - 52 + 71 - 38 + 13$

23. $93 - (23 + 45)$

24. $67 - (-49 + 27)$

25. $46 - (-63 - 24)$

26. $-82 - (-12 - 93)$

27. $(43 - 71) - (-71 - 52)$

28. $(63 - 84) - (-19 + 63)$

29. $-(97 - 38) - (38 + 43)$

30. $-(-81 - 59) - (59 - 15)$

31. $-12x + 7 - 4x - 10$

32. $3y - 4 - 11 - 7y$

33. $14 - 9z^2 - 22 - 12z^2$

34. $-15w^2 - 12w + 6w - 18w^2$

35. $-13m^2 - 9m + 8 - 10m^2 - 14$

36. $3n - 5n^3 - 7n + 7n^2 - 4n^3$

**B**

37. $(-11a - 7) - (-1 + 4a)$

38. $(13b - 9) - (-7b - 15)$

39. $-(16p + 9) - (5p - 6)$

40. $-(6q - 14) - (8q + 5)$

41. $(8m - 5n) - (-18n + 5m)$

42. $(-7h + 8g) - (12g + 11h)$

43. $-(7x^3 + 4x) - (4x^3 - 11x)$

44. $(3y^2 - 10y) - (-5y - 10y^2)$

45. $-9z^2 - (13z^2 + 2z) - (-6z - 7)$

46. $-(7w + 2w^2) - (19w^2 - 5w^3) - 9w^3$

**Give the reason that justifies each statement in the given proof.**

47. Prove: For all real numbers $a$ and $b$, $(a + b) - b = a$.

PROOF

1. $a$ and $b$ are real numbers.
2. $(a + b) - b = (a + b) + (-b)$
3. $\quad = a + [b + (-b)]$
4. $\quad = a + 0$
5. $\quad = a$
6. $\therefore (a + b) - b = a$

48. Prove: For all real numbers $a$ and $b$, $-(a - b) = -a + b$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-(a - b) = -[a + (-b)]$
3. $\quad = -a + [-(-b)]$
4. $\quad = -a + b$
5. $\therefore -(a - b) = -a + b$

**Write a direct proof of each theorem.**

**C**

49. For all real numbers $a$ and $b$, $(a - b) + b = a$.
50. For all real numbers $a$ and $b$, $-(-a - b) = a + b$.
51. For all real numbers $a$, $b$, and $c$, $-[(a + b) + c] = (-a - b) - c$.
52. For all real numbers $a$, $b$, and $c$, $a - (b - c) = (a - b) + c$.

**A set of numbers is said to be *closed under subtraction* if all possible differences between members of the set are also members of the set. Tell whether each set is closed or not closed under subtraction.**

53. $\{0\}$
54. $\{1\}$
55. $\{0, 1\}$
56. $\{-1, 0, 1\}$
57. {the natural numbers}
58. {the integers}
59. {the even integers}
60. {the odd integers}
61. {the rational numbers}
62. {the real numbers}

# Problems

**a. Express the answer to each question as the difference between two real numbers and compute this difference.**

**b. Interpret the sign of the difference (positive or negative) and answer the question.**

EXAMPLE What is the difference in altitude between Mount McKinley, Alaska, which is 6194 m above sea level, and Death Valley, California, which is 86 m below sea level?

SOLUTION 1 a. $6194 - (-86) = 6194 + 86 = 6280$

b. Mount McKinley is 6280 m higher than Death Valley.

SOLUTION 2 a. $-86 - 6194 = -86 + (-6194) = -6280$

b. Death Valley is 6280 m lower than Mount McKinley.

A

1. Carlos rode the elevator from the 29th floor to the parking garage 2 floors below street level. How many floors did he ride the elevator?
2. Tania rode her bicycle from her home 55 blocks south of Main Street to a store 36 blocks north of Main Street. How many blocks did she ride?
3. The German mathematician Emmy Noether was born in 1882 and died after her birthday in 1935. What was her age in years when she died?
4. If the Greek mathematician Pythagoras was born in 572 B.C. and died after his birthday in 497 B.C., what was his age in years when he died?
5. The highest temperature recorded in the world is 58°C, while the lowest temperature recorded is −88.3°C. What is the difference between these record high and low temperatures?
6. Mercury melts at a temperature of −38.86°C, but it does not boil until it is at a temperature of 356.86°C. What is the difference between the melting and boiling points of mercury?
7. When the wind is blowing at 72 km/h, an actual temperature of 2°C can feel like a temperature of −17°C. What is the difference between these temperatures?
8. An actual temperature of 32°C can feel like a temperature of 50°C when the relative humidity is 90%. What is the difference between these temperatures?

**Use what you know about subtraction of positive and negative numbers to answer the question.**

B 9. Mount Everest rises 8848 m above sea level, which is 19,763 m higher than the floor of the Marianas Trench in the Pacific Ocean. What is the depth of the Marianas Trench?

10. The melting point of nitrogen is 14.21°C lower than its boiling point. If its boiling point is −195.79°C, what is its melting point?

11. There is a difference of approximately 52° latitude between Point Barrow, Alaska and Ka Lae, Hawaii. If Ka Lae is located at 19° north latitude, what is the latitude of Point Barrow?

## Computer Exercises For students with computer experience

1. Write a program that will print in chart form the values of the following expressions when you input values for $a$ and $b$.

   $|a| - |b| \qquad |b| - |a| \qquad |a - b| \qquad |b - a|$

   RUN the program for several test values of $a$ and $b$. Which two expressions have equal values for all real values of $a$ and $b$?

2. Write a program that will compute the distance between any two points on a number line when you input their coordinates.

## Self-Test 2

| VOCABULARY | | |
|---|---|---|
| | identity element for addition (p. 68) | conclusion (p. 78) |
| | additive inverse (p. 68) | direct proof (p. 78) |
| | theorem (p. 78) | subtraction (p. 82) |
| | hypothesis (p. 78) | difference (p. 82) |

**Simplify.**

1. $-7 + 2$ 2. $-3 + (-6)$ 3. $5 + (-5)$ *Obj. 1, p. 67*

4. $-9 + 17 + 35 + (-4)$ 5. $3a + (-7)b + (-8)a$ *Obj. 2, p. 67*

**Give the reason that justifies each statement.**

6. $x + 0 = x$ 7. $x + (-x) = 0$ 8. $-(-x) = x$ *Obj. 3, p. 67*

**Simplify.**

9. $-90 - (-36)$ 10. $(4n + 3) - (-2n - 7)$ *Obj. 4, p. 67*

**Check your answers with those at the back of the book.**

# *Multiplication and Division*

*OBJECTIVES for Sections 2-9 through 2-11:*
1. *To multiply real numbers and to simplify variable expressions involving multiplication.*
2. *To use multiplicative inverses and to simplify variable expressions involving multiplicative inverses.*
3. *To divide real numbers and to simplify variable expressions involving division.*

## 2–9 Rules for Multiplication

When you multiply any given real number by 1, the product is identical to the given number. For example,

$$5 \times 1 = 5 \qquad \text{and} \qquad 1 \times 5 = 5.$$

Thus 1 is called the **identity element for multiplication.**

### *Identity Axiom for Multiplication*

There is a unique real number 1, $1 \neq 0$, such that, for every real number $a$,

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a.$$

The equations

$$5 \times 0 = 0 \qquad \text{and} \qquad 0 \times 5 = 0$$

illustrate the *multiplicative property of zero*: When one of the factors of a product is zero, the product itself is zero. The following is a formal statement of this property.

### *Multiplicative Property of Zero*

For every real number $a$,

$$a \cdot 0 = 0 \quad \text{and} \quad 0 \cdot a = 0.$$

This property is a theorem that can be proved as follows.

*Plan*: By the identity axiom for addition, there is a *unique* real number 0 such that $a \cdot 0 + 0 = a \cdot 0$. If it can be shown that $a \cdot 0 + a \cdot 0 = a \cdot 0$, then by the identity axiom for addition $a \cdot 0$ must itself be equal to 0.

PROOF

| *Statements* | *Reasons* |
| --- | --- |
| 1. $a$ is a real number. | Hypothesis |
| 2. $0 = 0 + 0$ | Identity axiom for addition |
| 3. $a(0 + 0) = a \cdot 0$ | Substitution principle |
| 4. $a \cdot 0 + a \cdot 0 = a \cdot 0$ | Distributive axiom |
| 5. $\therefore a \cdot 0 = 0$ | Identity axiom for addition |
| 6. $a \cdot 0 = 0 \cdot a$ | Commutative axiom for multiplication |
| 7. $\therefore 0 \cdot a = 0$ | Transitive axiom of equality |

Would you guess that $a(-1) = -a$? You might arrive at this guess by noticing that

$$2 \times (-1) = (-1) + (-1) = -2,$$
$$3 \times (-1) = (-1) + (-1) + (-1) = -3,$$
$$4 \times (-1) = (-1) + (-1) + (-1) + (-1) = -4,$$

and so on. These examples suggest the *multiplicative property of −1*: multiplying any real number by −1 produces the opposite of the number. You can state this property formally as follows.

### *Multiplicative Property of −1*

For every real number $a$,

$$a(-1) = -a \quad \text{and} \quad (-1)a = -a.$$

This property will be proved as a theorem in Exercise 28 on page 91. Note that a special case of this property occurs when the value of $a$ is −1:

$$(-1)(-1) = -(-1) = 1$$

The multiplicative property of −1 and the identity axiom for multiplication are used frequently in simplifying variable expressions.

EXAMPLE 1 Simplify.

**a.** $x(-x)$ **b.** $(-y)(-y)$ **c.** $-2z^3 + z^3$

SOLUTION **a.** $x(-x) = x[(-1)x] = (-1)(x \cdot x) = (-1)x^2 = -x^2$

**b.** $(-y)(-y) = [(-1)y][(-1)y] = [(-1)(-1)](y \cdot y) = 1 \cdot y^2 = y^2$

**c.** $-2z^3 + z^3 = -2 \cdot z^3 + 1 \cdot z^3 = (-2 + 1)z^3 = (-1)z^3 = -z^3$

You can compute the product of *any* two real numbers by using the multiplicative property of −1 along with the familiar multiplication facts for positive numbers and with the commutative and associative axioms.

Consider the following examples.

$$4(3) = 12$$
$$(-4)3 = [(-1)4]3 = (-1)[(4)(3)] = (-1)12 = -12$$
$$4(-3) = 4[3(-1)] = [(4)(3)](-1) = 12(-1) = -12$$
$$(-4)(-3) = [(-1)4][(-1)3] = [(-1)(-1)][(4)(3)] = (1)12 = 12$$

These examples suggest the following theorem, which will be proved in Exercises 61, 63, and 64 on pages 92–93.

### *Property of Opposites in Products*

For all real numbers $a$ and $b$:

$$(-a)b = -ab \qquad a(-b) = -ab \qquad (-a)(-b) = ab$$

Practice in computing products of real numbers may lead you to discover the following rules.

### *Rules for Multiplication of Positive and Negative Numbers*

1. The product of a positive number and a negative number is a negative number.
2. The product of two positive numbers or of two negative numbers is a positive number.
3. The absolute value of the product of two real numbers is equal to the product of the absolute values of the numbers. That is, for all real numbers $a$ and $b$, $|ab| = |a| \cdot |b|$.

When a product has more than two factors, pairing the negative numbers should lead you to find that

the product of an *even* number of negative numbers is *positive;*
the product of an *odd* number of negative numbers is *negative.*

**EXAMPLE 2** Tell whether the expression names a positive number, a negative number, or zero. Then simplify the expression.

**a.** $-7(-5)(-2)$ **b.** $5^3(-2)^2(0)$ **c.** $(-1)^9(-2)^3(7)$

**SOLUTION** **a.** negative; $-70$ **b.** zero; 0 **c.** positive; 56

**EXAMPLE 3** Simplify. **a.** $(-4a)(-6a)$ **b.** $-7(b - 2c)$

**SOLUTION**

**a.** $(-4a)(-6a) = [(-4)a][(-6)a] = [(-4)(-6)](a \cdot a) = 24a^2$

**b.** $-7(b - 2c) = -7[b + (-2)c] = -7b + (-7)(-2)c = -7b + 14c$

## *Oral Exercises*

**Simplify.**

1. $(-9)4$
2. $(6)(-5)$
3. $(-3)(-7)$
4. $(-4)(-8)$
5. $(-1)(-9)(5)$
6. $(-7)(-1)(-4)$
7. $(-3)(-4)(-2)$
8. $(-5)(2)(-8)$
9. $(19)(0)(41)$
10. $(18)(-11)(0)$
11. $6(-a)$
12. $(-13)(-z)$
13. $2(-5x)$
14. $(-6)(-9y)$
15. $(-8a)(-7b)$
16. $(-6p)(3q)(2r)$
17. $(-10m)(7m)$
18. $(-4n)(5n)(-2n)$
19. $-12g + g$
20. $-10h - h$
21. $-6j + 7j$
22. $-11k + 10k$
23. $4v - 3v$
24. $5w - 6w$
25. $(-a)(-a)$
26. $-b + (-b)$
27. $-c - (-c)$

**Replace each _?_ with the reason that justifies the statement to its left.**

28. Prove: For all real numbers $a$, $a(-1) = -a$ and $(-1)a = -a$.
*Plan*: By the axiom of additive inverses, the additive inverse of the real number $a$ is the *unique* real number $-a$. If it can be shown that $a + a(-1) = 0$, then by the axiom of additive inverses $a(-1)$ is the additive inverse of $a$ and is therefore equal to $-a$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ is a real number. | _?_ |
| 2. $a + a(-1) = a \cdot 1 + a(-1)$ | Identity axiom for multiplication |
| 3. $= a[1 + (-1)]$ | _?_ |
| 4. $= a \cdot 0$ | _?_ |
| 5. $= 0$ | Multiplicative property of zero |
| 6. $a + a(-1) = 0$ | _?_ |
| 7. $\therefore a(-1) = -a$ | _?_ |
| 8. $a(-1) = (-1)a$ | _?_ |
| 9. $\therefore (-1)a = -a$ | _?_ |

## *Written Exercises*

**Simplify.**

A

1. $(17)(-35)(-1)$
2. $(-22)(-1)(-29)$
3. $(-36)(0)(-14)$
4. $(-2)(-16)(31)$
5. $(-6)(-2)(15)(-3)$
6. $(-8)(19)(-7)(0)$
7. $(-2)(-3)(-5)^2$
8. $(-2)^3(-3)(-5)$
9. $(-2)^3(-3)^2(11)$
10. $(-2)(-3)^3(-7)^2$
11. $(-1)^6(-2)(-5)^3$
12. $(-1)^7(-2)^4(7)$

**Simplify.**

13. $27(-4) + 27(-6)$
14. $-98(3) - 2(3)$
15. $-5(-33) + 5(-33)$
16. $-12(-8) - 12(8)$
17. $-1(-56) - 9(-56)$
18. $-83(11) - 83(-1)$
19. $-49(7) + (-7)$
20. $-12 + 12(-39)$
21. $-22 - 29(22)$
22. $-15(-21) - 15$
23. $-89(11) + 89$
24. $-16(41) + 16$
25. $(-x)(-x)(-x)$
26. $(-y)(-y)(-y)(-y)$
27. $(-9a)(4a)(-2a)$
28. $(-6b)(-8b)(-2)$
29. $(-10p)(-8q)(-r)$
30. $(-x)(6y)(-14z)$
31. $-2(5m + 3)$
32. $(-6n + 13)(-3)$
33. $(-10j - 4k)7$
34. $-8(9g - 4h)$
35. $-3(-x + 2y)$
36. $(-9u - v)(-11)$

**B**

37. $2w - 3x - w + x$
38. $-9y - z + 8y - 2z$
39. $-12j^2 + 7j - 8j + j^2$
40. $-6k + 9k^3 + k^3 + 5k$
41. $-a^2 + 9a - 2a^2 - 9 - 10a$
42. $-b^2 - b + 7b + 2b^2 - 6b$
43. $m + 5n - (3m + 4n)$
44. $7p + q - (7q - 6p)$
45. $7r + 3(s - 2r)$
46. $-2(5u - v) - v$
47. $-5(-c + d) + 6c - 4d$
48. $-11g + h - 4(-3g - 2h)$
49. $4(-a + 2b) + 3(-2a - 3b)$
50. $7(-c - 2d) + 3(2c - 5d)$
51. $5(3x - 4y) - 8(2x + y)$
52. $-2(2v - 11w) - 3(v + 7w)$
53. $-7(p - 3q) - 5(4q - 3p)$
54. $4(-7r - 2s) - 3(-3s - 4r)$

**Evaluate each expression when $a = -2$ and $b = -5$.**

55. $2[-3(a - 4b) - 9b] + 7a - 5b$
56. $[2(-3a + b) + 8a]3 - 5a - 7b$
57. $-2[2(2a - 5b) + b] + 2(5a - 9b)$
58. $[-2(3a - b) + 2a]3 - 2(-6a + 5b)$
59. $2[a - 3(2a - b)] - 3[-4b - 3(a - 2b)]$
60. $-3[2a - 2(5a - 2b)] + 2[-8a - 2(2a - 3b)]$

**Give the reason that justifies each statement in the given proof.**

61. Prove: For all real numbers $a$ and $b$, $a(-b) = -ab$.

PROOF

1. $a$ and $b$ are real numbers.
2. $-b$ is a real number.
3. $a(-b) = a[b(-1)]$
4. $\quad = (ab)(-1)$
5. $\quad = -ab$
6. $\therefore a(-b) = -ab$

62. Prove: For all real numbers $a$, $b$, and $c$, $a(b - c) = ab - ac$.

PROOF

1. $a$, $b$, and $c$ are real numbers.
2. $a(b - c) = a[b + (-c)]$
3. $\quad = ab + [a(-c)]$
4. $\quad = ab + (-ac)$
5. $\quad = ab - ac$
6. $\therefore a(b - c) = ab - ac$

**Write a direct proof of each theorem.**

C 63. For all real numbers $a$ and $b$, $(-a)b = -ab$.

64. For all real numbers $a$ and $b$, $(-a)(-b) = ab$.

65. For all real numbers $a$, $b$, and $c$, $-a(b + c) = -ab - ac$.

66. For all real numbers $a$, $b$, and $c$, $-a(b - c) = ac - ab$.

67. For all real numbers $a$, $b$, $c$, and $d$, $a[(b - c) - d] = ab - (ac + ad)$.

68. For all real numbers $a$, $b$, $c$, and $d$, $a[b - (c - d)] = (ab - ac) + ad$.

## *Computer Exercises* For students with computer experience

**Write a program that will allow you to input any pair of real numbers and will tell you, *without adding*, whether their sum is positive, negative, or zero. RUN the program for each of the following pairs of numbers.**

| | | | |
|---|---|---|---|
| 1. 10, 2 | 2. $-10$, $-2$ | 3. $-10$, 2 | 4. 10, $-2$ |
| 5. 10, $-10$ | 6. 0, $-10$ | 7. 2, 0 | 8. 0, 0 |

**Write a program that will allow you to input any pair of real numbers and will tell you, *without multiplying*, whether their product is positive, negative, or zero. RUN the program for each of the following pairs of numbers.**

| | | | |
|---|---|---|---|
| 9. 5, 7 | 10. $-5$, 7 | 11. $-5$, $-7$ | 12. $-5$, 0 |

# 2–10 The Multiplicative Inverse of a Real Number

Two numbers whose product is 1 are called **multiplicative inverses,** or **reciprocals,** of each other. For example:

4 and $\frac{1}{4}$ are multiplicative inverses because $4 \times \frac{1}{4} = 1$.
$-\frac{3}{5}$ and $-\frac{5}{3}$ are multiplicative inverses because $(-\frac{3}{5})(-\frac{5}{3}) = 1$.
5 and 0.2 are multiplicative inverses because $5(0.2) = 1$.
1 is its own multiplicative inverse because $1 \times 1 = 1$.
$(-1)$ is its own multiplicative inverse because $(-1)(-1) = 1$.
0 has no multiplicative inverse because the product of 0 and any real number is 0, *not* 1.

The symbol for the multiplicative inverse of a *nonzero* real number $a$ is $\frac{1}{a}$. It is assumed that every real number except 0 has a multiplicative inverse. This assumption is stated formally as the following axiom.

### *Axiom of Multiplicative Inverses*

For every nonzero real number $a$, there is a unique real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1.$$

**EXAMPLE 1** Simplify.

**a.** $(6 \cdot 7)\frac{1}{7}$ **b.** $(-24)\frac{1}{8}$

**SOLUTION** **a.** $(6 \cdot 7)\frac{1}{7} = 6\left(7 \cdot \frac{1}{7}\right) = 6 \cdot 1 = 6$

**b.** $(-24)\frac{1}{8} = (-3 \cdot 8)\frac{1}{8} = (-3)\left(8 \cdot \frac{1}{8}\right) = (-3)(1) = -3$

Example 1 suggests the following theorem, which will be proved in Exercise 51 on page 98.

***Theorem.*** For all real numbers $a$ and all nonzero real numbers $b$,

$$(ab)\frac{1}{b} = a.$$

This theorem can be useful in simplifying variable expressions.

**EXAMPLE 2** Simplify.

**a.** $(-7m^3)\left(-\frac{1}{7}\right)$ **b.** $\frac{1}{5}(45a - 70b)$

**SOLUTION** **a.** $(-7m^3)\left(-\frac{1}{7}\right) = \left[(-7)\left(-\frac{1}{7}\right)\right]m^3 = 1 \cdot m^3 = m^3$

**b.**
$$\begin{aligned}\frac{1}{5}(45a - 70b) &= \frac{1}{5}(45a) - \frac{1}{5}(70b)\\ &= \frac{1}{5}(5 \cdot 9a) - \frac{1}{5}(5 \cdot 14b)\\ &= \left(\frac{1}{5} \cdot 5\right)9a - \left(\frac{1}{5} \cdot 5\right)14b\\ &= (1)9a - (1)14b\\ &= 9a - 14b\end{aligned}$$

You can use the axiom of multiplicative inverses to show that, for every nonzero real number $a$, $(-a)\left(-\frac{1}{a}\right) = 1$. Therefore, $-a$ and $-\frac{1}{a}$ are multiplicative inverses. This fact can be stated formally as follows.

**Theorem.** For all nonzero real numbers $a$,

$$\frac{1}{-a} = -\frac{1}{a}.$$

A proof of this theorem is outlined in Exercise 17 on page 96.

Now consider the following example.

EXAMPLE 3 Simplify.

a. $(4 \cdot 5)\left(\frac{1}{4} \cdot \frac{1}{5}\right)$ b. $(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right)$, $a \neq 0$, $b \neq 0$

SOLUTION a. $(4 \cdot 5)\left(\frac{1}{4} \cdot \frac{1}{5}\right) = \left(4 \cdot \frac{1}{4}\right)\left(5 \cdot \frac{1}{5}\right) = (1)(1) = 1$

b. $(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right) = \left(a \cdot \frac{1}{a}\right)\left(b \cdot \frac{1}{b}\right) = (1)(1) = 1$

Part (b) of Example 3 suggests that, for nonzero real numbers $a$ and $b$, the product of $ab$ and $\frac{1}{a} \cdot \frac{1}{b}$ is 1. Therefore, $\frac{1}{a} \cdot \frac{1}{b}$ is the multiplicative inverse, or reciprocal, of $ab$. This example suggests the following property, which will be proved as a theorem in Exercise 18 on pages 96–97.

### *Property of the Reciprocal of a Product*

The reciprocal of a product of nonzero real numbers is equal to the product of the reciprocals of the numbers. That is, for all nonzero real numbers $a$ and $b$,

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}.$$

EXAMPLE 4 Simplify.

a. $\frac{1}{4} \cdot \frac{1}{-5}$ b. $-48v\left(\frac{1}{4v}\right)$, $v \neq 0$

SOLUTION a. $\frac{1}{4} \cdot \frac{1}{-5} = \frac{1}{4(-5)} = \frac{1}{-20} = -\frac{1}{20}$

b.
$$\begin{aligned} -48v\left(\frac{1}{4v}\right) &= [(-12)4v]\left(\frac{1}{4} \cdot \frac{1}{v}\right) \\ &= (-12)\left(4 \cdot \frac{1}{4}\right)\left(v \cdot \frac{1}{v}\right) \\ &= (-12)(1)(1) \\ &= -12 \end{aligned}$$

## Oral Exercises

**State the multiplicative inverse of each number.**

1. $7$
2. $-7$
3. $-1$
4. $1$
5. $\frac{1}{8}$
6. $-\frac{3}{8}$
7. $-\frac{9}{8}$
8. $1\frac{5}{8}$
9. $-0.3$
10. $1.7$
11. $s,\ s \neq 0$
12. $-t,\ t \neq 0$
13. $-\frac{1}{x},\ x \neq 0$
14. $\frac{3}{y},\ y \neq 0$
15. $\frac{z}{2},\ z \neq 0$
16. $-\frac{u}{v},\ u \neq 0,\ v \neq 0$

**Replace each _?_ with the reason that justifies the statement to its left.**

17. Prove: For all nonzero real numbers $a$, $\frac{1}{-a} = -\frac{1}{a}$.

*Plan*: If it can be shown that $(-a)\left(-\frac{1}{a}\right) = 1$, then by the axiom of multiplicative inverses $-\frac{1}{a}$ is the multiplicative inverse of $-a$ and is therefore equal to $\frac{1}{-a}$.

PROOF

| *Statements* | *Reasons* |
|---|---|
| 1. $a$ is a nonzero real number. | _?_ |
| 2. $\frac{1}{a}$ is a real number. | _?_ |
| 3. $-a$ and $-\frac{1}{a}$ are real numbers. | _?_ |
| 4. $(-a)\left(-\frac{1}{a}\right) = a \cdot \frac{1}{a}$ | Property of opposites in products |
| 5. $= 1$ | _?_ |
| 6. $(-a)\left(-\frac{1}{a}\right) = 1$ | _?_ |
| 7. $-\frac{1}{a} = \frac{1}{-a}$ | Axiom of multiplicative inverses |
| 8. $\therefore \frac{1}{-a} = -\frac{1}{a}$ | _?_ |

18. Prove: For all nonzero real numbers $a$ and $b$, $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

*Plan*: If it can be shown that $(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$, then by the axiom of multiplicative inverses $\left(\frac{1}{a} \cdot \frac{1}{b}\right)$ is the multiplicative inverse of $ab$ and is therefore equal to $\frac{1}{ab}$.

PROOF

| Statements | Reasons |
|---|---|
| 1. $a$ and $b$ are nonzero real numbers. | ? |
| 2. $\frac{1}{a}$ and $\frac{1}{b}$ are real numbers. | Axiom of multiplicative inverses |
| 3. $(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right) = \left(a \cdot \frac{1}{a}\right)\left(b \cdot \frac{1}{b}\right)$ | ? |
| 4. $= 1 \cdot 1$ | ? |
| 5. $= 1$ | Identity axiom for multiplication |
| 6. $(ab)\left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$ | Transitive axiom of equality |
| 7. $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ | ? |
| 8. $\therefore \frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$ | ? |

## Written Exercises

**Simplify.**

A

**1.** $\frac{1}{2} \cdot \frac{1}{9}$

**2.** $\frac{1}{3} \cdot \frac{1}{-5}$

**3.** $\frac{1}{-7} \cdot \frac{1}{4}$

**4.** $\frac{1}{-8} \cdot \frac{1}{-3}$

**5.** $\frac{1}{5}(-75)$

**6.** $-\frac{1}{4}(92)$

**7.** $-72\left(-\frac{1}{3}\right)$

**8.** $-100\left(\frac{1}{25}\right)$

**9.** $\frac{1}{12}(-9)(-4)$

**10.** $-5\left(-\frac{1}{10}\right)(-16)$

**11.** $-56\left(-\frac{1}{7}\right)\left(-\frac{1}{8}\right)$

**12.** $\frac{1}{5}(-70)\left(-\frac{1}{14}\right)$

**13.** $-\frac{1}{9}(-54)\left(-\frac{1}{2}\right)$

**14.** $-\frac{1}{3}\left(-\frac{1}{4}\right)(96)$

**15.** $-\frac{1}{8}(-8jk)$

**16.** $7gh\left(-\frac{1}{7}\right)$

**17.** $-\frac{1}{3}(81a^3)$

**18.** $-\frac{1}{4}(-56c^2)$

**19.** $\frac{1}{n}(-18mn),\ n \neq 0$

**20.** $-\frac{1}{y}(-24xyz),\ y \neq 0$

**21.** $\frac{1}{3p}(-12pq),\ p \neq 0$

**22.** $28rs^3\left(-\frac{1}{2r}\right),\ r \neq 0$

**23.** $-\frac{1}{4g}(-60gh^2)\left(\frac{1}{3}\right),\ g \neq 0$

**24.** $-\frac{1}{2}(-84cd)\left(-\frac{1}{6c}\right),\ c \neq 0$

**25.** $\frac{1}{3}(-15a + 33)$

**26.** $\frac{1}{4}(-36b - 52)$

**27.** $-\frac{1}{6}(24c - 36d)$

**28.** $(-48g + 88h)\left(-\frac{1}{8}\right)$

**29.** $-35\left(\frac{1}{5}j + \frac{1}{7}k\right)$

**30.** $\left(-\frac{1}{2}m - \frac{1}{3}n\right)(-42)$

**Simplify.**

**B** 31. $\frac{1}{9}(-45w - 9z) + \frac{1}{4}(44w - 32z)$

32. $\frac{1}{4}(20x - 8y) + \frac{1}{3}(-27x - 30y)$

33. $12\left(\frac{1}{2}s - \frac{1}{3}t\right) - 14\left(-\frac{1}{7}s + \frac{1}{2}t\right)$

34. $-3\left(\frac{1}{3}c + \frac{1}{3}d\right) - 10\left(\frac{1}{2}c - \frac{1}{5}d\right)$

35. $-\frac{1}{5}(5m^2 + 15m - 40) + \frac{1}{2}(-8m^2 - 6m - 30)$

36. $\frac{1}{6}(12n^2 - 6n + 18) - \frac{1}{3}(-12 + 6n - 6n^2)$

37. $-\frac{1}{4}(16w^2 - 12w) + \frac{1}{3}(6w^2 - 3w) - \frac{1}{2}(8w^2 + 4w)$

38. $\frac{1}{2}(-4z^3 - 2z^2) - \frac{1}{6}(6z^3 - 12z^2) + \frac{1}{3}(9z^3 - 15z^2)$

39. $-\frac{1}{2}[3(2r - 4s)] - \frac{1}{3}[2(6r + 9s)] + \frac{1}{4}[5(4r - 8s)]$

40. $\frac{1}{4}[2(-2j + 6k) + 6(2j - 4k) - 3(4j + 12k)]$

41. $18\left[\frac{1}{3}(5a - 2b)\right] - 12\left[\frac{1}{2}(3a + 7b)\right] - 16\left[\frac{1}{4}(2a - b)\right]$

42. $-6\left[\frac{1}{3}(p - 2q) + \frac{1}{2}(5p + 3q) - \frac{1}{6}(7p - q)\right]$

**Solve.**

43. $\frac{1}{n} = -\frac{1}{4}$ 44. $\frac{1}{n} = -5$ 45. $\frac{1}{n} = 2\frac{1}{2}$ 46. $\frac{1}{n} = 1.75$

47. $\frac{1}{n} = 0$ 48. $\frac{1}{n} = n$ 49. $\frac{1}{n} = -n$ 50. $\frac{1}{n} = n^2$

**Write a direct proof of each theorem.**

**C** 51. For all real numbers $a$ and all nonzero real numbers $b$, $(ab)\left(\frac{1}{b}\right) = a$.

52. For all real numbers $b$ and all nonzero real numbers $a$, $(ab)\left(\frac{1}{a}\right) = b$.

53. For all real numbers $a$ and all nonzero real numbers $b$, $-\frac{1}{b}(ab) = -a$.

54. For all real numbers $b$ and all nonzero real numbers $a$, $-\frac{1}{a}(-ab) = b$.

55. For all nonzero real numbers $a$ and $b$, $-\frac{1}{a}\left(-\frac{1}{b}\right) = \frac{1}{ab}$.

56. For all nonzero real numbers $a$ and $b$, $\frac{1}{a}\left(-\frac{1}{b}\right) = -\frac{1}{ab}$.

# 2–11 Dividing Real Numbers

The first column below lists a few examples of dividing by 3.

The second column lists related examples of multiplying by $\frac{1}{3}$.

| Dividing by 3 | Multiplying by $\frac{1}{3}$ |
|---|---|
| $3 \div 3 = 1$ | $3 \times \frac{1}{3} = 1$ |
| $6 \div 3 = 2$ | $6 \times \frac{1}{3} = 2$ |
| $9 \div 3 = 3$ | $9 \times \frac{1}{3} = 3$ |
| $12 \div 3 = 4$ | $12 \times \frac{1}{3} = 4$ |

Comparing the entries in the two columns shows that *dividing* by 3 gives the same result as *multiplying by the reciprocal* of 3. This relationship between multiplication and division suggests the following definition.

### *Definition of Division*

For all real numbers $a$ and all *nonzero* real numbers $b$, the **quotient** denoted as $a \div b$ is defined by:

$$a \div b = a \cdot \frac{1}{b}$$

That is, to divide $a$ by $b$, multiply $a$ by the reciprocal of $b$.

Note that a quotient is often represented as a fraction.

$$a \div b = \frac{a}{b}$$

You can use the definition of division to replace any quotient with a product. For example:

$$\frac{56}{7} = 56 \times \frac{1}{7} = 8$$

$$\frac{56}{-7} = 56\left(-\frac{1}{7}\right) = -8$$

$$\frac{-56}{7} = -56 \times \frac{1}{7} = -8$$

$$\frac{-56}{-7} = -56\left(-\frac{1}{7}\right) = 8$$

Division of real numbers is *not* commutative. For example:

$$12 \div 3 = 4 \qquad \text{but} \qquad 3 \div 12 = 0.25$$

Nor is division of real numbers associative. For example:

$$(36 \div 6) \div 2 = 6 \div 2 = 3 \qquad \text{but} \qquad 36 \div (6 \div 2) = 36 \div 3 = 12$$

On the other hand, note the following.

$$\frac{6 + 15}{3} = \frac{21}{3} = 7 \qquad \text{and} \qquad \frac{6}{3} + \frac{15}{3} = 2 + 5 = 7$$

$$\frac{20 - 8}{2} = \frac{12}{2} = 6 \qquad \text{and} \qquad \frac{20}{2} - \frac{8}{2} = 10 - 4 = 6$$

These examples illustrate the fact that *division is distributive with respect to both addition and subtraction*, which will be proved in Exercises 42 and 44 on page 102.

For all real numbers $a$, $b$, and $c$ such that $c \neq 0$,

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \quad \text{and} \quad \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}.$$

Why can you never divide by zero? Using the relationship between division and multiplication, dividing by 0 would mean multiplying by the reciprocal of 0. As you have seen, 0 has no reciprocal. Therefore, *division by zero has no meaning in the set of real numbers.*

You cannot divide zero by zero, but can you divide zero by any other number? Consider these examples:

$$\frac{0}{7} = 0 \times \frac{1}{7} = 0 \qquad \frac{0}{-4} = 0\left(-\frac{1}{4}\right) = 0$$

*When zero is divided by any nonzero number, the quotient is zero.* That is, for any nonzero real number $a$,

$$0 \div a = 0 \cdot \frac{1}{a} = 0.$$

EXAMPLE Simplify.

a. $-40 \div \frac{1}{8}$ b. $\frac{45a}{-3}$ c. $8z^2 \div \left(-\frac{2}{3}\right)$

SOLUTION

a. $-40 \div \frac{1}{8} = -40 \times 8 = -320$

b. $\frac{45a}{-3} = 45a\left(-\frac{1}{3}\right) = \left[45\left(-\frac{1}{3}\right)\right]a = -15a$

c. $8z^2 \div \left(-\frac{2}{3}\right) = 8z^2\left(-\frac{3}{2}\right) = \left[8\left(-\frac{3}{2}\right)\right]z^2 = -12z^2$

## *Oral Exercises*

**Simplify.**

1. $\frac{20}{4}$
2. $\frac{20}{-4}$
3. $\frac{-4}{20}$
4. $\frac{-4}{-20}$
5. $\frac{0}{12}$
6. $\frac{0}{-12}$
7. $\frac{12}{-12}$
8. $\frac{-12}{-12}$
9. $\frac{x}{-1}$
10. $\frac{-x}{-1}$
11. $\frac{-x}{-x}, x \neq 0$
12. $\frac{x}{-x}, x \neq 0$

**Read each quotient as a product. Then simplify.**

13. $8 \div \frac{1}{4}$
14. $16 \div \left(-\frac{1}{4}\right)$
15. $-12 \div \frac{3}{4}$
16. $-1 \div \left(-\frac{1}{10}\right)$
17. $-9a \div \frac{1}{3}$
18. $-12b \div \left(-\frac{2}{3}\right)$
19. $6 \div \frac{1}{c}, c \neq 0$
20. $7d \div (-d), d \neq 0$

**Replace each _?_ with one of the words *positive*, *negative*, or *zero* to make a true statement.**

21. The quotient when a positive number is divided by a negative number is always _?_.
22. The quotient when a negative number is divided by a negative number is always _?_.
23. The quotient when zero is divided by a negative number is always _?_.
24. In a quotient, the divisor can never be _?_.

## *Written Exercises*

**Simplify.**

A

1. $160 \div (-5)$
2. $-124 \div (-4)$
3. $-600 \div 24$
4. $0 \div (-18)$
5. $-15 \div \left(-\frac{1}{5}\right)$
6. $18 \div \left(-\frac{1}{3}\right)$
7. $\frac{1}{7} \div \left(-\frac{3}{7}\right)$
8. $-\frac{3}{8} \div \left(-\frac{2}{5}\right)$
9. $\frac{87jk}{-3}$
10. $\frac{-95m^2}{-5}$
11. $\frac{-108n}{9n}, n \neq 0$
12. $\frac{156pq}{-13p}, p \neq 0$
13. $-68w^3 \div \frac{1}{4}$
14. $-22uv \div \left(-\frac{1}{11}\right)$
15. $-\frac{7}{8}r \div \left(-\frac{1}{16}\right)$
16. $\frac{4}{5}st \div \left(-\frac{1}{10}\right)$

**The *average* of a set of numbers is the quotient when the sum of the numbers is divided by the number of numbers in the set. Find the average of each set of numbers.**

17. $\{42, 17, 51, 37, 48\}$
18. $\{-29, 14, 3, -11, -19, 24\}$
19. $\{32, -47, -16, 19, 50, -38\}$
20. $\{5, -1, 3, -4, 1, 1, -7, -2\}$

**Evaluate each expression when $x = -3$, $y = -1$, $z = 2$, and $w = 6$.**

21. a. $\frac{xy}{w}$ b. $\frac{-xy}{-w}$

22. a. $\frac{-x}{wz}$ b. $\frac{x}{-wz}$

23. a. $\frac{x+z}{y}$ b. $\frac{-x-z}{-y}$

24. a. $\frac{x-w}{y}$ b. $\frac{w-x}{y}$

25. a. $\frac{y+w}{x-z}$ b. $\frac{w+y}{z-x}$

26. a. $\frac{x-y}{z-w}$ b. $\frac{y-x}{w-z}$

27. a. $\frac{-x-w}{z-y}$ b. $\frac{x+w}{y-z}$

28. a. $\frac{-w-x}{y-z}$ b. $-\frac{x+w}{z-y}$

**Evaluate each expression when $a = -4$, $b = -2$, $c = -1$, and $d = 6$.**

**B** 29. $\frac{a^2}{-c}$

30. $\frac{d^2}{a}$

31. $\frac{b^3}{-a}$

32. $\frac{c^3}{b}$

33. $\frac{(b+c)^2}{-d}$

34. $\frac{(b-c)^2}{-ac}$

35. $\frac{a^2+b^2}{-a-c}$

36. $\frac{c^2-d^2}{a+b+c}$

37. $\frac{d^2-6c}{a-2}$

38. $\frac{5a-b^2}{-c-4}$

39. $\frac{3a-7b}{c^2+3}$

40. $\frac{-4b-9c}{b^2+1}$

**Give the reason that justifies each statement in the given proof.**

41. Prove: For all real numbers $a$ and $b$ such that $b \neq 0$, $(ab) \div b = a$.

PROOF

1. $a$ and $b$ are real numbers such that $b \neq 0$.
2. $(ab) \div b = (ab) \cdot \frac{1}{b}$
3. $= a\left(b \cdot \frac{1}{b}\right)$
4. $= a \cdot 1$
5. $= a$
6. $\therefore (ab) \div b = a$

42. Prove: For all real numbers $a$, $b$, and $c$ such that $c \neq 0$, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

PROOF

1. $a$, $b$, and $c$ are real numbers such that $c \neq 0$.
2. $\frac{a+b}{c} = (a+b)\frac{1}{c}$
3. $= a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$
4. $= \frac{a}{c} + \frac{b}{c}$
5. $\therefore \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

**Write a direct proof of each theorem.**

**C** 43. For all real numbers $a$ and $b$ such that $b \neq 0$, $(a \div b)b = a$.

44. For all real numbers $a$, $b$, and $c$ such that $c \neq 0$, $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.

45. For all real numbers $a$ such that $a \neq 0$, $\frac{a}{a} = 1$.

46. For all real numbers $a$ such that $a \neq 0$, $\frac{-a}{a} = -1$.

47. For all real numbers $a$ and $b$ such that $b \neq 0$, $\frac{a+b}{b} = 1 + \frac{a}{b}$.

48. For all real numbers $a$ and $b$ such that $a \neq 0$, $\frac{a-b}{a} = 1 - \frac{b}{a}$.

**A set of numbers is said to be *closed under division* if all possible quotients when a member of the set is divided by a nonzero member of the set are also members of the set. Tell whether each set is closed or not closed under division.**

49. $\{1\}$

50. $\{-1\}$

51. $\{-1, 0, 1\}$

52. $\{\frac{1}{2}, 1, 2\}$

53. {the natural numbers}

54. {the integers}

55. {the positive real numbers}

56. {the negative real numbers}

57. {the rational numbers}

58. {the real numbers}

## Computer Exercises For students with computer experience

1. Write a program that will print the average of any two real numbers that you input.

2. Modify the program that you wrote for Exercise 1 to allow you to input two *or more* real numbers.

3. Write a program that will allow you to input the coordinates of any two points on a number line and will print the coordinate of the point that is located exactly halfway between the given points.

## Self-Test 3

VOCABULARY identity element for multiplication (p. 88)
multiplicative inverse or reciprocal (p. 93)
division (p. 99)
quotient (p. 99)

**Simplify.**

1. $(-2)^2(-11)$
2. $(-7x)(-9x)$
3. $7(-y + 1) - 5y$
4. $4z^3 - (9z + 3z^3)$

*Obj. 1, p. 88*

5. $\frac{1}{-7} \cdot \frac{1}{-3}$
6. $\frac{1}{6}(-90w)$
7. $-\frac{1}{2}(14p - 2q)$

*Obj. 2, p. 88*

8. $65 \div (-5)$
9. $-\frac{4}{5} \div \left(-\frac{1}{2}\right)$
10. $\frac{-32n}{8}$

*Obj. 3, p. 88*

**Check your answers with those at the back of the book.**

# Chapter Summary

1. The *additive inverse*, or the *opposite*, of the real number $a$ is denoted by $-a$. The positive number of any pair of opposite nonzero real numbers is called the *absolute value* of each number in the pair. The absolute value of the real number $a$ is denoted by $|a|$.

2. The *multiplicative inverse* of the real number $a$ is denoted by $\frac{1}{a}$.

3. *Axioms*, or *postulates*, are statements about the real numbers that are assumed to be true. The following axioms are assumed to be true for all real values of each variable except as noted.

| | | |
|---|---|---|
| *Axioms of equality* | *Reflexive axiom*: $a = a$<br>*Symmetric axiom*: If $a = b$, then $b = a$.<br>*Transitive axiom*: If $a = b$ and $b = c$, then $a = c$. | |
| | *Addition* | *Multiplication* |
| *Axioms of closure* | $a + b$ is a unique real number. | $ab$ is a unique real number. |
| *Commutative axioms* | $a + b = b + a$ | $ab = ba$ |
| *Associative axioms* | $(a + b) + c = a + (b + c)$ | $(ab)c = a(bc)$ |
| *Identity axioms* | $a + 0 = a$ and $0 + a = a$ | $a \cdot 1 = a$ and $1 \cdot a = a$ |
| *Axioms of inverses* | $a + (-a) = 0$ and $-a + a = 0$ | $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$, if $a \neq 0$ |
| *Distributive axiom* | $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ | |

4. *Theorems* are statements about the real numbers that can be proved to be true. The *hypothesis* of a theorem states what is assumed to be true, and the *conclusion* states something that follows logically from this assumption. By reasoning from the hypothesis to the conclusion, you can give a *direct proof* of a theorem. The following properties of the real numbers are theorems that can be proved to be true for all real values of each variable except as noted.

   *Cancellation property of opposites*: $-(-a) = a$
   *Property of the opposite of a sum*: $-(a + b) = -a + (-b)$

| | |
|---|---|
| *Multiplicative property of zero:* | $a \cdot 0 = 0$ and $0 \cdot a = 0$ |
| *Multiplicative property of −1:* | $a(-1) = -a$ and $(-1)a = -a$ |
| *Property of opposites in products:* | $-a(b) = -ab$; $a(-b) = -ab$; $-a(-b) = ab$ |
| *Property of the reciprocal of a product:* | $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$, if $a \neq 0$, $b \neq 0$ |

5. A number line can be used to find the sum of two real numbers.

6. Subtraction and division are defined as follows:

$$a - b = a + (-b) \qquad a \div b = a \cdot \frac{1}{b}, \text{ if } b \neq 0$$

## Chapter Review

**Write the letter of the correct answer.**

1. Which of the following statements is false? *2-1*
   a. For each whole number $n$, $n^2 + 1 > 0$.
   b. There exists an integer $k$ such that $k < 0$.
   c. For every natural number $z$, $z^2 > 1$.
   d. For some real number $c$, $c + 3 < 4$.

2. Name the axiom of equality that is illustrated by the following: *2-2*
   "If $a + 5 = b$ and $b = 2a$, then $a + 5 = 2a$."
   a. reflexive
   b. symmetric
   c. transitive
   d. It illustrates none of the axioms; the conclusion is false.

**Simplify.**

3. $(3k + 8) + (4m + 7)$
   a. $7km + 15$ b. $12km + 15$ c. $11k + 11m$ d. $3k + 4m + 15$

4. $5p + 9q + 7p$ *2-3*
   a. $21pq$ b. $35p + 9q$ c. $12p + 9q$ d. $12p^2 + 9q$

5. $5y + 3(y + 2) + 10$
   a. $8y + 16$ b. $8y + 12$ c. $10y + 10$ d. $16y^2 + 10$

6. $-(117 - 49)$ *2-4*
   a. 166 b. −166 c. 68 d. −68

7. $9|-8| - 2|-7|$
   a. −58 b. 58 c. −86 d. 86

8. $-4 + [3 + (-5)]$ *2-5*
   a. −6 b. −12 c. 2 d. 6

9. Solve $-2 + u = 11$.

a. $\{-9\}$  b. $\{9\}$  c. $\{13\}$  d. $\{-13\}$

10. Simplify $-17 + 28 + (-56) + 19$. 2-6

a. $-48$  b. $-26$  c. $86$  d. $26$

11. The temperature at the end of one school day was 6°C. The temperature fell 10°C overnight, rose 15°C by noon, then fell 10°C by the end of the school day. What was the temperature at the end of the second school day?

a. 15°C  b. 11°C  c. 1°C  d. $-11$°C

12. Give the reason that justifies the following statement: 2-7
$-(x + y) + x = -(y + x) + x$

a. property of the opposite of a sum
b. cancellation property of opposites
c. associative axiom for addition
d. commutative axiom for addition

**Simplify.**

13. $-38 - (-14)$ 2-8

a. $-52$  b. $52$  c. $24$  d. $-24$

14. $(14 - 2a) - (5 - 6a)$

a. $9 - 8a$  b. $9 - 4a$  c. $9 + 4a$  d. $19 + 4a$

15. $(-3v)(2v)(-5v)$ 2-9

a. $-6v$  b. $-6v^3$  c. $-30v^3$  d. $30v^3$

16. $-4m + n - (5m + 3n)$

a. $-9m + 4n$  b. $-9m^2 + 4n^2$  c. $m - 2n$  d. $-9m - 2n$

17. $\frac{1}{16}(-12)(-20)$ 2-10

a. $15$  b. $2$  c. $-2$  d. $-15$

18. $\left(\frac{1}{4}a - \frac{1}{3}b\right)(-48)$

a. $12a - 16b$  b. $-12a + 16b$  c. $4ab$  d. $-4ab$

19. $\frac{-36d}{-4}$ 2-11

a. $-9d$  b. $-9$  c. $9d$  d. $9$

20. $-\frac{4}{5}h \div \left(-\frac{3}{10}\right)$

a. $\frac{8}{3}h$  b. $-\frac{8}{3}h$  c. $\frac{3}{8}h$  d. $-\frac{3}{8}h$

## Chapter Test

1. Find a value of the variable that makes this statement true: "There exists a natural number $r$ such that $r(r - 5) = 0$." 2-1

2. Name the axiom that is illustrated by the following: $(g + 8) + 11 = g + (8 + 11)$ 2-2

**Simplify.**

3. $12 + 46 + 18 + 94$
4. $(2a)(16b)(5c)$
5. $5c^2 + 6c + 8c^2 + 9 + 12c$
6. $4(2d + 1) + 3(5d + 2)$ 2-3
7. $-[-(74 - 58)]$
8. $|-11| - |7 - 3|$ 2-4
9. $-14 + 3$
10. $[5 + (-8)] + (-7)$ 2-5

11. Solve $12 + t = 4$.

12. Simplify $-[47 + (-10)] + (-35) + 66$. 2-6

13. In the first four months of this year, Tami's baseball card collection increased by 12 cards, decreased by 19 cards, increased by 3 cards, and then decreased by 8 cards. If she had 185 cards at the end of that time, how many cards did she have at the beginning of the year?

14. Give the reason that justifies the following statement: $(x + y) + (-x) = (y + x) + (-x)$ 2-7

**Simplify.**

15. $33 - (-9 + 7)$
16. $8w - (5w - 3) - (2 - 7w)$ 2-8
17. $(-4t)(3t)(-6t)$
18. $(-5r - s)(-9) - 10s$ 2-9
19. $-\frac{1}{2}\left(-\frac{2}{3}\right)(-30)$
20. $-\frac{1}{3}(-6p - 15q)$ 2-10
21. $\frac{-48xy}{16x}$
22. $\frac{3}{8}ab \div \left(-\frac{6}{5}\right)$ 2-11

## Mixed Review

**Simplify.**

1. $-43 + 25$
2. $18 - (-56)$
3. $(-28)(-5)$
4. $-26 \div 4$
5. $-15a - a$
6. $-2b + b$
7. $-18c \div \left(-\frac{1}{2}\right)$
8. $(6d)(-9d)$
9. $3.26 + 4.3 + 2.74 + 6.7$
10. $16 - 28 + 42 - 64 + 87$
11. $(11 - 3)9 \div 3(2) - 2$
12. $(8^2 - 2^3)2 - 2(6 - 2)^2 + 12$

**Solve if $n \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.**

13. $9 - n = 16$
14. $2n + 7 = 15$
15. $3n = 4$
16. $2 = \frac{n}{2}$
17. $n < -2$
18. $n + 1 > 5$
19. $-n \geq 0$
20. $|n| < 2$

**Evaluate each expression when $a = -2$, $b = 3$, and $c = 10$.**

21. $-a(b - c)$
22. $\frac{a - b}{c}$
23. $a^2 - 2bc$
24. $\frac{5a - b^2}{-c - 9}$

**Graph each inequality on a number line.**

25. $x \leq 3$
26. $x + 2 > 5$
27. $-1 < x < 4$
28. $|x| > 4$

**Tell whether each statement is true or false.**

29. {the natural numbers} = {the whole numbers greater than 0}
30. {the irrational numbers} = {the real numbers that are not rational}
31. {the rational numbers} $\not\subset$ {the integers}
32. {the integers} $\subset$ {the natural numbers}

33. Write a variable expression for the following word phrase: the total value in cents of $n$ nickels, $q$ quarters, and $x$ dollars.
34. Write a mathematical sentence for the following word sentence: The product when the sum of a number $x$ and two is multiplied by six is not equal to the square of $x$.
35. An airplane on a test flight was flying at an altitude of 9200 m. The airplane dropped 800 m, leveled off, rose 1100 m, and then dropped 500 m. What was its new altitude?
36. The temperature at 6:00 one morning was −3°C, which was 9°C lower than the temperature at midnight the night before. What had the temperature been at midnight?

## ON THE CALCULATOR

**Use a calculator to find the reciprocal of each number. You may wish to use the reciprocal key if the calculator has one.**

1. 16
2. −1250
3. −0.0625
4. 0.00032

**Use a calculator to evaluate each expression when $a = 8$, $b = -12$, $c = 5$, and $d = -4$.**

5. $\frac{1}{a} \cdot \frac{1}{c}$
6. $\frac{1}{a} + \frac{1}{d}$
7. $\frac{1}{a + d}$
8. $\frac{1}{bc + d}$

# PREPARING FOR COLLEGE ENTRANCE EXAMS

**Strategy for Success:** When you are taking a college entrance exam, it is important to work quickly, but not so quickly that you lose accuracy. In particular, be sure to read the directions, the questions, and the answer choices very carefully. You may wish to underline important words such as *not, exactly, false, never,* and *except*. Cross out answer choices that are clearly incorrect. Mark the answer sheet carefully, and check the numbering after every few questions to avoid misplaced markings.

**Decide which is the best of the choices given and write the corresponding letter on your answer sheet.**

1. Name the coordinate of the point that is halfway between the points whose coordinates are $-3$ and $4$ on a number line.
   **(A)** $-1$ **(B)** $1$ **(C)** $-3\frac{1}{2}$ **(D)** $-\frac{1}{2}$ **(E)** $\frac{1}{2}$
2. Which of the following statements must be true?
   I. $\emptyset = \{0\}$ II. $5 \notin \{2, 4, 6\}$ III. $\{1, 3, 5, 7\} \subset \mathcal{R}$
   **(A)** I only **(B)** II only **(C)** III only **(D)** II and III only **(E)** I, II, and III
3. Which of the following statements must be true?
   I. $2^5 > 5^2$ II. $10^3 = 30$ III. $3^4 = 81$
   **(A)** I only **(B)** II only **(C)** III only **(D)** I and III only **(E)** I, II, and III
4. Which word sentence is represented by $5n + 10d = 90$?
   **(A)** The total value of five nickels and ten dimes is ninety cents.
   **(B)** The total number of five nickels and ten dimes is ninety coins.
   **(C)** The total value of $n$ nickels and $d$ dimes is ninety cents.
   **(D)** The total value of $n$ nickels and $d$ dimes is ninety dollars.
   **(E)** The total number of $n$ nickels and $d$ dimes is ninety coins.
5. Solve $-16 = 8 + 8 + a$.
   **(A)** $\{0\}$ **(B)** $\{32\}$ **(C)** $\{-32\}$ **(D)** $\{-16\}$ **(E)** $\mathcal{R}$
6. Simplify $-(-3|-5|) + (-|-7 + 2|) + |-20|$.
   **(A)** 30 **(B)** 0 **(C)** 40 **(D)** 44 **(E)** $-30$
7. For which of the following is $3m - m + 7 = 7 + 2m$ true?
   **(A)** exactly one real number $m$
   **(B)** exactly two real numbers $m$
   **(C)** all real numbers $m$
   **(D)** some, but not all, real numbers $m$
   **(E)** no real numbers $m$
8. An operation $*$ is defined as $a * b = 2a + b + 1$ for all real numbers $a$ and $b$. Which of the following are properties of $*$?
   I. closure II. commutativity III. associativity
   **(A)** I only **(B)** II only **(C)** III only **(D)** I and II only **(E)** I, II, and III

## Temperature Scales

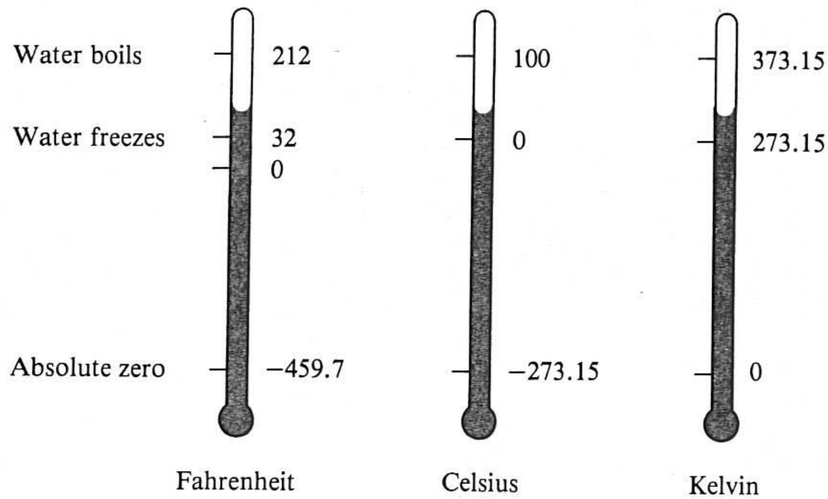
How hot or cold will it be today? Weather forecasters give you this information in terms of temperature readings. *Temperature* is a measure of how hot or cold something is.

Temperature can be measured with a thermometer. A common type of thermometer is one that contains a liquid, usually mercury or alcohol, in a narrow glass tube. The volumes of these liquids increase when the temperature increases and decrease when the temperature decreases. Therefore, when there is a change in temperature there is also a change in the height of the liquid in the thermometer. In order to determine temperature by observing the height of a liquid in a tube, a thermometer must have a *temperature scale* marked on it. A temperature scale can be considered as a type of number line. Different types of thermometers have different scales.

Anders Celsius, a Swedish scientist who lived from 1701 to 1744, devised a temperature scale that today bears his name. Celsius called the temperature at which water freezes 0 degrees and the temperature at which water boils 100 degrees. He divided the space between these two points into 100 equal parts, and each part represented one *degree*. One of these degrees is now referred to as one degree Celsius (1°C). Celsius extended the scale above 100°C and below 0°C, keeping the size of the degree the same. Temperatures below 0°C are indicated by negative numbers.

A few years before the development of the Celsius thermometer, a German instrument maker, Gabriel Daniel Fahrenheit, designed a thermometer using a different scale. He chose the zero point to be the lowest temperature obtainable with a mixture of ice and common salt. The second point he chose was the body temperature of a healthy person. At first he divided the space between these two points into twelve equal divisions. Soon, however, he wanted to be able to take more accurate readings with his thermometer. Therefore, he divided each of the twelve parts into eight equal parts, resulting in 96 divisions between the two fixed points. Each of these divisions represented one degree, which is now referred to as one degree Fahrenheit (1°F). On the Fahrenheit scale, 32°F corresponds to the freezing point of water and 212°F corresponds to the boiling point of water.

Another important temperature scale is the *absolute*, or *Kelvin*, scale. It was named for the 19th century British scientist, Lord Kelvin, and used the concept that there is a lower limit to how cold any substance can be. This "lowest possible temperature" is referred to as *absolute zero*. Both theory and experiment indicate that absolute zero is approximately −273.15°C. On the Kelvin scale, absolute zero is the zero point. The divisions are the same size as on the Celsius scale, but they are called *kelvins* (K) rather than degrees Celsius. On this scale, water freezes at 273.15 K and boils at 373.15 K. This scale is often used in scientific work.




## Exercises

1. A person reads an outside thermometer early one evening and finds the temperature to be 14°C. Early the next morning the same thermometer reads −11°C. How much did the temperature drop overnight?
2. A certain antifreeze solution for an automobile's cooling system freezes at a temperature that is 45°C below the temperature at which water freezes. What is the freezing point of this antifreeze solution in degrees Celsius? in Kelvins?
3. Mercury freezes at a temperature of approximately 234 K and boils at approximately 629 K. What is the temperature range, expressed in degrees Celsius, to which a mercury thermometer is limited?
4. A student made a thermometer and devised a scale for it. The scale was divided into 200 equal parts between the freezing point and the boiling point of water. The freezing point was then assigned the value of 50 degrees, and the boiling point was assigned 250 degrees. What temperature on the Celsius scale corresponds to a reading of 150 degrees on the student's scale? *Hint*: You may want to draw and compare simple diagrams of the two scales.