Chapter 12

Trigonometry and Vectors

Trigonometry

OBJECTIVES for Sections 12-1 through 12-4:
1. To picture an angle as a rotation.
2. To determine the sine, cosine, and tangent of an angle in standard position.
3. To use trigonometric tables to find the sine, cosine, and tangent of an angle.
4. To solve right triangles.

12–1 Angles

When an airport radar antenna revolves, as shown in Figure 1, you can think of its beam as generating an *angle*. For example, since the measure of a right angle is 90°, when the antenna turns through one complete revolution the beam generates an angle of measure 360°. In two revolutions it generates an angle of measure 720°, and so on.

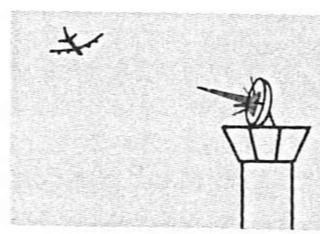


Figure 1

In geometry an **angle** is defined as the union of two rays that have the same endpoint. To develop the extended notion of an angle described above, you can call the ray of the angle that is at the starting position of the generating ray the **initial side** of the angle. The ray of the angle that is at the final position of the generating ray is then called the **terminal side** of the angle. Thus, a **directed angle** is defined as the union of two ordered rays with a common endpoint, called the **vertex** of the angle, *together with a rotation* from the initial side to the terminal side, as shown in Figure 2.

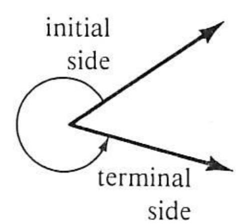


Figure 2

If two or more angles have the same initial side *and* the same terminal side, they are called **coterminal angles.** Figure 3 shows a pair of coterminal angles.

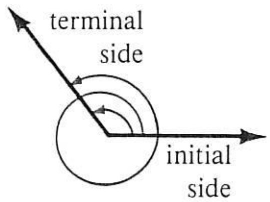


Figure 3

The number of degrees through which a ray rotates in turning from the initial side to the terminal side is the **measure** of the angle. Ordinarily, the measure is considered to be *positive* if the rotation is counterclockwise, and *negative* if the rotation is clockwise. In Figure 4, you can use a *protractor* to verify that $m\angle AOB$ (read "the measure of angle A, O, B") is approximately 145°, and that $m\angle COD$ is approximately −30°.

Figure 4

On a coordinate plane, an angle that has its vertex at the origin and has the positive x-axis as its initial side is said to be in **standard position.** In Figure 5, $\angle A$ and $\angle B$ are in standard position.

Recall from Section 5-1 that the x- and y-axes separate a coordinate plane into four quadrants, which are numbered as shown in Figure 5. The terminal side of an angle in standard position determines the quadrant in which the angle is said to *be*, or to *lie*. For example, $\angle A$ in Figure 5 lies in the fourth quadrant, or Quadrant IV. Similarly, an angle in standard position having a measure of 200° lies in the third quadrant, or Quadrant III.

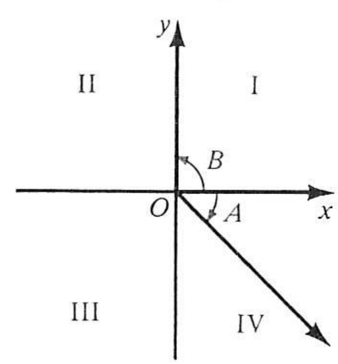
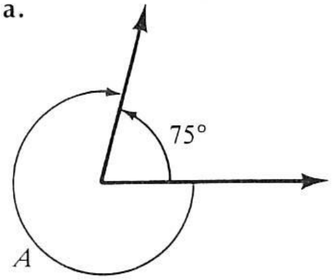


Figure 5

An angle in standard position whose terminal side coincides with an axis is called a **quadrantal angle.** For example, $\angle B$ in Figure 5 is a quadrantal angle; its terminal side lies on the positive y-axis. In fact, any angle in standard position having a measure of 0°, 90°, 180°, 270°, 360°, −90°, −180°, and so on, is a quadrantal angle. A quadrantal angle does *not* lie in any quadrant.

EXAMPLE 1 Determine the measure of $\angle A$ in each figure.

a.



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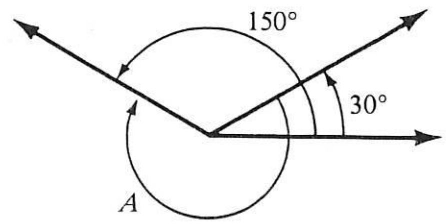

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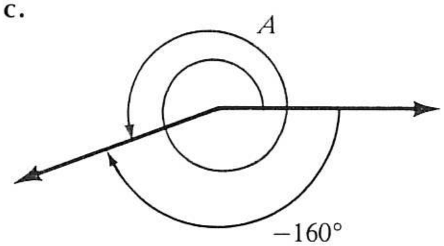
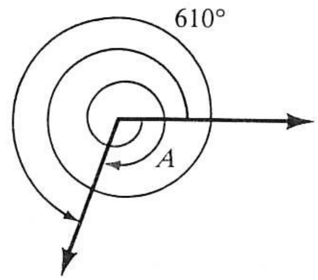


d.



SOLUTION The measures of coterminal angles differ by multiples of 360°.

a. $m\angle A$ is negative.

$$m\angle A = -(360° - 75°) = -285°$$

b. $m\angle A$ is negative.

$$m\angle A = -[360° - (150° - 30°)] = -240°$$

c. $m\angle A$ is positive.

$$m\angle A = (360° \times 2) - 160° = 720° - 160° = 560°$$

d. An angle of 610° is coterminal with an angle of $(610 - 360)°$, or 250°.

$$m\angle A = -[(360° \times 2) - 250°] = -470°$$

EXAMPLE 2 Assume that $\angle A$ is in standard position and that $0° \leq m\angle A < 360°$. Find $m\angle A$ so that each set of conditions is satisfied.

a. $\angle A$ is a quadrantal angle and $100° < m\angle A < 215°$.

b. $\angle A$ is coterminal with an angle measuring 550°.

c. $\angle A$ is coterminal with an angle measuring $-220°$.

SOLUTION a. The terminal side of $\angle A$ lies on an axis between Quadrant II and Quadrant III. Therefore, $m\angle A = 180°$.

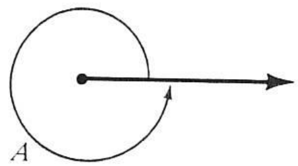
b. $m\angle A = (550 - 360)° = 190°$

c. $m\angle A = (-220 + 360)° = 140°$

Oral Exercises

Find $m\angle A$.

1.



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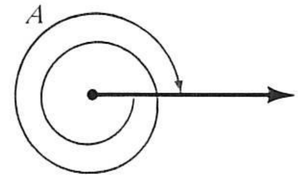

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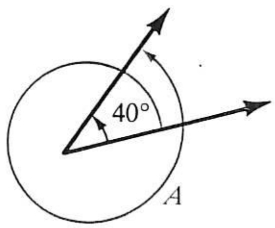

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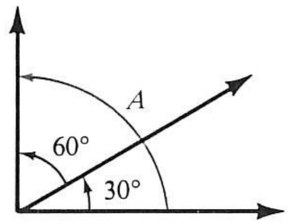

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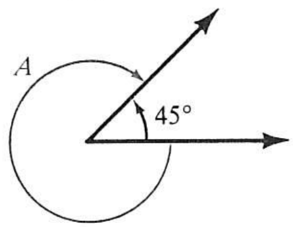

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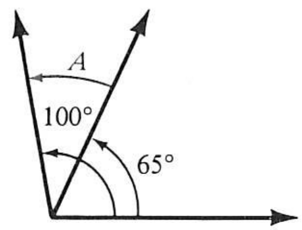


Find $m\angle A$, given that $\angle A$ is coterminal with the angle whose measure is given and $0° \le m\angle A < 360°$.

7. 380° **8.** −40° **9.** −90° **10.** 560° **11.** −300°

12. −10° **13.** −200° **14.** 720° **15.** 410° **16.** 800°

17–26. In Exercises 7–16 above, name the quadrant, if any, in which an angle with the given measure would lie if it were in standard position.

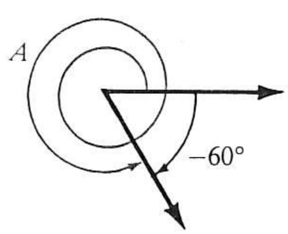
Find the measure of the quadrantal angle A that satisfies the given conditions.

27. $45° < m\angle A < 135°$ **28.** $-200° < m\angle A < -100°$ **29.** $200° < m\angle A < 300°$

Written Exercises

Find $m\angle A$.

A **1.**



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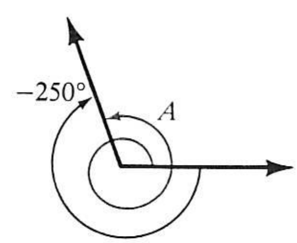

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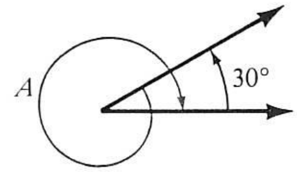

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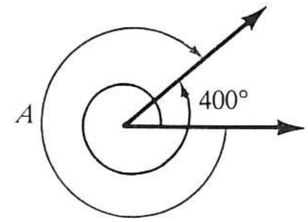

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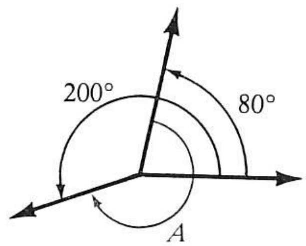

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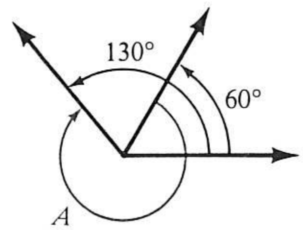

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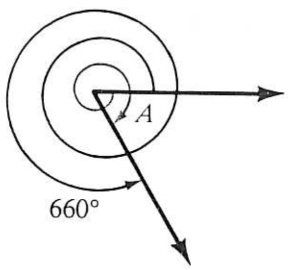

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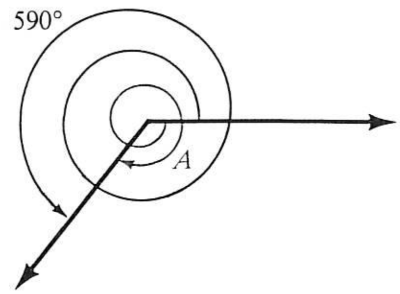


In Exercises 9–16 assume that $\angle A$ is in standard position and that $0° \leq m\angle A < 360°$.

9. If $m\angle A > 180°$ and $\angle A$ is not a quadrantal angle, then $\angle A$ is in either Quadrant _?_ or _?_.
10. If $90° < m\angle A < 180°$, then $\angle A$ is in Quadrant _?_.
11. If $\angle A$ is in Quadrant I, then _?_ $< m\angle A <$ _?_.
12. If $\angle A$ is in Quadrant IV, then _?_ $< m\angle A <$ _?_.
13. If $\angle A$ is a quadrantal angle and $180° < m\angle A < 360°$, then $m\angle A =$ _?_.
14. If $\angle A$ is a quadrantal angle and $50° < m\angle A < 200°$, then either $m\angle A =$ _?_ or $m\angle A =$ _?_.
15. If $\angle A$ is coterminal with an angle measuring $-40°$, then $m\angle A =$ _?_.
16. If $\angle A$ is coterminal with an angle measuring $550°$, then $m\angle A =$ _?_.

B 17. If $\angle A$ is coterminal with an angle measuring $-520°$, and if $400° < m\angle A < 600°$, then $m\angle A =$ _?_.

18. If $\angle A$ is coterminal with an angle measuring $1010°$, and if $-500° < m\angle A < -100°$, then $m\angle A =$ _?_.

In Exercises 19–24 assume that all angles are in standard position.

19. If $180° < |m\angle R| < 270°$, then $\angle R$ is in either Quadrant __?__ or __?__.

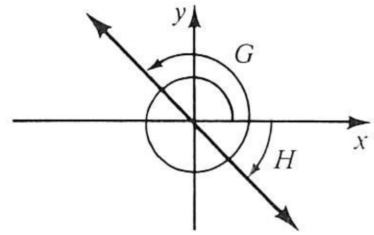
20. If $0° < |m\angle S| < 90°$, then $\angle S$ is in either Quadrant __?__ or __?__.

21. If $|m\angle T| < 90°$ and $\angle T$ is not in Quadrant I or IV, then $m\angle T =$ __?__.

22. If $|90° - m\angle U| < 90°$ and $\angle U$ is not in Quadrant I or II, then $m\angle U =$ __?__.

23. If the terminal sides of $\angle G$ and $\angle H$ are collinear, $m\angle G = 495°$, and $|m\angle H| < 90°$, then $m\angle H =$ __?__.

24. If the terminal sides of $\angle J$ and $\angle K$ are collinear, $m\angle J = -340°$, and $90° < |m\angle K| < 180°$, then $m\angle K =$ __?__.



Ex. 23

C **25.** If $\angle R$ lies in Quadrant IV and $m\angle R = 3m\angle Q$, then $\angle Q$ lies in Quadrant __?__ if $0° < m\angle R < 360°$ and in Quadrant __?__ if $-360° < m\angle R < 0°$.

26. Given that $180° < 5m\angle E < 270°$ and $240° < 6m\angle E < 360°$, then __?__ $< m\angle E <$ __?__.

27. A wagon wheel makes 20 revolutions per minute. Through how many degrees does a spoke of the wheel turn in 1 s? 5 s? 2 min?

28. The beam of a radar antenna sweeps through an angle of 54° each second. How many revolutions does it make in 5 min?

29. The arm of a garden sprinkler rotates counterclockwise at a constant rate. If the arm makes one rotation in 0.1 s, through how many degrees does it turn in 5 s?

30. A carnival ride turns through 24° in 1.5 s. How long does it take the ride to complete 12 revolutions?

Computer Exercises For students with computer experience

1. Write a program that will allow you to input a positive integer n and will compute a positive integer d, $0 < d \leq 360$, such that angles of measure $n°$ and $d°$ are coterminal.

2. Modify the program that you wrote for Exercise 1 so that it will allow you to input a negative integer. The output should still be an integer d such that $0 < d \leq 360$.

3. Write a program that will allow you to input the coordinates of a point and will determine in which quadrant or on which axis the angle lies whose terminal side in standard position passes through the given point. If the angle lies on an axis, the output should specify whether it lies on the positive or negative side of the axis.

12–2 Trigonometric Functions

A circle whose radius is one *unit* long and whose center is at the origin of a coordinate plane is called a **unit circle.** From the distance formula derived in Section 10-6, it follows that a point (x, y) is on the unit circle if and only if $\sqrt{x^2 + y^2} = 1$. Hence an equation of the unit circle is $x^2 + y^2 = 1$.

You can see from Figure 6 that the terminal side of each angle in standard position intersects the unit circle in exactly one point. For example, the terminal side of $\angle AOB$ intersects the circle in point B, and the terminal side of $\angle AOC$ intersects it in point C.

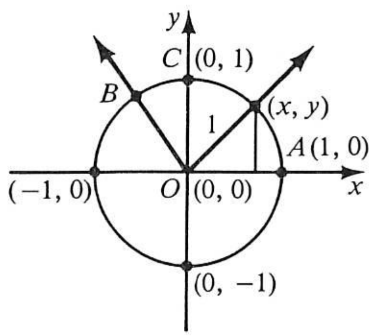


Figure 6

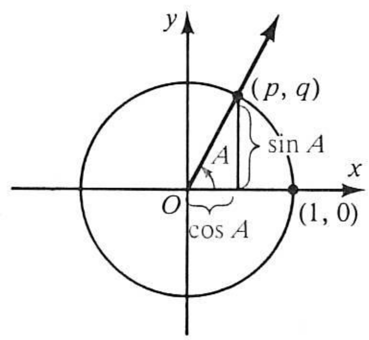


Figure 7

Special names are given to the ordinate and abscissa of the intersection point for each angle and also to the quotient of the ordinate and the abscissa. Namely, if (p, q) is the point of intersection of the unit circle and the terminal side of an angle A in standard position (Figure 7), then

$\sin A = q$ (read "the **sine** of A is equal to q"),
$\cos A = p$ (read "the **cosine** of A is equal to p"),

and, if $p \neq 0$,

$\tan A = \frac{q}{p}$ (read "the **tangent** of A is equal to q divided by p").

These definitions tell you what is meant by the sine, cosine, and tangent of an angle in standard position. However, any angle can be put into standard position by choosing the coordinate system so that the origin is the vertex of the angle and the positive x-axis is the initial side of the angle. Therefore, you can speak of the sine, cosine, and tangent of an angle whether or not it is given in standard position. You should realize that the terminal side of an angle is determined when its initial side and its measure are known.

The notation "tan 45°" means "the tangent of an angle of measure 45°." However, in the notation "tan A," the letter A does not denote a measure, but rather an angle, even though the symbol $\angle$ is omitted.

The definitions of the numbers sin A, cos A, and tan A suggest the following three functions.

$$\text{sine:} \quad \angle A \longrightarrow \sin A$$
$$\text{cosine:} \quad \angle A \longrightarrow \cos A$$
$$\text{tangent:} \quad \angle A \longrightarrow \tan A$$

Since the numbers sin A and cos A are defined for every angle A, the sine and cosine functions have the same domain, namely the set of all angles. Further, since sin A and cos A are coordinates of points on the unit circle, these functions also have the same range, namely the set of all real numbers between -1 and 1, inclusive.

Notice that, for an angle A in standard position, tan A is not defined if the terminal side is on the y-axis because division by zero is not defined. Therefore, the domain of the tangent function is the set of all angles whose degree measure is *not* an odd multiple of 90. The range of the tangent function is the set of all real numbers.

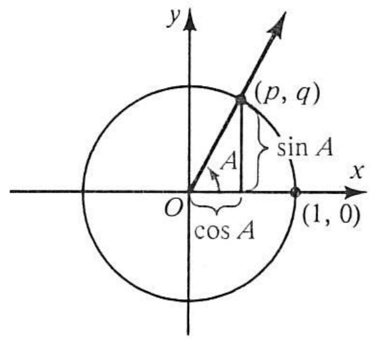


Figure 7

Sine, cosine, and tangent are examples of **trigonometric functions**. The word *trigonometric* comes from two Greek words: *trigonon* (triangle) and *metron* (measure).

The following are two useful trigonometric formulas:

1. For each point (p, q) on the unit circle,

$$q^2 + p^2 = 1.$$

But, as in Figure 7, $q = \sin A$ and $p = \cos A$; therefore

$$(\sin A)^2 + (\cos A)^2 = 1.$$

In place of the symbols $(\sin A)^2$ and $(\cos A)^2$, you ordinarily write $\sin^2 A$ and $\cos^2 A$, respectively. Thus,

$$\sin^2 A + \cos^2 A = 1.$$

2. If $\cos A \neq 0$, then

$$\tan A = \frac{\sin A}{\cos A},$$

because $\tan A = \frac{q}{p}$, and $q = \sin A$ and $p = \cos A$.

EXAMPLE 1 Give the sine, cosine, and tangent of an angle A in standard position whose terminal side contains the given point on the unit circle.

a. $\left(\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\right)$

b. $(-1, 0)$

c. $(0, 1)$

SOLUTION

a. $\sin A = -\frac{2\sqrt{5}}{5}$; $\cos A = \frac{\sqrt{5}}{5}$; $\tan A = \frac{\sin A}{\cos A} = \frac{-\frac{2\sqrt{5}}{5}}{\frac{\sqrt{5}}{5}} = -2$

b. $\sin A = 0$; $\cos A = -1$; $\tan A = \frac{\sin A}{\cos A} = \frac{0}{-1} = 0$

c. $\sin A = 1$; $\cos A = 0$; $\tan A$ is undefined.

When the quadrant of an angle A is given, and also the value of $\sin A$, $\cos A$, or $\tan A$, you can find the remaining two values, using the formulas $\sin^2 A + \cos^2 A = 1$ and $\tan A = \frac{\sin A}{\cos A}$ as needed.

EXAMPLE 2 Find $\cos A$ and $\tan A$, given that angle A lies in Quadrant III, and that $\sin A = -\frac{\sqrt{2}}{2}$.

SOLUTION Substitute $-\frac{\sqrt{2}}{2}$ for $\sin A$ in the formula

$$\sin^2 A + \cos^2 A = 1.$$

$$\left(-\frac{\sqrt{2}}{2}\right)^2 + \cos^2 A = 1$$

$$\cos^2 A = 1 - \frac{2}{4} = \frac{1}{2}$$

$$\cos A = \frac{\sqrt{2}}{2} \quad \text{or} \quad \cos A = -\frac{\sqrt{2}}{2}$$

Since angle A is in Quadrant III, $\cos A$ is negative.

Thus, $\cos A = -\frac{\sqrt{2}}{2}$.

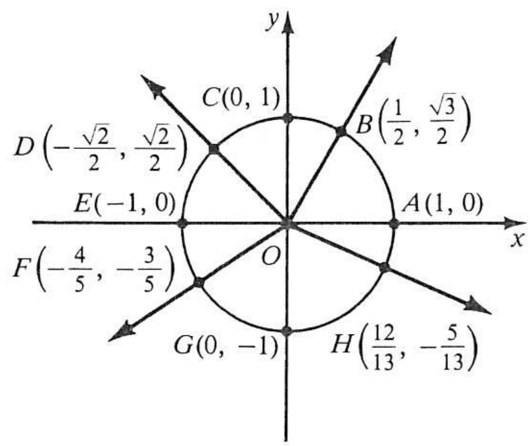
Then $\tan A = \frac{\sin A}{\cos A} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$.

$\therefore \cos A = -\frac{\sqrt{2}}{2}$ and $\tan A = 1$.

Oral Exercises

In the figure below, state the sine, cosine, and tangent (if it exists) of the angle in standard position whose terminal side contains the given point.

1. A
2. B
3. C
4. D
5. E
6. F
7. G
8. H



Use the figure above to state the value (if it exists) of each of the following.

9. $\sin 0°$
10. $\cos 0°$
11. $\sin 90°$
12. $\cos 90°$
13. $\tan 0°$
14. $\tan 90°$
15. $\cos 180°$
16. $\sin 270°$
17. $\cos 360°$
18. $\sin 180°$
19. $\cos 450°$
20. $\tan 180°$
21. $\sin(-90°)$
22. $\cos(-180°)$
23. $\tan(-360°)$
24. $\sin(-540°)$

Explain why there is no angle A for which the given statement is true.

25. $\sin A = 2$
26. $\cos A = -1.5$

Written Exercises

Give the sine, cosine, and tangent of an angle A in standard position whose terminal side contains the given point on the unit circle.

A

1. $\left(\frac{3}{5}, \frac{4}{5}\right)$
2. $\left(\frac{4}{5}, \frac{3}{5}\right)$
3. $\left(-\frac{4}{5}, \frac{3}{5}\right)$
4. $\left(\frac{3}{5}, -\frac{4}{5}\right)$
5. $\left(\frac{5}{13}, \frac{12}{13}\right)$
6. $\left(-\frac{5}{13}, \frac{12}{13}\right)$
7. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
8. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
9. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
10. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
11. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
12. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

In Exercises 13–20 the quadrant of angle A is given, and also the value of $\sin A$, $\cos A$, or $\tan A$. Find the remaining two values.

13. $\cos A = \frac{\sqrt{3}}{2}$; Quadrant I

14. $\sin A = \frac{3}{5}$; Quadrant I

15. $\sin A = -\frac{\sqrt{2}}{2}$; Quadrant IV

16. $\cos A = -\frac{4}{5}$; Quadrant II

17. $\tan A = 1$; Quadrant III

18. $\tan A = -\sqrt{3}$; Quadrant IV

19. $\cos A = -\frac{12}{13}$; Quadrant II

20. $\sin A = -\frac{5}{13}$; Quadrant III

21. Copy and complete the following chart.

	Quadrant I	*Quadrant II*	*Quadrant III*	*Quadrant IV*
sin *A*	positive	?	?	?
cos *A*	?	negative	?	?
tan *A*	?	?	?	?

Find $\sin A$, $\cos A$, and $\tan A$ such that the given conditions are satisfied.

B 22. $\sin A = \cos A$ and $\sin A < 0$

23. $\sin A = \tan A$ and $\cos A > 0$

Using the trigonometric formulas given on page 586, show why there is no angle A for which the given statement is true.

24. $\sin A = 1$ and $\cos A = -1$

25. $\sin A = 0$ and $\cos A = 0$

26. $\sin A < 0$, $\cos A > 0$, and $\tan A > 0$

27. $\sin A < 0$, $\cos A < 0$, and $\tan A < 0$

28. $0 < \tan A < \sin A$

29. $\sin A < \tan A < 0$

In each of Exercises 30–33, k is a positive real number. Find the sine, cosine, and tangent of the angle A in standard position whose terminal side contains the given point. (*Hint*: Find the distance of the given point from the origin. Then solve for "p" and "q" by using the fact that lengths of corresponding sides of similar triangles have equal ratios.)

30. $(3k, 4k)$

31. $(5k, 12k)$

32. $(k, k\sqrt{3})$

33. $(k\sqrt{2}, k\sqrt{2})$

Find the value of k if the terminal side of angle A in standard position contains the point whose coordinates are given. (*Hint:* Use corresponding sides of similar triangles.)

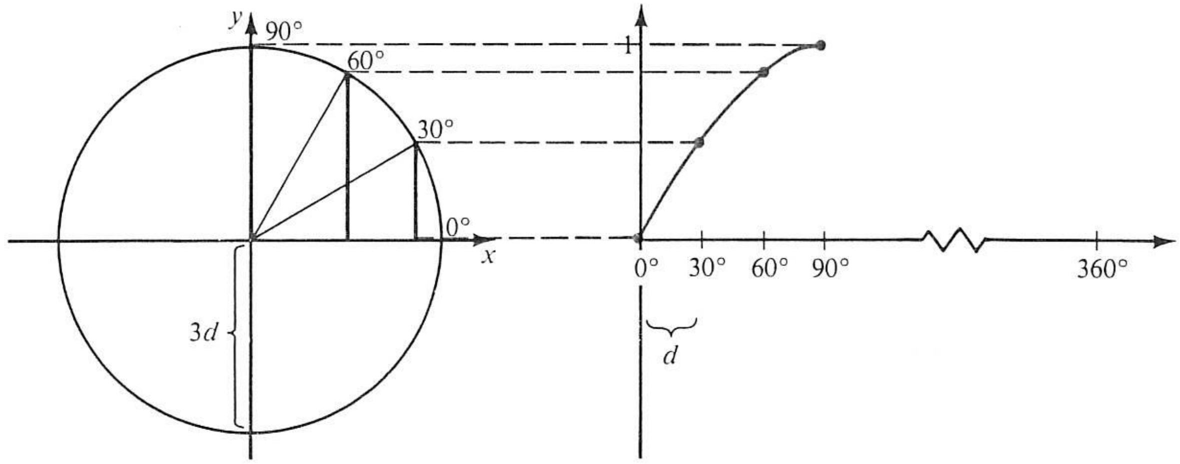
34. $(k, 4)$, if $\tan A = 3$

35. $(3, k)$, if $\tan A = \frac{1}{2}$

36. $(6, 3k)$, if $\tan A = -1$

37. $(6, -2k)$, if $\tan A = -\frac{5}{12}$

C 38. Draw a unit circle. Then use a protractor to find the intersections of the circle with the terminal sides of angles at 30° intervals: 30°, 60°, 90°, . . . , 360°. Accurately transfer to a function graph the lengths of line segments for sin A found from the unit circle diagram. To maintain a reasonable scale in the graph, make the length of the unit circle radius three times as long as whatever unit you choose for the 30° intervals on the function graph. Connect the plotted points with a smooth curve. Make a similar graph for cos A.



Unit Circle

Sine Function Graph

12–3 Trigonometric Tables

You can use the table on page 684 to find approximations for sin A, cos A, and tan A for angles *in the first quadrant* when you know the measure of angle A in degrees. A part of that table is shown below.

$m\angle A$	sin A	cos A	tan A
16°	0.2756	0.9613	0.2867
17°	0.2924	0.9563	0.3057
18°	0.3090	0.9511	0.3249

For example, to find an approximation for tan 17° from this table, you use the entry where the *row* containing 17° intersects the *column* headed "tan A." That entry is shown here in color and is 0.3057. The word *approximation* is used in describing the entries in this table because, except for sin 30°, tan 45°, cos 60°, sin 90°, and cos 90°, the entries represent rational-number approximations to irrational numbers.

If you are given a four-digit decimal as a value of sin A, cos A, or tan A, you can reverse the procedure just described to find an approximation for $m\angle A$. For example, if you are given cos $A = 0.9511$, you locate 0.9511 in the column headed "cos A" and read $m\angle A$ from the left-hand column in the same (horizontal) row. You find $m\angle A = 18°$.

If a given value for sin A, cos A, or tan A is not an entry in the table, you can use the nearest table entry.

EXAMPLE If sin $A = 0.2806$, find the measure of angle A to the nearest degree.

SOLUTION There is no entry for 0.2806 in the "sin A" column of the table.
The nearest entries are 0.2756 and 0.2924.
Find which of the entries is closer to 0.2806.

$$0.2924 - 0.2806 = 0.0118$$
$$0.2806 - 0.2756 = 0.0050$$

The entry 0.2806 is closer to 0.2756.

$\therefore m\angle A = 16°$ to the nearest degree, or $m\angle A \approx 16°$.

Oral Exercises

Use the table on page 684 to find the required value. Then tell whether or not this value is exact.

1. sin 20°
2. cos 30°
3. tan 60°
4. sin 30°
5. cos 45°
6. tan 18°
7. sin 90°
8. cos 60°

Use the table on page 684 to find the measure of angle A, $1° \leq m\angle A \leq 90°$, to the nearest degree.

9. sin $A = 0.2079$
10. cos $A = 0.4067$
11. tan $A = 5.6713$
12. sin $A = 0.9934$
13. cos $A = 0.4142$
14. tan $A = 42.9628$

For Exercises 15–23, refer to the table on page 684.

For which $m\angle A$ is each of the following statements true?

15. sin 40° = cos A
16. cos 10° = sin A
17. sin 25° = cos A
18. cos 60° = sin A
19. sin A = cos A
20. sin A = cos $2A$
21. The greater the degree measure of $\angle A$, the __?__ (greater/lesser) is the value of sin A.
22. The greater the degree measure of $\angle A$, the __?__ (greater/lesser) is the value of cos A.
23. The greater the degree measure of $\angle A$, the __?__ (greater/lesser) is the value of tan A.

Written Exercises

Use the table on page 684 to find the required value.

A

1. $\tan 75°$	2. $\cos 51°$	3. $\sin 45°$	4. $\cos 16°$	5. $\tan 65°$	6. $\sin 35°$
7. $\cos 55°$	8. $\tan 89°$	9. $\sin 81°$	10. $\tan 6°$	11. $\cos 73°$	12. $\sin 60°$

Use the table on page 684 to find the measure of angle A, $1° \leq m\angle A \leq 90°$, to the nearest degree.

13. $\sin A = 0.2425$	14. $\cos A = 0.2214$	15. $\cos A = 0.1099$
16. $\tan A = 0.3757$	17. $\cos A = 0.5533$	18. $\sin A = 0.6130$
19. $\tan A = 2.8415$	20. $\cos A = 0.8811$	21. $\sin A = 0.9723$
22. $\tan A = 16.8228$	23. $\tan A = 0.5120$	24. $\sin A = 0.3812$

Suppose that you have a table with sine values only for $1° \leq m\angle A \leq 90°$. What value would you look up to determine each of the following?

B

25. $\cos 45° = \sin$ __?__	26. $\cos 30° = \sin$ __?__	27. $\cos 70° = \sin$ __?__
28. $\cos 5° = \sin$ __?__	29. $\cos 60° = \sin$ __?__	30. $\cos R = \sin$ __?__

31. To the nearest degree, find $m\angle A$ if $\cos A = 3 \sin A$.

Suppose that you have a table with sine values only for $1° \leq m\angle A \leq 90°$. How might you use this table to determine each of the following?

C

32. $\tan 40°$	33. $\tan 10°$	34. $\tan S$

Computer Exercises For students with computer experience

***Note*: In the programs that you write for the following exercises, you will need to include a statement of the form**

```
LET A = A * 3.14159/180
```

before using A as the *argument* of any of the functions SIN, COS, or TAN. This is necessary because a computer is programmed to use a system of angle measurement that is different from degree measure.

1. Write a program that will print in chart form the sine, cosine, and tangent of any angle whose degree measure you input. Be sure to account for angles whose tangent is undefined.
2. Write a program that will print a table of the sines of angles from 0° through 90° in increments of 3°.

12–4 Solving Triangles

Suppose that you know the coordinates (x, y), $x \neq 0$, of some point B, other than the origin, on the terminal side of an angle A in standard position. Can you find sin A, cos A, and tan A?

Figure 8 pictures an angle A in standard position. It also pictures points (p, q), $p \neq 0$, and $B(x, y)$, $x \neq 0$, on the terminal side of A; (p, q) is on the unit circle, and $B(x, y)$ is not on the unit circle.

You learned in Section 5-7 that the slope of a line is constant. The slope of line OB is given by $\frac{q}{p}$ and also by $\frac{y}{x}$. Therefore,

$$\frac{q}{p} = \frac{y}{x}.$$

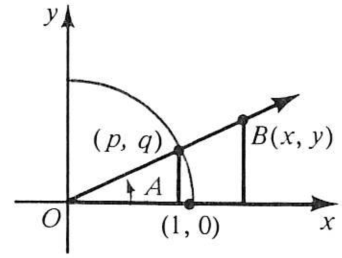


Figure 8

But by definition of tangent, $\tan A = \frac{q}{p}$, and so

$$\tan A = \frac{y}{x}.$$

Since $\frac{q}{p} = \frac{y}{x}$, you have $q = \frac{y}{x}p$, and since $\sin A = q$ and $\cos A = p$, you have

$$\sin A = \frac{y}{x}\cos A.$$

Replacing sin A with $\frac{y}{x}\cos A$ in $\sin^2 A + \cos^2 A = 1$, you have:

$$\left(\frac{y}{x}\right)^2 \cos^2 A + \cos^2 A = 1$$

$$\left(\frac{y^2}{x^2} + 1\right)\cos^2 A = 1$$

$$\cos^2 A = \frac{1}{\frac{x^2 + y^2}{x^2}} = \frac{x^2}{x^2 + y^2}$$

Therefore:

$$\cos A = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{or} \quad \cos A = \frac{-x}{\sqrt{x^2 + y^2}}$$

Since $\sqrt{x^2 + y^2} > 0$ and, for $\angle A$ in any quadrant, cos A and x are both positive or both negative, only the first of these equations applies.

$$\therefore \cos A = \frac{x}{\sqrt{x^2 + y^2}}$$

Since $\sin A = \frac{y}{x}\cos A$, you also have $\sin A = \frac{y}{x}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$, that is:

$$\sin A = \frac{y}{\sqrt{x^2 + y^2}}$$

For the case $x = 0$, you know that the terminal side of the angle is on the y-axis, and the formulas on the preceding page give $\sin A = 1$ and $\cos A = 0$ if $y > 0$, and $\sin A = -1$ and $\cos A = 0$ if $y < 0$.

For the case $y = 0$, the terminal side of the angle is on the x-axis, and the formulas give $\sin A = 0$ and $\cos A = 1$ if $x > 0$, and $\sin A = 0$ and $\cos A = -1$ if $x < 0$. These are also the values resulting from the definitions of sine and cosine on page 585.

Thus the formulas for $\sin A$, $\cos A$, and $\tan A$ shown on page 585 hold in every case, except that $\tan A$ is not defined if $x = 0$.

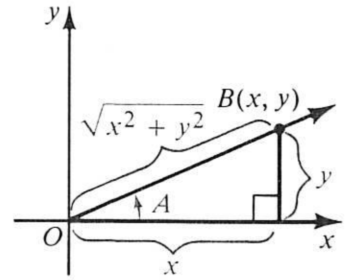


Figure 9

Notice that $\sqrt{x^2 + y^2}$ is the distance from the origin to point $B(x, y)$, because $\overline{OB}$ is the hypotenuse of a right triangle having remaining sides of lengths $|x|$ and $|y|$, as shown in Figure 9.

EXAMPLE 1 Find $\sin A$, $\cos A$, and $\tan A$ for an angle A in standard position whose terminal side contains the point $(3, -2)$.

SOLUTION Since $x = 3$ and $y = -2$:

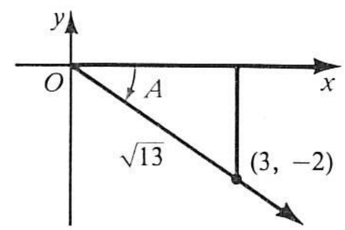
$$\sqrt{x^2 + y^2} = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

Then:

$$\sin A = \frac{y}{\sqrt{x^2 + y^2}} = \frac{-2}{\sqrt{13}}, \text{ or } -\frac{2}{\sqrt{13}}$$

$$\cos A = \frac{x}{\sqrt{x^2 + y^2}} = \frac{3}{\sqrt{13}}$$

$$\tan A = \frac{y}{x} = \frac{-2}{3}, \text{ or } -\frac{2}{3}$$



Any right triangle ACB in which $\angle C$ is the right angle can be placed so that $\angle A$ is in standard position, $\overline{AC}$ lies along the x-axis, and the point B is in the first quadrant, as shown in Figure 10. Then by applying the formulas on the preceding page, you arrive at the following.

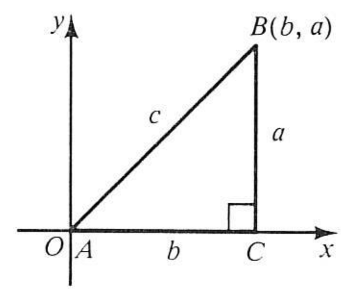


Figure 10

$$\tan A = \frac{a}{b} = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$$

$$\sin A = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{c} = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}$$

$$\cos A = \frac{b}{\sqrt{a^2 + b^2}} = \frac{b}{c} = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}}$$

Finding measures, or approximations to measures, of angles or sides of a triangle when measures of other angles or sides are given is called **solving the triangle.** It is customary in labeling triangles to use capital letters for vertices and the corresponding lowercase letters for the lengths of the sides opposite the vertices, as shown in Figure 11. The capital letters are also used to denote the angles of the triangle. When A, B, and C and a, b, and c are used in this way, the right angle is ordinarily labeled C, and the length of the hypotenuse is represented by c.

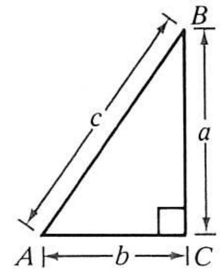


Figure 11

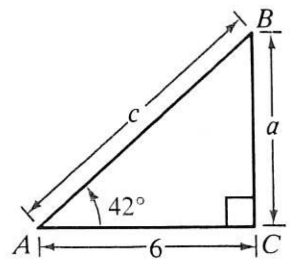
EXAMPLE 2 Solve the right triangle ABC given that $m\angle A = 42°$ and $b = 6$. Give angle measures to the nearest degree and lengths to the nearest tenth of a unit.

SOLUTION Make a sketch and label it. To find $m\angle B$, notice that

$$m\angle A + m\angle B = 90°.$$

Therefore:

$$42° + m\angle B = 90°$$
$$m\angle B = 48°$$



To find a, use the fact that $\tan A = \frac{a}{b}$, or $a = b \tan A$. Thus,

$$a = 6 \tan 42°.$$

From the table on page 684 you find that $\tan 42° \approx 0.9004$, and so

$$a \approx 6 \times 0.9004 \approx 5.4.$$

To find c, use

$$\frac{b}{c} = \sin B, \qquad \text{or} \qquad c = \frac{b}{\sin B} = \frac{6}{\sin 48°}.$$

From the table, $\sin 48° \approx 0.7431$, and so

$$c \approx \frac{6}{0.7431} \approx 8.1.$$

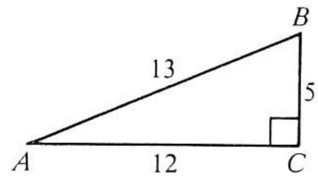
As a check, you might verify that the values of a, b, and c satisfy the Pythagorean Theorem. However, since the values of a and c were rounded, in this case you can only check that the value of $a^2 + b^2$ is *approximately* equal to the value of c^2.

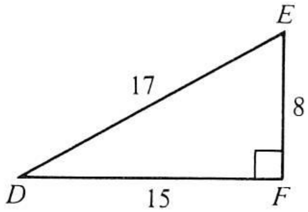
$$a^2 + b^2 \approx c^2$$
$$5.4^2 + 6^2 \overset{?}{\approx} 8.1^2$$
$$29.16 + 36 \overset{?}{\approx} 65.61$$
$$65.16 \approx 65.61 \quad \checkmark$$

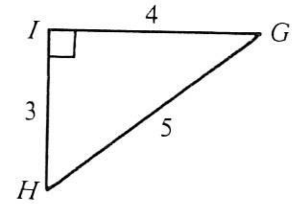
$\therefore m\angle B = 48°, a \approx 5.4, c \approx 8.1$

Oral Exercises

Exercises 1–10 refer to the three triangles below. Find the required value.



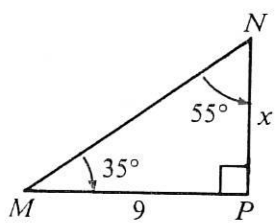




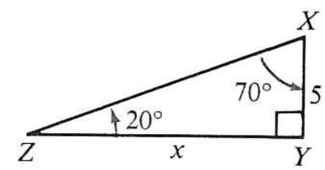
1. $\sin A$ 2. $\cos B$ 3. $\tan E$ 4. $\cos D$ 5. $\sin G$
6. $\cos H$ 7. $\tan G$ 8. $\sin D$ 9. $\tan B$ 10. $\sin H$

11. In triangle MNP below, would it be more convenient to find an approximation for x using $\tan 35°$ or using $\tan 55°$? Why?
12. In triangle XYZ below, would it be more convenient to find an approximation for x using $\tan 20°$ or using $\tan 70°$? Why?

Ex. 11



Ex. 12

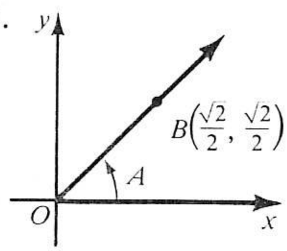


Written Exercises

In Exercises 1–6 find the following.

a. the length of $\overline{OB}$ **b.** $\sin A$ **c.** $\cos A$ **d.** $\tan A$

A 1.



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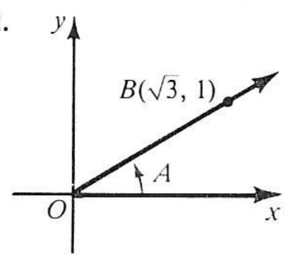

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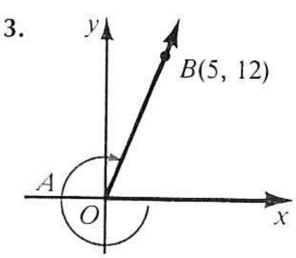

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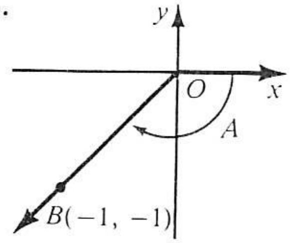

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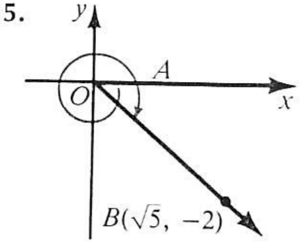

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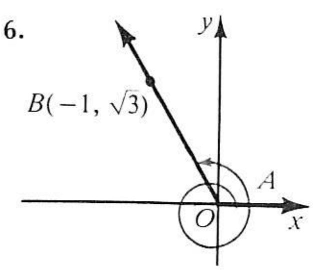
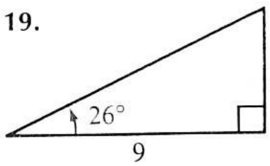


Find $\sin A$, $\cos A$, and $\tan A$ for an angle A in standard position whose terminal side contains the given point.

7. $(5, 5)$
8. $(1, \sqrt{3})$
9. $(-\sqrt{3}, 1)$
10. $(-3, 4)$
11. $(-\sqrt{3}, \sqrt{3})$
12. $(-5, -12)$
13. $(12, 5)$
14. $(0, -4)$
15. $(-3, 0)$
16. $(2, 2)$
17. $(-4, -3)$
18. $(\sqrt{5}, -\sqrt{5})$

Find an approximation for the length x or the measure of $\angle A$. Use the table on page 684 as necessary. Give angle measures to the nearest degree and lengths to the nearest tenth of a unit.

19.


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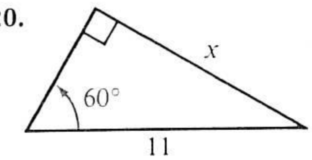

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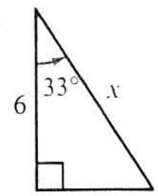

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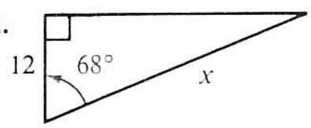

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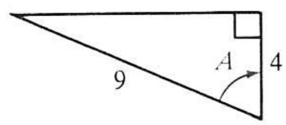

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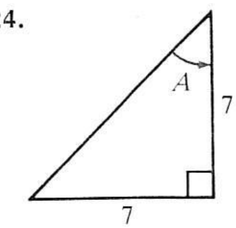

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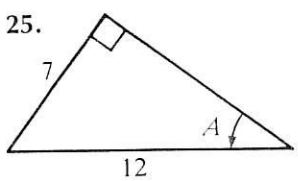

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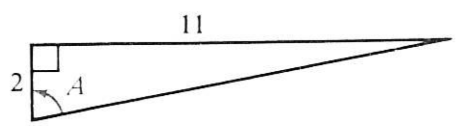
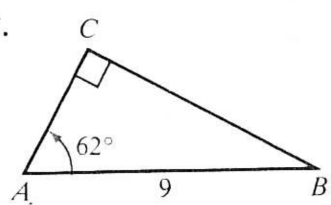


In Exercises 27–34 solve the right triangle. Use the table on page 684 as necessary. Give angle measures to the nearest degree and lengths to the nearest tenth of a unit.

B 27.


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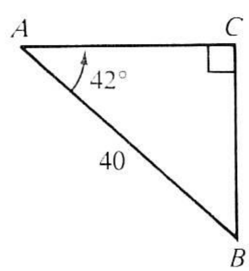

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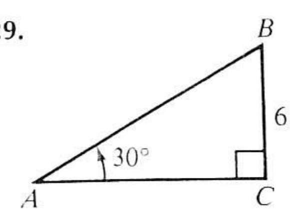

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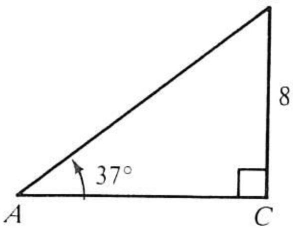

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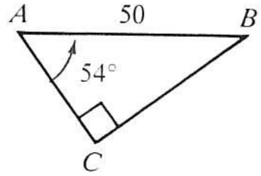

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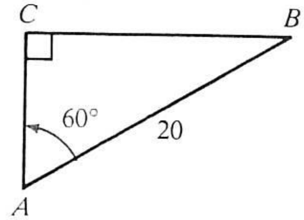

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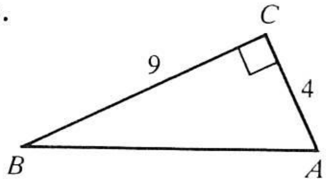

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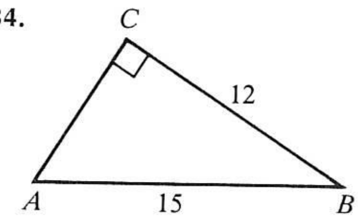
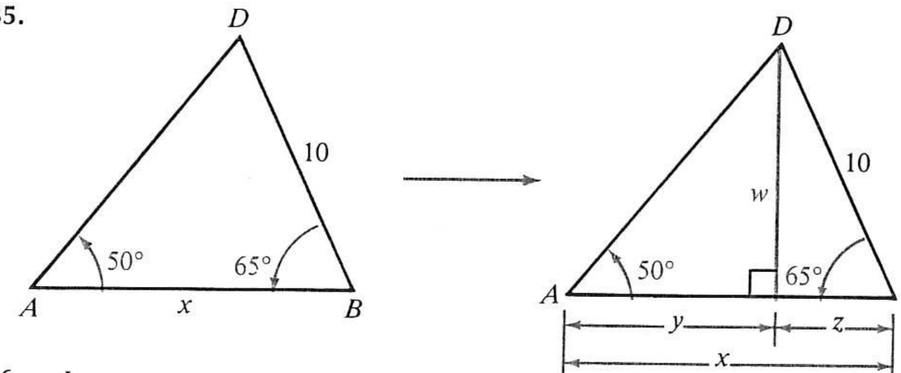


In Exercises 35–39 copy the figure, and by drawing an additional line segment introduce some right triangles to help you find a value for x to the nearest tenth of a unit. (*Hint for Exercise 35*: Draw a segment from D perpendicular to $\overline{AB}$. Find z, then y, and note that $x = y + z$.)

C 35.



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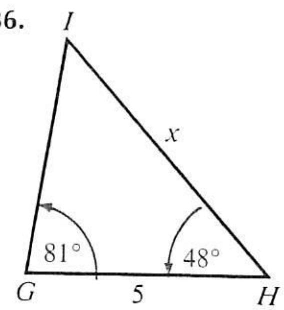
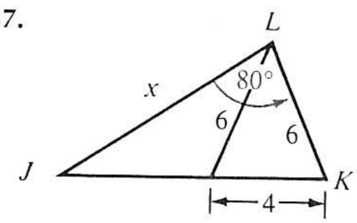


37.



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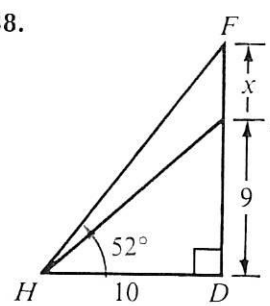

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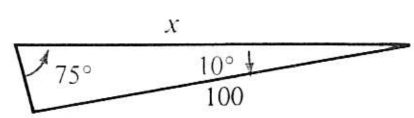
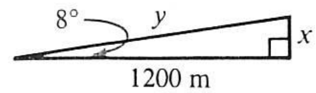


Problems

Solve each problem using the table on page 684 as necessary. Give angle measures to the nearest degree and lengths to the nearest tenth of a unit.

A

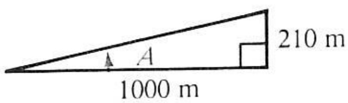
1. After takeoff, an airplane maintained a flight angle of 8° with the ground as shown at the right. Find its elevation x after it had covered a ground distance of 1200 m.



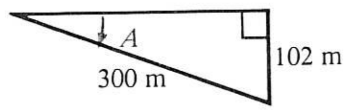
Exs. 1, 2

2. For the airplane in Problem 1, find the distance y it traveled in the air along the flight path while covering the ground distance of 1200 m.

3. If an airplane attains an elevation of 210 m after takeoff while covering a ground distance of 1000 m, as shown below, what is the degree measure of its flight angle A with the ground?



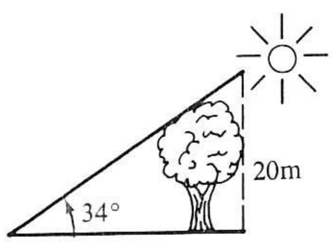
Ex. 3



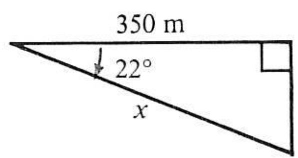
Ex. 4

4. Diving at a constant angle A, as shown above, a submarine descends 102 m while traveling 300 m. Find the degree measure of A.

5. A tree is 20 m tall. When the angle of elevation of the sun is 34°, as shown below, how long is the shadow of the tree?



Ex. 5

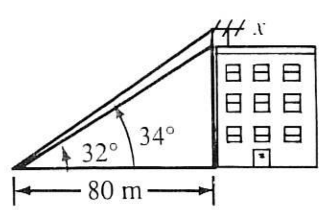


Ex. 6

6. A submarine maintains a diving angle of 22° as shown above. How far has it traveled when it is directly under a point 350 m along the surface from the point where it submerged?

B

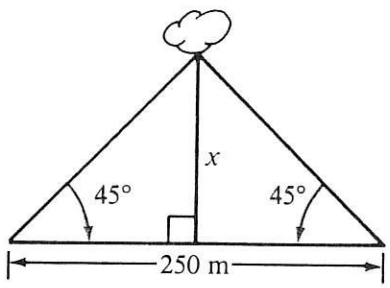
7. At a point 80 m from a building, the angle of elevation of the top of the building is 32° and the angle of elevation of the top of a television antenna at the edge of the building is 34°, as shown at the right. What is the height x of the antenna?



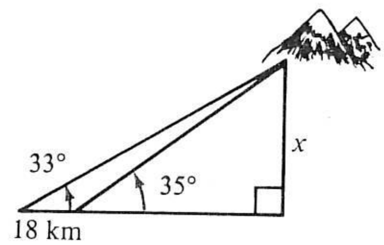
Ex. 7

8. Two surveying transits are located 250 m apart. Both transits are sighted on the same rock. For each transit, the angle between the line of sight to the rock and the line of sight to the other transit is 45°. What is the distance x from the rock to the line connecting the two transits?

9. If the transits described in Problem 8 are 120 m apart, and if each angle described in Problem 8 measures 65°, how far is the rock from the line connecting the transits?



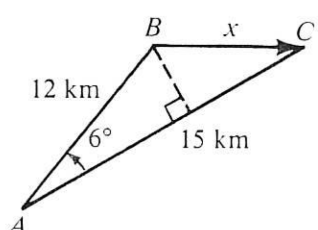
Exs. 8, 9



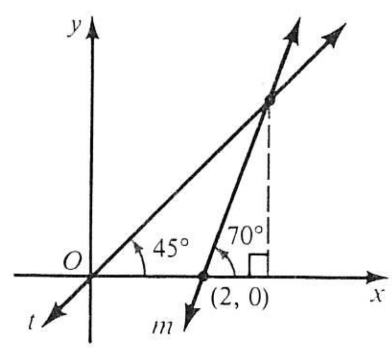
Ex. 10

10. At one point along a straight road the direction toward Mount Tanner makes an angle of 33° with the direction of the road. At another point 18 km farther along the road, the angle is 35°. Find the perpendicular distance x of Mount Tanner from the road.

C 11. A radar set at a point A sights a UFO at point B and tracks it to a point C along a straight and level path $\overline{BC}$. The distance from A to B is 12 km and the distance from A to C is 15 km. If $m\angle BAC$ is 6°, what is the distance from B to C? How fast was the UFO traveling (in km/h) if it took 10 s to move from B to C? (*Hint*: Use the dashed segment in the diagram.)



Ex. 11



Ex. 12

12. On a coordinate plane, line t passes through the origin and forms a 45° angle with the positive x-axis. Line m passes through the point (2, 0) and forms a 70° angle with the positive x-axis. Find the coordinates of the point of intersection of lines t and m.

Self-Test 1

VOCABULARY

angle (p. 579)
initial side (p. 579)
terminal side (p. 579)
directed angle (p. 579)
vertex of an angle (p. 579)
coterminal angles (p. 580)
measure of an angle (p. 580)
standard position (p. 580)
quadrantal angle (p. 580)
unit circle (p. 585)
sine of an angle (p. 585)
cosine of an angle (p. 585)
tangent of an angle (p. 585)
trigonometric functions (p. 586)
solve a triangle (p. 595)

Find the measure of $\angle A$, if $\angle A$ is coterminal with the given angle and $0° \leq m\angle A < 360°$. *Obj. 1, p. 579*

1. 750°
2. −240°
3. 395°

If $\angle A$ is in standard position, and the terminal side of $\angle A$ contains the point $(-3, -4)$, determine the following. *Obj. 2, p. 579*

4. $\sin A$
5. $\cos A$
6. $\tan A$

Use the table on page 684 to find the following. *Obj. 3, p. 579*

7. sin 78°
8. cos 51°
9. tan 27°
10. Solve the right triangle ABC given that $m\angle A = 48°$ and $b = 8$. Give lengths to the nearest tenth of a unit. *Obj. 4, p. 579*

Check your answers with those at the back of the book.

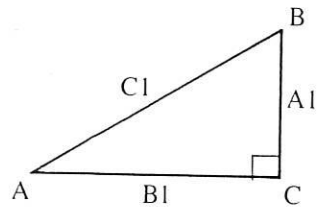
ON THE CALCULATOR

Many calculators have sine, cosine, and tangent keys that can be used in evaluating expressions involving trigonometric functions. Such calculators often have features that allow you to work with angles measured either in degrees or radians. (One *radian* is equal to the quotient when 180° is divided by π.)

Use a calculator to evaluate each expression to the nearest ten-thousandth. (Use the calculator's *degree mode*.)

1. sin 45°
2. cos 71°
3. tan 3°
4. 2 cos 22°
5. $\frac{1}{2}$ tan 84°
6. $(\sin 51°)^2$
7. $(\sin 40°)^2 + (\cos 40°)^2$
8. $(\cos 76°)^2 + (\sin 76°)^2$

PROGRAMMING IN BASIC



The program given on these two pages can be used in solving right triangles. The variables A, B, C, A1, B1, and C1 refer to the angles and sides of a right triangle as labeled on the triangle at the right. The program will output values for all the angles and sides, except the right angle, when any of the following pairs are input.

A, A1   A, B1   A, C1   A1, B1   A1, C1

Note that the trigonometric functions used in the BASIC language are defined for angles measured in *radians*, where

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}, \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} \text{ radians}.$$

Thus, the program allows you to input an angle measure in degrees, but it uses $\pi \approx 3.14159$ to express this angle measure in radians.

Notice, too, that the program makes use of the BASIC function ATN(X). This function gives the radian measure of the angle between $-90°$ and $90°$ whose tangent is X.

```
10  PRINT "TO SOLVE A RIGHT TRIANGLE"
20  PRINT " WITH RIGHT ANGLE C AND"
30  PRINT " GIVEN SET (1) OR SET (2):"
40  PRINT " (1) A, A1 OR A, B1 OR A, C1"
50  PRINT " (2) A1, B1 OR A1, C1"
60  PRINT "SELECT SET NO. (1) OR (2)";
70  INPUT N
80  LET P = 3.14159
90  IF N = 2 THEN 320
95  REM *FOR ANGLE A AND A SIDE
100  PRINT "INPUT A";
110  INPUT A
120  LET B = 90 - A
130  LET A0 = A * P/180
140  PRINT "SELECT (1) A1 OR (2) B1 OR (3) C1";
150  INPUT N
160  ON N GOTO 170, 220, 270
165  REM *FOR A1
170  PRINT "INPUT A1";
180  INPUT A1
190  LET B1 = A1/TAN(A0)
200  LET C1 = A1/SIN(A0)
210  GOTO 480
215  REM *FOR B1
```

If the computer has a built-in value for π, use that. (refers to line 80)

```
220  PRINT "INPUT B1";
230  INPUT B1
240  LET A1 = B1 * TAN(A0)
250  LET C1 = B1/COS(A0)
260  GOTO 480
265  REM *FOR C1
270  PRINT "INPUT C1";
280  INPUT C1
290  LET A1 = C1 * SIN(A0)
300  LET B1 = C1 * COS(A0)
310  GOTO 480
315  REM *FOR TWO SIDES
320  PRINT "INPUT A1";
330  INPUT A1
340  PRINT "SELECT (1) B1 OR (2) C1";
350  INPUT N
360  ON N GOTO 370, 430
365  REM *FOR B1
370  PRINT "INPUT B1";
380  INPUT B1
390  LET A = (180/P) * ATN(A1/B1)
400  LET B = 90 - A
410  LET C1 = SQR(A1*A1 + B1*B1)
420  GOTO 480
425  REM *FOR C1
430  PRINT "INPUT C1";
440  INPUT C1
450  LET B1 = SQR(C1*C1 - A1*A1)
460  LET A = (180/P) * ATN(A1/B1)
470  LET B = 90 - A
475  REM *OUTPUT
480  PRINT
490  PRINT "ALL PARTS OF THE TRIANGLE:"
500  PRINT "A = ";INT(100*A + 0.5)/100      }
510  PRINT "B = ";INT(100*B + 0.5)/100      }  Values are
520  PRINT "A1 = ";INT(100*A1 + 0.5)/100    }  rounded to
530  PRINT "B1 = ";INT(100*B1 + 0.5)/100    }  hundredths.
540  PRINT "C1 = ";INT(100*C1 + 0.5)/100    }
550  END
```

Exercises

Type in the program as given. Then RUN the program for the following pairs of parts.

1. A = 30°; A1 = 0.5

2. A = 60°; B1 = 5

3. A = 30°; C1 = 20

4. A1 = 3; B1 = 4

5. A1 = 3; C1 = 5

6. A1 = 1; B1 = 1

Vectors

OBJECTIVES for Sections 12-5 through 12-7:
1. *To sketch a vector in standard position on a coordinate plane.*
2. *To use the distance formula to find the norm of a vector.*
3. *To find the norm of the resultant of two given vectors.*
4. *To use vectors to solve problems.*

12–5 Vectors in the Plane

The red arrow, or *directed line segment,* in Figure 12 represents a *vector quantity,* namely, the displacement of a boat that has traveled 9 km northeast from port. Such an arrow indicating both a *magnitude* and a *direction* is called a **vector**. Force and velocity are other examples of vector quantities.

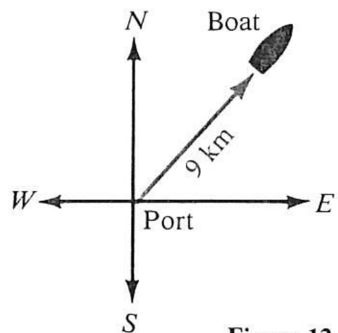


Figure 12

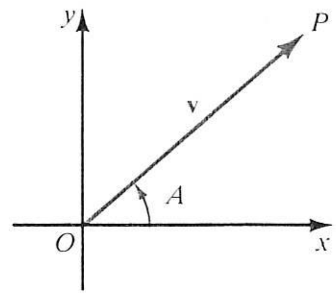
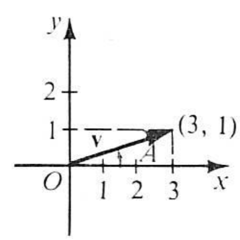


Figure 13

Vectors are usually designated by lowercase letters in bold type. For example, Figure 13 shows a vector **v** in *standard position* on a coordinate plane, that is, with **initial point** at the origin. The **terminal point** P is at the tip of the arrowhead.

The magnitude, or **norm**, of a vector **v** is represented by the symbol $\|\mathbf{v}\|$. The **direction** of a vector in standard position is the measure of the angle of rotation from the positive x-axis to the vector. The direction of vector **v** in Figure 13 is $m\angle A$.

EXAMPLE 1 For the vector **v** shown at the right, find $\|\mathbf{v}\|$ to the nearest tenth and its direction to the nearest degree.



SOLUTION From the Pythagorean Theorem:

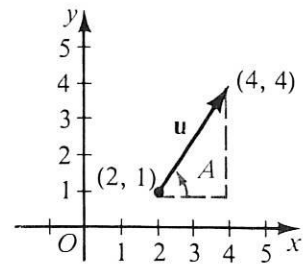
$$\|\mathbf{v}\|^2 = 3^2 + 1^2 = 10$$
$$\|\mathbf{v}\| = \sqrt{10} \approx 3.2$$

Then to find the direction of **v**, you observe that $\tan A = \frac{1}{3} \approx 0.3333$.

From the table on page 684 and the fact that $\angle A$ is in the first quadrant, you find that $m\angle A \approx 18°$.

$\therefore \|\mathbf{v}\| \approx 3.2$ and the direction of $\mathbf{v} \approx 18°$.

You can find the norm and direction of a vector located anywhere in the plane if you know the coordinates of its initial and terminal points.



EXAMPLE 2 For the vector **u** shown at the right, find $\|\mathbf{u}\|$ to the nearest tenth and its direction to the nearest degree.

SOLUTION Using the distance formula:

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{(4 - 2)^2 + (4 - 1)^2}\\ &= \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}\\ &\approx 3.6\end{aligned}$$

Further, to find the direction:

$$\tan A = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{2} = 1.5$$

Thus, by the table on page 684:

$$m\angle A \approx 56°$$

$\therefore \|\mathbf{u}\| \approx 3.6$ and the direction of $\mathbf{u} \approx 56°$.

For any vector **v**, the length of the *horizontal* displacement from the initial to the terminal point is called the **x-component** of **v**. Similarly, the length of the *vertical* displacement is called the the **y-component** of **v**. For example, in Figure 14 the x- and y-components of **v** are 1 and 3, respectively. For the vector **u**, the x-component is $4 - 2$, or 2, and the y-component is $4 - 1$, or 3.

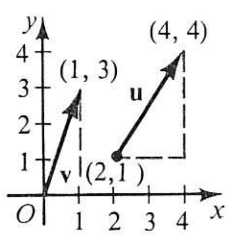
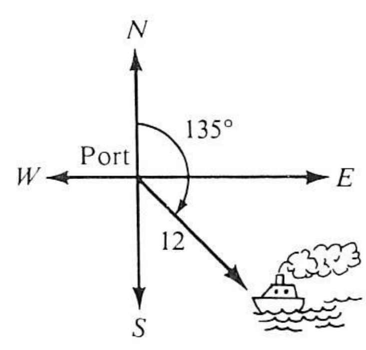


Figure 14

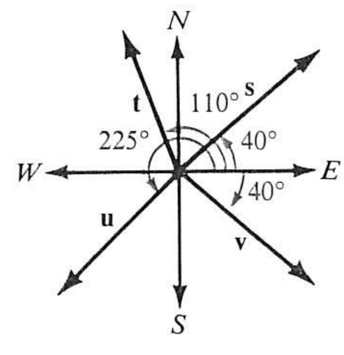
Note that a navigator gives the *bearing*, or *heading*, of a vector by measuring *clockwise* the angle from north to the vector. Notice that this is different from the definition of *direction* of a vector on the preceding page. Thus, in the diagram at the right, the bearing and magnitude of the ship's displacement from port are 135° and 12, respectively.



Oral Exercises

Exercises 1–4 refer to the figure at the right. Give the bearing of each vector.

1. **v** 2. **t** 3. **s** 4. **u**



Exs. 1–4

State the norm if the terminal point of a vector in standard position has the given coordinates.

5. $(5, 5)$ 6. $(1, 4)$ 7. $(0, -6)$

8. $(-2, -5)$ 9. $(3, x)$ 10. (j, k)

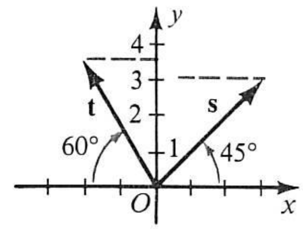
Written Exercises

Sketch a vector of the given magnitude and direction in standard position on a coordinate plane.

A 1. $\|\mathbf{v}\| = 3$; direction $60°$ 2. $\|\mathbf{v}\| = 5$; direction $140°$

3. $\|\mathbf{v}\| = 6$; direction $260°$ 4. $\|\mathbf{v}\| = 2$; direction $300°$

Exercises 5–8 refer to the figure at the right. Find the required value.



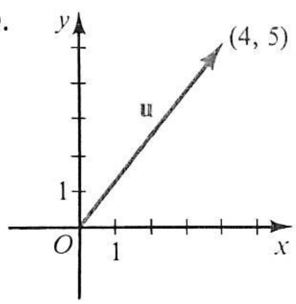
Exs. 5–8

5. the norm of **s** 6. the norm of **t**

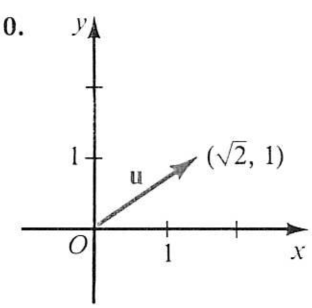
7. the x-component of **s** 8. the x-component of **t**

Find the norm and direction of the vector u.

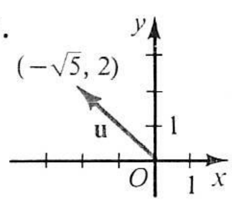
9.



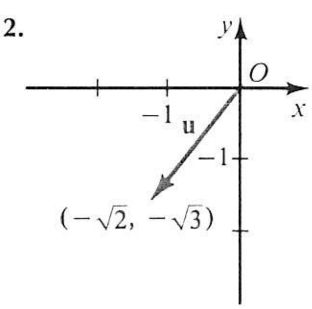
10.



11.



12.



13. A ship travels 8 km due east and then 8 km due north.
   a. What is the bearing of the ship from its initial point?
   b. If the ship next travels 16 km due west, what is the final bearing from its initial point?

14. An airplane flies 400 km due south from an airport and then 400 km due east.
   a. What is the bearing of the airplane from its starting point?
   b. If the airplane next flies 1100 km due north, what is the final bearing from its starting point?

Find the norm of u to the nearest tenth and the direction of u to the nearest degree.

B 15. **u** is in standard position, terminal point is $(-3, 5)$.

16. **u** is in standard position, terminal point is $(-3, -3)$.

17. **u** has initial point $(3, 2)$ and terminal point $(5, 6)$.

18. **u** has initial point $(2, 6)$ and terminal point $(6, 1)$.

C 19–22. What are the navigational bearings of the vectors described in Exercises 15–18?

Norbert Rillieux

1806–1894

Born in New Orleans, Norbert Rillieux was an engineer and inventor. He was educated in France, where he later taught applied mechanics.

In 1840, Rillieux returned to Louisiana, where he worked as an engineer in a sugar refinery. There he invented and patented a *multiple effect vacuum pan evaporator*, with which water could be removed from sugar cane without damaging the sugar. This invention increased the yield of sugar and reduced the cost of production. Today, the principle of multiple effect evaporation is still used in the manufacture of sugar, soap, glue, and condensed milk.

When Rillieux tried to interest New Orleans officials in advanced sewage-disposal methods, his ideas were rejected. Disheartened, Rillieux left the United States in 1854. He spent the rest of his life in Paris, where he adapted his evaporator for use in the sugar beet industry in France.

12–6 The Sum of Two Vectors

Figure 15 pictures two successive displacements in the plane, **s** and **t**. From the figure you can see that the x- and y-components of **s** are 3 and 2, and those of **t** are 1 and 4. The total resulting displacement from O is represented by the vector **v**, with initial point $O(0, 0)$ and terminal point $P(4, 6)$.

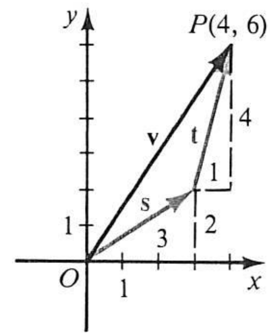


Figure 15

The x-component of **v** is the sum of the x-components of **s** and **t**, namely, $3 + 1$, or 4. The y-component of **v** is the sum of the y-components of **s** and **t**, namely, $2 + 4$, or 6. The vector, **v** is called the **sum**, or **resultant**, of the vectors **s** and **t**.

Vectors that have the same magnitude and direction are called **equivalent vectors**. Figure 16 shows five equivalent vectors, each having a norm of 2 units and directed 60° counterclockwise from the positive x-direction.

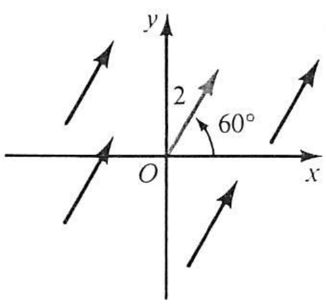


Figure 16

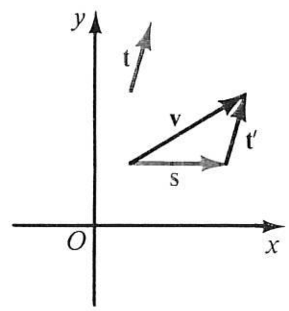
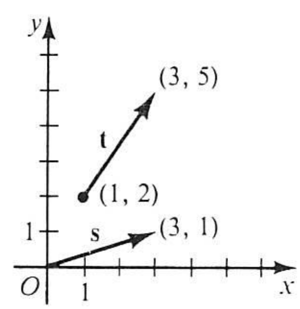


Figure 17

Figure 17 shows two vectors **s** and **t** in the plane. In order to find the sum of **s** and **t**, you would first draw a vector **t′** equivalent to **t** and positioned as shown, with its tail at the head of vector **s**. Then **v** represents the sum **s** + **t**.

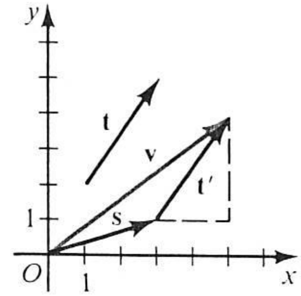
EXAMPLE Draw a vector diagram showing the sum **v** of **s** and **t**, and find the x- and y-components and the norm of **v**.



SOLUTION The red vector, **v**, represents **s** + **t**. The x-component of **s** is 3, and that of **t** is $3 - 1$, or 2. Hence the x-component of **v** is $3 + 2$, or 5.

The y-component of **s** is 1, and that of **t** is $5 - 2$, or 3. Hence the y-component of **v** is $1 + 3$, or 4.

Then



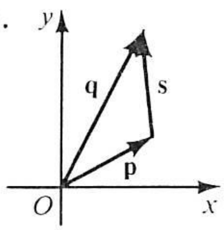
$$\|\mathbf{v}\| = \sqrt{5^2 + 4^2} = \sqrt{41}.$$

$\therefore$ the x-component of **v** is 5, the y-component is 4, and the norm is $\sqrt{41}$.

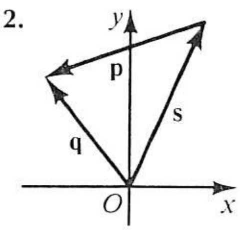
Oral Exercises

Tell which of the vectors p, q, or s is the resultant of the other two.

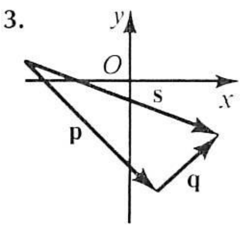
1.



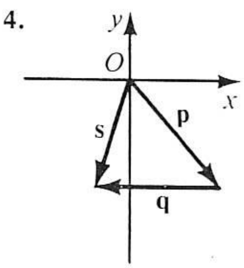
2.



3.



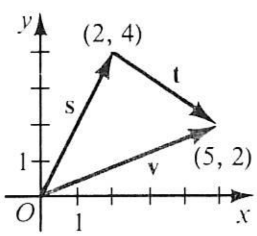
4.



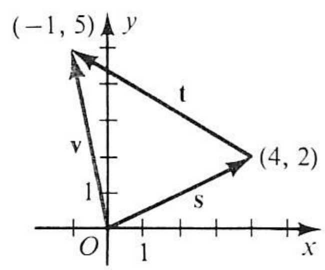
In Exercises 5–8, v is the sum of s and t. Name the following.

a. the x- and y-components of **t** **b.** $\|\mathbf{v}\|$

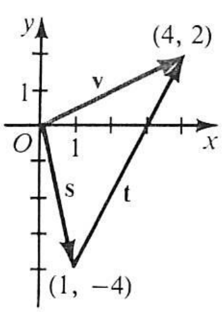
5.



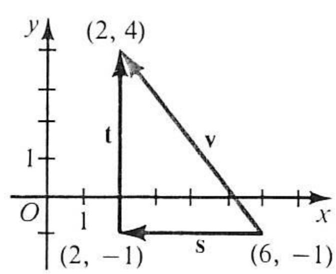
6.



7.

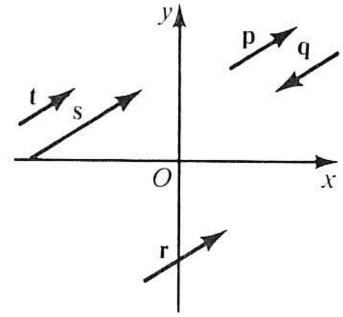


8.



Exercises 9 and 10 refer to the figure at the right.

9. Tell which vectors appear to be equivalent, and give a reason for your answer.
10. If you were to add **p** and **q**, what would be the magnitude of the vector for the sum?
11. What are the coordinates of the terminal point of the vector in standard position that is equivalent to the vector from (2, 3) to (5, 1)?



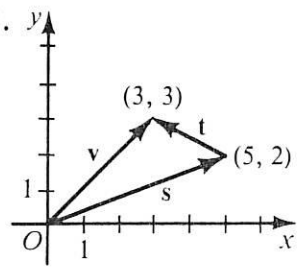
Exs. 9, 10

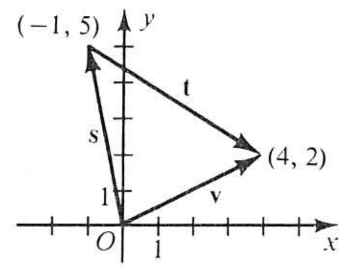
Written Exercises

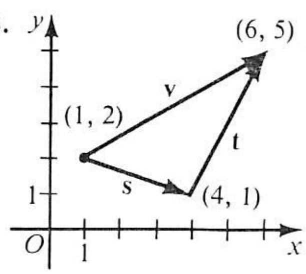
Determine the following.

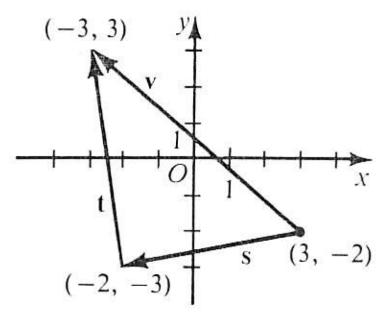
a. the *x*- and *y*-components of t **b. the norm of the resultant vector v**

A

1. 

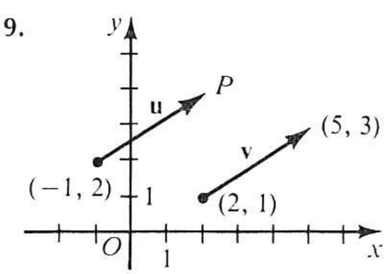
2. 

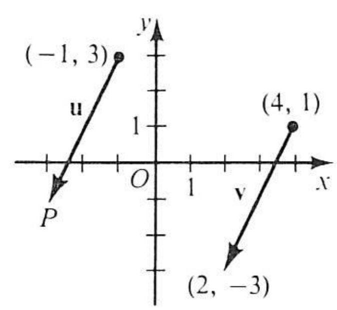
3. 

4. 

5–8. Find the direction of vector **v** in each of Exercises 1–4, to the nearest degree.

In Exercises 9 and 10, the vectors u and v are equivalent. Find the coordinates of the terminal point *P* of the vector u.

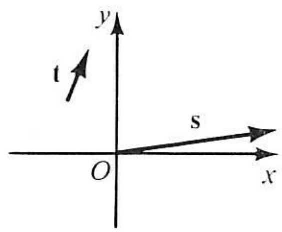
9. 

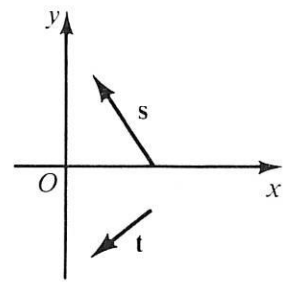
10. 

In Exercises 11 and 12:

a. Make a copy of the given vector diagram and on it show the sum **u** = **s** + **t** and the sum **v** = **t** + **s**.

b. What can you conclude about **u** and **v**?

11. 

12. 

Determine p and q so that the given points will be the initial and terminal points, respectively, of a vector equivalent to the vector from (5, 2) to (1, 5).

B 13. $(-3, 1), (p, q)$

14. $(p, q), (-3, 1)$

15. $(-1, q), (p, 7)$

16. $(p, 3), (-5, q)$

Using the norms and directions of vectors **u** and **v** as given, sketch **u**, **v**, and their resultant **u** + **v**. Then find the norm of **u** + **v** and its direction to the nearest degree.

C 17. **u**: 10; 0°
v: 4; 90°

18. **u**: 6; 45°
v: 5; 315°

19. **u**: 8; 30°
v: 6; 120°

20. Is vector addition commutative or associative? Support your answer with vector diagrams. Use vectors **r**, **s**, and **t**, with x- and y-components of a, b; c, d; and e, f respectively.

Computer Exercises For students with computer experience

1. Write a program that will determine the x- and y-components of a vector when you input a positive number that represents its norm and a positive degree measure that represents its direction with respect to the positive x-axis. (*Hint*: Use the sine and cosine functions. Remember to use a statement of the form LET A = A * 3.14159/180, as discussed on page 592, before using the sine or cosine of an angle.)
2. Modify the program that you wrote for Exercise 1 so that it allows you to input a heading in degrees rather than the direction of a vector in standard position.

12–7 Applications of Vectors

Figure 18 pictures the resultant force **v** when two different forces, **s** and **t**, are operating on an object at a point P. Physicists refer to the fact that the resultant force can be represented in this way as the *parallelogram law of forces*, because the resultant force **v** can be represented as the diagonal of the parallelogram formed by **s** and **t** and the equivalent vectors **s**′ and **t**′. You can see from the figure that addition of vectors is a commutative operation since

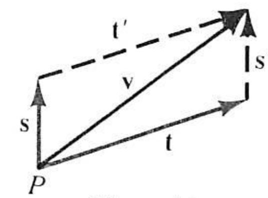


Figure 18

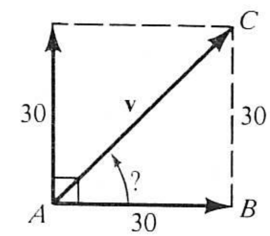
$$\mathbf{v} = \mathbf{s} + \mathbf{t}' = \mathbf{t} + \mathbf{s}',$$

or

$$\mathbf{v} = \mathbf{s} + \mathbf{t} \ = \mathbf{t} + \mathbf{s}.$$

EXAMPLE 1 Find the magnitude and the direction of the resultant force when an object is acted on at the point A by two forces at right angles to each other, each of magnitude 30 N (newtons).

SOLUTION First draw a *vector diagram*, such as the one shown at the right.



Since the two forces acting at A are at right angles to each other, you can use the Pythagorean Theorem to determine $\|\mathbf{v}\|$.

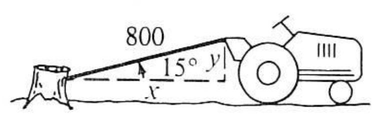
$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{30^2 + 30^2}\\ &= \sqrt{1800} = 30\sqrt{2}\end{aligned}$$

Since the two forces are equal and at right angles, triangle ABC in the diagram is an isosceles right triangle.

$\therefore$ the magnitude of **v** is $30\sqrt{2}$, and $m\angle CAB = 45°$.

EXAMPLE 2 A tractor pulls with a force of 800 N on a cable attached to an old tree stump. If the cable makes an angle of 15° with the ground, what are the horizontal component x and the vertical component y of the force, to the nearest tenth of a newton?

SOLUTION From the diagram, you can see that



$$\cos 15° = \frac{x}{800} \quad \text{and} \quad \sin 15° = \frac{y}{800}$$

so that

$$x = 800 \cos 15° \approx 800(0.9659) = 772.72,$$

and

$$y = 800 \sin 15° \approx 800(0.2588) = 207.04.$$

$\therefore x \approx 772.7$ N and $y \approx 207.0$ N

Problems

Give magnitudes to the nearest tenth and angle measures to the nearest degree.

A

1. A weather balloon is released vertically at a speed of 6 m/s in a wind blowing horizontally at a speed of 3 m/s. What angle does the path of the balloon form with the level ground?
2. Laurie walks across fields from her home, first 2 km due north and then 1 km due east. To return home by the shortest path, how far and on what heading must she walk?
3. A boat moves due east at a speed of 12 km/h in a current flowing due south at a rate of 2 km/h. Describe the speed and bearing of the boat over the surface of the earth.
4. Two people push an object across a smooth surface. One pushes with a force of 96 N due south and the other with a force of 120 N due west. Describe the force acting on the object.
5. From a tractor equipped with a winch, a cable is attached to a large rock. If the cable makes a 40° angle with the ground, what is the magnitude of the force applied vertically to the rock when a force of magnitude 4500 N is applied along the cable?
6. For the tractor in Problem 5, suppose that the cable makes a 35° angle with the ground and a force of 5000 N is applied along the cable. What is the magnitude of the force applied horizontally to the rock?

B

7. A ship sails 90 km due north from a harbor, and then turns 30° toward the east and sails 80 km. How far from the harbor is the ship at that time, and what is the bearing from the harbor to the ship?
8. A kite string makes an angle of 48° with the horizontal. If the string is going out at the rate of 2 m/s, how fast is the kite rising?
9. An airplane flies at an air speed of 400 km/h on a bearing of 60° through a wind blowing due north at 50 km/h. Find the speed and bearing of the plane with respect to the ground.
10. By sitting in the center of a hammock, Mark exerts a force of 880 N downward. Find the pull on each supporting rope if each is 45° from the horizontal.

C

11. A river flows from north to south. To cross from the east bank directly to the west bank a boat captain must keep on a course of 275°. If the trip takes 15 min when the boat travels 14.8 km/h, how wide is the river? What is the speed of the current?

Self-Test 2

VOCABULARY

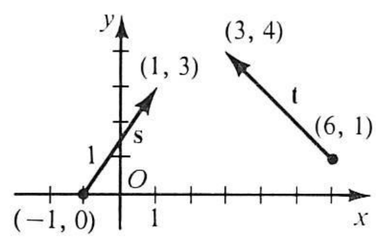
- vector (p. 604)
- initial point (p. 604)
- terminal point (p. 604)
- norm of a vector (p. 604)
- direction of a vector (p. 604)
- sum or resultant of vectors (p. 608)
- equivalent vectors (p. 608)

1. Sketch the vector **v** in standard position with $\|\mathbf{v}\| = 5$ and direction 195°. *Obj. 1, p. 604*

2. Find the norm to the nearest tenth and direction to the nearest degree of a vector **v** whose initial and terminal points have the coordinates (3, 2) and (6, 7), respectively. *Obj. 2, p. 604*

3. Find the norm of the resultant of **s** and **t** in the diagram at the right. *Obj. 3, p. 604*

4. A swimmer heads due east at a speed of 3 km/h in a current that flows due south at a speed of 2 km/h. Describe the speed and bearing of the swimmer over the surface of the earth. *Obj. 4, p. 604*



Check your answers with those at the back of the book.

Chapter Summary

1. A *directed angle* is the union of two ordered rays with the same endpoint, together with a rotation from one ray, called the initial side, to the other ray, called the terminal side.
2. The number of degrees through which a ray rotates from the initial side to the terminal side is the *measure* of an angle.
3. An angle with its vertex at the origin and with the positive x-axis as its initial side is in *standard position*.
4. If the terminal side of $\angle A$ in standard position intersects the unit circle at (p, q), then the *cosine* of $\angle A$ ($\cos A$) is p, the *sine* of $\angle A$ ($\sin A$) is q, and the *tangent* of $\angle A$ ($\tan A$) is $\frac{q}{p}$. The sine, cosine, and tangent of an angle in standard position can be determined from any point on the terminal side.
5. Given the measures of certain sides and angles of a triangle, *solving a triangle* means to find its remaining sides and angles.

6. A *vector* is a directed line segment having both magnitude and direction. The *magnitude*, or *norm*, of a vector, $\|\mathbf{v}\|$, is its length. The *direction* of a vector in standard position is the measure of the angle of rotation from the positive x-axis to the vector.
7. *Equivalent vectors* have the same magnitude and direction.
8. The *sum*, or *resultant*, of two vectors is found by adding the x- and y-components of the vectors.

Chapter Review

Write the letter of the correct answer.

1. Which angle is *not* coterminal with an angle of 160° in standard position? 12-1

   a. −200° b. 660° c. 880° d. −560°

2. If angle A is *not* a quadrantal angle, and $600° < m\angle A < 700°$, then the terminal side of angle A is in Quadrant _?_.

   a. I or II b. II or III c. III or IV d. I or IV

3. If $\sin A < 0$ and $\cos A < 0$ and angle A is in standard position, then the terminal side of angle A is in Quadrant _?_. 12-2

   a. I b. II c. III d. IV

4. The terminal side of $\angle A$ in standard position contains the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle. Then $\tan A =$ _?_.

   a. 1 b. −1 c. $\sqrt{2}$ d. $-\sqrt{2}$

Use the table on page 684 to complete Exercises 5 and 6.

5. $\cos 75° =$ _?_ 12-3

   a. 0.2588 b. 0.9659 c. 3.7321 d. 0.9695

6. sin _?_ = 0.5592

   a. 56° b. 33° c. 34° d. 35°

7. The terminal side of $\angle A$ is in Quadrant II and $\cos A = -\frac{4}{5}$. Then $\sin A =$ _?_.

   a. $-\frac{3}{5}$ b. $\frac{3}{5}$ c. $\frac{3}{4}$ d. $-\frac{3}{4}$

8. The terminal side of $\angle A$ in standard position contains the point $(-5, -12)$. Then $\cos A =$ _?_. 12-4

   a. $-\frac{5}{13}$ b. $-\frac{12}{13}$ c. $\frac{5}{13}$ d. $\frac{12}{13}$

9. In right triangle ABC, $m\angle A = 70°$ and $c = 18$. Find b to the nearest tenth.

   a. 49.5 b. 16.9 c. 11.8 d. 6.2

10. If a vector has initial point (2, 1) and terminal point (4, 5), find its direction to the nearest degree. 12-5

a. 27° b. 63° c. 60° d. 56°

11. If vector **v** is in standard position, with terminal point (−8, 15), then $\|\mathbf{v}\| = $ ___?___.

a. 12 b. $\sqrt{161}$ c. 7 d. 17

12. If vectors **s** and **t** are in standard position with terminal points (−2, 2) and (5, 2) respectively, and $\mathbf{v} = \mathbf{s} + \mathbf{t}$, then $\|\mathbf{v}\| = $ ___?___. 12-6

a. $\sqrt{3}$ b. 4 c. 5 d. $\sqrt{10}$

13. If vector **p** has initial point (1, 3) and terminal point (−1, 5), vector **q** has initial point (2, 4) and terminal point (6, 6), and $\mathbf{v} = \mathbf{p} + \mathbf{q}$, then the x-component of vector **v** is ___?___.

a. 2 b. −2 c. 4 d. 6

14. A boat moves due west at a speed of 10 km/h in a current flowing south at the rate of 2 km/h. Then the bearing of the boat over the surface of the earth is ___?___. 12-7

a. 11° b. 79° c. 191° d. 259°

Chapter Test

1. If A is a quadrantal angle and $-200° \le m\angle A \le -100°$, find $m\angle A$. 12-1
2. If $\angle A$ is in standard position, $180° \le m\angle A \le 270°$, and $\angle A$ is coterminal with an angle of −465°, find $m\angle A$.
3. Evaluate tan 270°. 12-2
4. Angle A is in standard position and its terminal side contains the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Find tan A.
5. If $1° \le m\angle A \le 90°$ and $\cos A = 0.7777$, find $m\angle A$ to the nearest degree. 12-3
6. In right triangle ABC, $c = 25$ and $b = 16$. Find $m\angle A$ to the nearest degree. 12-4
7. At a point 25 m from a building, the angle of elevation of the top of the building is 72°. How tall is the building?
8. If a vector in standard position has direction 140° and y-component 30, find its norm. 12-5
9. If vectors **s** and **t** are in standard position with terminal points (4, 2) and (5, 2) respectively, and $\mathbf{v} = \mathbf{s} + \mathbf{t}$, find $\|\mathbf{v}\|$. 12-6
10. In order to cross a river, a canoeist leaves a dock and paddles due north at a speed of 6 km/h. The river current is flowing due east at a speed of 5 km/h. Describe the speed and bearing of the canoeist over the surface of the earth. 12-7

Mixed Review

Simplify.

1. $\frac{5x}{x^2 - 25} - \frac{x^2}{x^2 - 25}$

2. $\frac{4a}{a^2 - 4a - 5} + \frac{1}{a^2 + 2a + 1}$

3. $\frac{3n}{n + 4} + 2$

4. $\frac{6}{6 - m} - \frac{m}{m - 6}$

5. $\frac{x^{-4}y^3z^{-1}}{x^2y^{-1}}$

6. $\left(\frac{2^{-3}a^{-1}b^3}{2^{-1}a^2b^{-3}}\right)^{-2}$

7. $\frac{\frac{3}{m} - \frac{1}{n}}{\frac{1}{m} - \frac{3}{n}}$

8. $\frac{\frac{r}{r + s} + \frac{s}{r - s}}{\frac{r}{r - s} - \frac{s}{r + s}}$

9. $\sqrt{188}$

10. $\sqrt{x^6}$

11. $\sqrt{(y - 4)^2}$

12. $\sqrt[4]{243}$

13. $\sqrt{\frac{4}{5}}$

14. $\sqrt[3]{\frac{2}{3}}$

15. $\sqrt{18} + \sqrt{20} - \sqrt{50}$

16. $\sqrt{2x^3} \cdot \sqrt{14x^3}$

17. $\sqrt{18}(\sqrt{2} - \sqrt{10})$

18. $\frac{2}{1 - \sqrt{5}}$

Express in scientific notation.

19. 0.00000531

20. 37,000,000,000,000

21. $3000 \times 0.000003 \times 3$

22. $49{,}000{,}000 \div 0.0007$

23. $\frac{20{,}000 \times 0.00015}{300}$

24. $\frac{0.00054}{0.0000012 \times 9000}$

Solve each system of equations.

25. $r - 4s = 3$
$-5r + 3s = 19$

26. $m = 15 - 2n$
$3m - 2n = 5$

27. $x - y = 1$
$-8x + 3y = 32$

28. $2a - 3b = 6$
$a = 4$

29. $3j = k$
$3j - k = 4$

30. $v - w = 3$
$2v = 2w + 6$

31. $15a - 4b = 8$
$-7a + 8b = 7$

32. $6r = 5s - 8$
$2r + 5s = -16$

33. Express $\frac{11}{12}$ as a repeating or terminating decimal.

34. Express $0.34\overline{3}$ as a fraction in simplest form.

In Exercises 35–40, $\angle A$ and $\angle B$ are the acute angles of a right triangle; a and b are the lengths of the sides opposite $\angle A$ and $\angle B$, respectively; and c is the length of the hypotenuse. Find the required measure, using a calculator or the tables on pages 683 and 684 as necessary. Give angle measures to the nearest degree and lengths to the nearest tenth of a unit.

35. If $a = 12$ and $b = 16$, find c.
36. If $b = 9$ and $c = 12$, find a.
37. If $m\angle A = 26°$ and $b = 10$, find a.
38. If $a = 3$ and $c = 5$, find $m\angle A$.
39. If $b = 5$ and $c = 8$, find $m\angle A$.
40. If $m\angle B = 28°$ and $a = 5$, find c.

Solve.

41. $a^2 - 14 = 0$
42. $5 - 2b^2 = 1$
43. $x^2 - 3x = 10$
44. $3y^2 - 2y = 12$
45. $\sqrt{m} - 9 = 0$
46. $\sqrt{n - 4} = 2$
47. $\sqrt{r} + r = 2$
48. $\sqrt[3]{8 - s} = -3$

49. Alicia has $1.15 in change in her wallet, consisting entirely of quarters, dimes, and nickels. She has one more nickel than quarters and three more dimes than quarters. How many of each type of coin does she have?
50. Jen and Kevin cycled 14 km in 30 min with the wind. They made the return trip in 35 min cycling against the same wind. Find their speed of cycling without the wind.
51. The measure of the diagonal of a certain rectangle is 22 cm long. If the length of the rectangle is 16 cm, find its width.
52. The measure in degrees of one of two supplementary angles is 20 more than half the measure of the other angle. Find the measures of the angles.
53. The sum of the digits of a two-digit number is 15. If 27 were subtracted from the number, the result would be the original number with its digits reversed. Find the original number.
54. When the dimensions of a rectangle that measures 2 cm by 5 cm are each increased by the same amount, the result is a rectangle whose area is 18 cm^2 greater than that of the original rectangle. Find the dimensions of the new rectangle.
55. The sum of the squares of two consecutive negative odd integers is 202. Find the integers.
56. The cost of a ticket to the school play was $3 for adults and $2 for students. If 900 tickets were sold and the total receipts from tickets were $2300, how many of each type of ticket was sold?

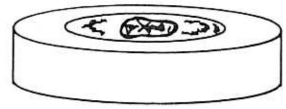
PREPARING FOR COLLEGE ENTRANCE EXAMS

Strategy for Success: In solving problems such as those involving length, width, area, perimeter, or relative position, you may find it helpful to draw a sketch. Use any available space in the test booklet. Be careful to make no assumptions in drawing the sketch, using only information that is specifically given in the problem.

Decide which is the best of the choices and write the corresponding letter on your answer sheet.

1. Which of the following are roots of the equation $x^2 - 2kx = -9$?
   I. $k + \sqrt{k - 3}$ II. $k + \sqrt{k^2 - 9}$ III. $k - \sqrt{k^2 - 9}$
   (A) I only **(B)** II only **(C)** III only **(D)** I and II only **(E)** II and III only
2. Sheila paddled a canoe 16 km up a stream and then returned. The round trip took 10 h. The current flowed at a rate of 3 km/h. How fast can Sheila paddle in still water?
   (A) 2 km/h **(B)** 4 km/h **(C)** 5 km/h **(D)** 6 km/h **(E)** 8 km/h
3. Which of the following is true for the function $f: x \longrightarrow 6 + x - x^2$?
   I. The graph of the function opens downward.
   II. The function has no zeros.
   III. The function has two real roots.
   (A) I only **(B)** II only **(C)** III only **(D)** I and II only **(E)** I and III only
4. Which of the following is true if $(n - 3)^2 = 8$?
   I. $n = 3 + \sqrt{8}$ II. $n = -3 + \sqrt{8}$ III. $n = 8 + \sqrt{3}$
   (A) I only **(B)** II only **(C)** III only **(D)** I, II, and III **(E)** none of these
5. Find $\|\mathbf{u}\|$ if the vector **u** is in standard position with terminal point $(-3, 4)$.
   (A) 5 **(B)** -5 **(C)** 25 **(D)** $\sqrt{6}$ **(E)** $-\sqrt{6}$
6. Which of the following is true for $A = \{(x, y): x = -2y^2\}$?
   (A) A is a function.
   (B) The minimum point of its graph is $(3, -4)$.
   (C) Its graph has a maximum point.
   (D) The graph opens to the left.
   (E) None of the above statements is true.
7. Find $\cos A$ for an angle A in standard position whose terminal side contains the point $(12, -5)$.
   (A) $\frac{13}{5}$ **(B)** $\frac{12}{13}$ **(C)** $\frac{5}{13}$ **(D)** $\frac{5}{12}$ **(E)** $-\frac{12}{13}$
8. Solve $\frac{9}{x^2} - \frac{12}{x} + 4 = 0$.
   (A) $\frac{2}{3}$ **(B)** $\frac{1}{2}$ **(C)** $2 + \sqrt{3}$ **(D)** $\frac{3}{2}$ **(E)** $2 - \sqrt{3}$

Contest Problems



1. Suppose that you are allowed to make five vertical cuts through a large circular wheel of cheese, like the one that is shown at the right. What is the greatest number of pieces of cheese that you can get?

2. Solve the system:
$$x^2 + y^2 = 25$$
$$4y^2 - 9x = 0$$

3. Solve the inequality $\frac{2m - 8}{m} < \frac{4}{m}$.

4. What is the next number in the following pattern?

   3, 5, 8, 13, 22, 39, 72, __?__

5. Find the least integral value of C for which the following equation has no real root.
$$2x^2 + 6x + C = 0$$

6. Determine how many pairs of prime numbers have a difference of 3.