Chapter 11

Quadratic Equations and Functions

Quadratic Equations

OBJECTIVES for Sections 11-1 through 11-3:
1. To solve quadratic equations by completing the square.
2. To solve quadratic equations by using the quadratic formula.
3. To use quadratic equations to solve problems.

11–1 Completing the Square

Recall from Section 7-10 that any equation that can be written equivalently in the form

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

is called a *quadratic equation*. In Chapter 7 you learned to solve some equations of this form by factoring. In this chapter you will learn methods that can be used to solve *any* quadratic equation.

As you learned in Chapter 10, you can solve a simple quadratic equation such as

$$x^2 = 9$$

by observing that it has two roots:

$$x = \sqrt{9} \quad \text{or} \quad x = -\sqrt{9}$$
$$x = 3 \quad \Big| \quad x = -3$$

Thus the solution set is $\{3, -3\}$.

Now consider the equation

$$(x - 2)^2 = 9.$$

You can solve this equation by noting that it also has two roots:

$$\begin{array}{rcl} x - 2 = \sqrt{9} & \text{or} & x - 2 = -\sqrt{9} \\ x - 2 = 3 & | & x - 2 = -3 \\ x = 5 & \text{or} & x = -1 \end{array}$$

Therefore, the solution set is $\{-1, 5\}$.

You can extend this method to solve quadratic equations such as those in the following examples.

EXAMPLE 1 Solve $2(y - 3)^2 = 50$.

SOLUTION

$$2(y - 3)^2 = 50$$
$$(y - 3)^2 = 25$$
$$\begin{array}{rcl} y - 3 = \sqrt{25} & \text{or} & y - 3 = -\sqrt{25} \\ y - 3 = 5 & | & y - 3 = -5 \\ y = 8 & \text{or} & y = -2 \end{array}$$

Check:

$$\begin{array}{rl} 2(y - 3)^2 = 50 & 2(y - 3)^2 = 50 \\ 2(8 - 3)^2 \stackrel{?}{=} 50 & 2(-2 - 3)^2 \stackrel{?}{=} 50 \\ 50 = 50 \quad \checkmark & 50 = 50 \quad \checkmark \end{array}$$

$\therefore$ the solution set is $\{8, -2\}$.

EXAMPLE 2 Solve $g^2 + 6g + 9 = 2$.

SOLUTION

$$g^2 + 6g + 9 = 2$$
$$(g + 3)^2 = 2$$
$$\begin{array}{rcl} g + 3 = \sqrt{2} & \text{or} & g + 3 = -\sqrt{2} \\ g = -3 + \sqrt{2} & | & g = -3 - \sqrt{2} \end{array}$$

Check:

$$g^2 + 6g + 9 = 2$$
$$(-3 + \sqrt{2})^2 + 6(-3 + \sqrt{2}) + 9 \stackrel{?}{=} 2$$
$$9 - 6\sqrt{2} + 2 - 18 + 6\sqrt{2} + 9 \stackrel{?}{=} 2$$
$$2 = 2 \quad \checkmark$$

$$g^2 + 6g + 9 = 2$$
$$(-3 - \sqrt{2})^2 + 6(-3 - \sqrt{2}) + 9 \stackrel{?}{=} 2$$
$$9 + 6\sqrt{2} + 2 - 18 - 6\sqrt{2} + 9 \stackrel{?}{=} 2$$
$$2 = 2 \quad \checkmark$$

$\therefore$ the solution set is $\{-3 + \sqrt{2}, -3 - \sqrt{2}\}$.

Examples 1 and 2 suggest that any quadratic equation can be solved readily if it can be written in the form $(x + p)^2 = q$. Examples 3 and 4 illustrate how to take advantage of this fact by completing the trinomial square that is suggested by the coefficient of the linear term. This method of solution is called **completing the square.**

EXAMPLE 3 Solve $x^2 - 10x + 15 = 0$ by completing the square.

SOLUTION

$$x^2 - 10x + 15 = 0$$

$$x^2 - 10x = -15 \quad \longleftarrow \text{\{Subtract 15 from both sides}$$

$$x^2 - 10x + (-5)^2 = -15 + (-5)^2 \quad \longleftarrow \text{\{Add the square of } \textit{half} \text{ the coefficient of } x \text{ to both sides}$$

$$= -15 + 25$$

$$(x - 5)^2 = 10 \quad \longleftarrow \text{\{Factor the trinomial square}$$

$$x - 5 = \sqrt{10} \quad \text{or} \quad x - 5 = -\sqrt{10}$$

$$x = 5 + \sqrt{10} \quad \Big| \quad x = 5 - \sqrt{10}$$

Checking the solutions is left to you.

$\therefore$ the solution set is $\{5 + \sqrt{10}, 5 - \sqrt{10}\}$.

EXAMPLE 4 Solve $d^2 + 5d = 5$ by completing the square.

SOLUTION

$$d^2 + 5d = 5$$

$$d^2 + 5d + \left(\frac{5}{2}\right)^2 = 5 + \left(\frac{5}{2}\right)^2 \quad \longleftarrow \text{\{Add the square of } \textit{half} \text{ the coefficient of } d \text{ to both sides}$$

$$= 5 + \frac{25}{4}$$

$$\left(d + \frac{5}{2}\right)^2 = \frac{45}{4} \quad \longleftarrow \text{\{Factor the trinomial square}$$

$$d + \frac{5}{2} = \sqrt{\frac{45}{4}} \quad \text{or} \quad d + \frac{5}{2} = -\sqrt{\frac{45}{4}}$$

$$d + \frac{5}{2} = \frac{3\sqrt{5}}{2} \quad \Big| \quad d + \frac{5}{2} = -\frac{3\sqrt{5}}{2}$$

$$d = -\frac{5}{2} + \frac{3\sqrt{5}}{2} \quad \Big| \quad d = -\frac{5}{2} - \frac{3\sqrt{5}}{2}$$

$$d = \frac{-5 + 3\sqrt{5}}{2} \quad \text{or} \quad d = \frac{-5 - 3\sqrt{5}}{2}$$

Check:

$$d^2 + 5d = 5$$

$$\left(\frac{-5 + 3\sqrt{5}}{2}\right)^2 + 5\left(\frac{-5 + 3\sqrt{5}}{2}\right) \stackrel{?}{=} 5$$

$$\frac{25 - 30\sqrt{5} + 45}{4} + \frac{-25 + 15\sqrt{5}}{2} \stackrel{?}{=} 5$$

$$25 - 30\sqrt{5} + 45 + 2(-25 + 15\sqrt{5}) \stackrel{?}{=} 20$$

$$25 - 30\sqrt{5} + 45 - 50 + 30\sqrt{5} \stackrel{?}{=} 20$$

$$20 = 20 \quad \checkmark$$

Checking the second solution is left to you.

$\therefore$ the solution set is $\left\{\frac{-5 + 3\sqrt{5}}{2}, \frac{-5 - 3\sqrt{5}}{2}\right\}$.

If the coefficient of the quadratic term of a quadratic equation is a number other than 1, you must divide each term in the equation by this coefficient before completing the square, as shown in Example 5.

EXAMPLE 5 Solve $3x^2 - 8x + 2 = 0$ by completing the square.

SOLUTION

$$3x^2 - 8x + 2 = 0$$

$$x^2 - \frac{8}{3}x + \frac{2}{3} = 0 \longleftarrow \left\{ \begin{array}{l} \text{First divide both sides by 3.} \\ \text{Then proceed to complete the square} \\ \text{as in the preceding examples.} \end{array} \right.$$

$$x^2 - \frac{8}{3}x = -\frac{2}{3}$$

$$x^2 - \frac{8}{3}x + \left(-\frac{4}{3}\right)^2 = -\frac{2}{3} + \left(-\frac{4}{3}\right)^2$$

$$\left(x - \frac{4}{3}\right)^2 = \frac{10}{9}$$

$$x - \frac{4}{3} = \sqrt{\frac{10}{9}} \quad \text{or} \quad x - \frac{4}{3} = -\sqrt{\frac{10}{9}}$$

$$x = \frac{4}{3} + \frac{\sqrt{10}}{3} \quad \Bigg| \quad x = \frac{4}{3} - \frac{\sqrt{10}}{3}$$

$$x = \frac{4 + \sqrt{10}}{3} \quad \text{or} \quad x = \frac{4 - \sqrt{10}}{3}$$

Checking the results is left to you.

$\therefore$ the solution set is $\left\{\frac{4 + \sqrt{10}}{3}, \frac{4 - \sqrt{10}}{3}\right\}$.

Notice that the solutions in Examples 2 through 5 are given as *exact solutions*, with all irrational solutions written in simplest form. If you use a calculator or the Table of Square Roots on page 683, you can also obtain *decimal approximations* of such solutions. For example, the solutions of the equation in Example 5 can be approximated as follows.

$$\frac{4 + \sqrt{10}}{3} \approx \frac{4 + 3.162}{3} = \frac{7.162}{3} = 2.387\bar{3} \approx 2.39$$

$$\frac{4 - \sqrt{10}}{3} \approx \frac{4 - 3.162}{3} = \frac{0.838}{3} = 0.279\bar{3} \approx 0.28$$

Throughout the rest of this chapter, irrational solutions should be approximated to the nearest hundredth unless otherwise specified.

Some quadratic equations have *no* solutions over the set of real numbers. For example, consider the equation

$$p^2 + 4p + 5 = 0.$$

By completing the square, you would obtain the equation

$$(p + 2)^2 = -1.$$

Since the square of a real number must be a *nonnegative* real number, the equation $p^2 + 4p + 5 = 0$ has no real-number solution.

Oral Exercises

Name the value(s) of q for which the expression is a trinomial square.

1. $x^2 + 10x + q$
2. $d^2 - 12d + q$
3. $b^2 + qb + 16$
4. $r^2 - qr + 81$
5. $a^2 - 5a + q$
6. $b^2 + 9b + q$
7. $y^2 + y + q$
8. $f^2 + \frac{2}{5}f + q$
9. $n^2 - qn + \frac{9}{16}$
10. $c^2 + qc + \frac{49}{64}$
11. $z^2 + mz + q$
12. $x^2 - \frac{r}{s}x + q$

Written Exercises

Solve by completing the square. First give the exact solution, expressing any irrational roots in simplest form. Then approximate irrational roots to the nearest hundredth.

A

1. $c^2 - 2c - 3 = 0$
2. $z^2 + 8z - 9 = 0$
3. $p^2 + 6p = -8$
4. $q^2 = 12q - 27$
5. $e^2 + 8e + 10 = 0$
6. $d^2 - 6d + 2 = 0$
7. $n^2 - 4n - 8 = 0$
8. $x^2 - 16x + 40 = 0$
9. $3m^2 - 12m + 9 = 0$
10. $5t^2 + 20t - 60 = 0$
11. $12x^2 + 84 = 96x$
12. $144 - 9y^2 = 54y$
13. $w^2 - w - 4 = 0$
14. $z^2 + z - 3 = 0$
15. $r^2 = 1 - 3r$
16. $p^2 - 5p = -5$
17. $z^2 - 9z + 10 = 0$
18. $b^2 + 17 = 11b$

Solve by completing the square. Give only the exact solutions, expressing any irrational roots in simplest form. If an equation has no real roots, so state.

B

19. $a^2 - \frac{1}{2}a - \frac{1}{2} = 0$
20. $b^2 - \frac{1}{8} = \frac{1}{4}b$
21. $s^2 = \frac{2}{3}s + \frac{4}{3}$
22. $m^2 - \frac{2}{5}m + \frac{1}{5} = 0$
23. $5x^2 + 30x - 25 = 0$
24. $8x^2 - 32x + 16 = 0$
25. $2n^2 + 6n + 5 = 0$
26. $3t^2 + 15t + 12 = 0$
27. $5y^2 + y - 3 = 0$
28. $4m^2 - m - 6 = 0$
29. $25x^2 + 20x + 1 = 0$
30. $18z^2 - 48z - 30 = 0$
31. $49x^2 - 42x + 10 = 0$
32. $12a^2 + 54a + 45 = 0$
33. $2 - \frac{1}{p - 1} = \frac{2}{p}$
34. $\frac{5}{x + 4} - 3 = \frac{9}{x - 4}$
35. $\frac{1}{u} + \frac{1}{u - 2} = 3$
36. $\frac{3}{c + 1} + \frac{4}{c - 1} = 2$

C

37. $(2x - 1)^2 - (x + 3) = -1$
38. $(x + 6)^2 + (3x - 5)^2 = 65$
39. $x^2 + x\sqrt{3} - 3 = 0$
40. $2y^2 - y\sqrt{5} + 1 = 0$
41. $kx^2 - \frac{2}{5}x - \frac{1}{5k} = 0$
42. $\frac{x^2}{6} - \frac{1}{2}kx = \frac{2}{3}k^2$
43. $x^2 + bx + c = 0$
44. $ax^2 + bx + c = 0$

Computer Exercises For students with computer experience

1. Write a program that will use the process of completing the square to rewrite an equation of the form $x^2 + bx = c$ in the form $(x - d)^2 = e$ when you input values for b and c. A sample output would be

$$(X - 3) \uparrow 2 = 7.$$

2. Modify the program you wrote for Exercise 1 so that it will rewrite an equation of the form $ax^2 + bx = c$ in the form $a(x - d)^2 = e$ when you input values for a, b, and c, $a \neq 0$.

11–2 The Quadratic Formula

Quadratic equations arise so frequently in mathematics that it is useful to have a formula that you can use to obtain their solutions directly from the coefficients. Such a formula can be derived by applying the method of completing the square to the general form of a quadratic equation, that is, $ax^2 + bx + c = 0$, $a \neq 0$.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \longleftarrow \{\text{Complete the square}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \longleftarrow \{\text{If } b^2 - 4ac \geq 0$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This open sentence is called the **quadratic formula.**

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, and $b^2 - 4ac \geq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1 Solve $2y^2 - 6y = -3$ using the quadratic formula.

SOLUTION First rewrite the given equation in the form $ax^2 + bx + c = 0$.

$$2y^2 - 6y = -3$$
$$2y^2 - 6y + 3 = 0$$

Then apply the formula $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, with $a = 2$, $b = -6$, $c = 3$.

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{2(2)}$$
$$= \frac{6 \pm \sqrt{36 - 24}}{4}$$
$$= \frac{6 \pm \sqrt{12}}{4}$$
$$= \frac{6 \pm 2\sqrt{3}}{4}$$
$$= \frac{3 \pm \sqrt{3}}{2}$$

Thus, $y = \dfrac{3 + \sqrt{3}}{2}$ or $y = \dfrac{3 - \sqrt{3}}{2}$.

Checking the results is left to you.

$\therefore$ the solution set is $\left\{\dfrac{3 + \sqrt{3}}{2}, \dfrac{3 - \sqrt{3}}{2}\right\}$.

Notice the role of the value of $b^2 - 4ac$ in the quadratic formula.

1. When $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ is a positive real number and the quadratic formula gives *two different* real roots of the equation.
2. When $b^2 - 4ac = 0$, $\sqrt{b^2 - 4ac} = 0$ and the quadratic formula gives *only one* root, called a **double root.**

$$\frac{-b \pm 0}{2a} = \frac{-b}{2a}$$

3. When $b^2 - 4ac < 0$, there is *no* real value for $\sqrt{b^2 - 4ac}$ and, hence, no real root.

Because the value of $b^2 - 4ac$ distinguishes, or *discriminates*, the three possibilities, it is called the **discriminant** of the quadratic equation.

EXAMPLE 2 Use the discriminant to determine the number of real roots of the equation $9x^2 = 12x - 5$.

SOLUTION First rewrite the given equation in the form $ax^2 + bx + c = 0$.

$$9x^2 = 12x - 5$$
$$9x^2 - 12x + 5 = 0$$

Then evaluate $b^2 - 4ac$, using $a = 9$, $b = -12$, and $c = 5$.

$$\begin{aligned} b^2 - 4ac &= (-12)^2 - 4(9)(5) \\ &= 144 - 180 \\ &= -36 \end{aligned}$$

Since $-36 < 0$, the equation has no real roots.

Oral Exercises

Give the values that you would use for *a*, *b*, and *c* in the quadratic formula.

1. $2u^2 - 3u + 1 = 0$
2. $5r^2 + r - 4 = 0$
3. $x^2 - x - 6 = 0$
4. $3x^2 + 5x = 2$
5. $s^2 + 5 = 3s$
6. $y^2 = 7y - 4$
7. $4m^2 - 7m = -1$
8. $x^2 + 13 = 4x$
9. $2w^2 = 5$
10. $6 - 3x^2 = 0$
11. $3q^2 = -4q$
12. $6z^2 = z$
13. $2x^2 - \frac{1}{5}x - \frac{11}{5} = 0$
14. $\frac{2}{3}x^2 + 4x + 5 = 0$
15. $\frac{e^2}{5} + 6e = 1$
16. $\frac{3f^2}{5} + 4 = 2f$
17. $9x^2 - x\sqrt{2} + 3 = 0$
18. $4y^2 - y\sqrt{13} - 2 = 0$

Written Exercises

Solve each equation using the quadratic formula. First give the exact solution, expressing any irrational roots in simplest form. Then approximate irrational roots to the nearest hundredth. If the equation has no real roots, so state.

A

1. $r^2 - 4r - 5 = 0$
2. $2s^2 - 8s + 6 = 0$
3. $x^2 + 3x = 4$
4. $9y^2 + 2y = -4$
5. $3a^2 - 10a = 8$
6. $4b^2 + 4b - 3 = 0$
7. $j^2 + 2j = 1$
8. $k^2 + 6k + 4 = 0$
9. $7m^2 - 6m + 1 = 0$
10. $n^2 + 5n + 2 = 0$
11. $8p^2 - 2p = -1$
12. $2q^2 + 8q = -5$

Use the discriminant to determine the number of real roots of each equation.

13. $m^2 + 4m - 4 = 0$
14. $2n^2 - 5n - 1 = 0$
15. $3x^2 - 6x + 30 = 0$
16. $x^2 - 4x + 5 = 0$
17. $v^2 = 4v - 1$
18. $4x^2 + 12x = -9$

19. $16x^2 + 1 = 8x$
20. $2t^2 = 5t - 3$
21. $5u^2 + 6u + 2 = 0$
22. $6x^2 - 4x = -1$
23. $15 - 3y^2 = 0$
24. $6p + 2 = -5p^2$

Solve each equation using the quadratic formula. Give only the exact solutions, expressing any irrational roots in simplest form.

B

25. $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$
26. $z^2 - \frac{1}{2}z - \frac{15}{16} = 0$
27. $\frac{m^2}{3} - 2m + \frac{5}{3} = 0$
28. $3y^2 - \frac{2}{5}y - \frac{1}{5} = 0$
29. $4x(x - 2) = 5(x - 1)$
30. $x(2x - 1) - 3(x + 4) = 0$
31. $\frac{3}{m^2} - \frac{8}{m} + 4 = 0$
32. $\frac{4z + 9}{z} + \frac{4}{z^2} = 0$
33. $\frac{3}{e - 2} + \frac{1}{e + 2} = 1$
34. $\frac{2}{n - 3} + \frac{5}{n + 3} = 1$
35. $\frac{6}{y + 1} = 1 - \frac{4}{y - 5}$
36. $\frac{5}{w - 2} - 2 = \frac{-1}{w + 3}$

C

37. $h^2 - 3 = h\sqrt{6}$
38. $r^2 + r\sqrt{3} = 3$
39. $v^2 - 2v\sqrt{5} = 3$
40. $x^2\sqrt{2} - x - 2\sqrt{2} = 0$
41. $2(c + 3)^2 = 5(c + 2)^2$
42. $(t + 1)^2 - 2(t - 2)^2 = 8$
43. Find a value of k so that the equation $kx^2 - 24x + 16 = 0$ has one real root.
44. Find a value of k so that the equation $25x^2 + kx + 4 = 0$ has one real root. Is there more than one such value of k?

Given the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac > 0$, show that the following statements are true.

45. The sum of the roots of the equation is $-\frac{b}{a}$.
46. The product of the roots of the equation is $\frac{c}{a}$.
47. Determine a quadratic equation whose roots are $2 - \sqrt{5}$ and $2 + \sqrt{5}$. (*Hint*: Use the results of Exercises 45 and 46.)
48. Determine a quadratic equation whose roots are $-3 \pm 2\sqrt{2}$.

Computer Exercises For students with computer experience

1. Write a program that will allow you to input the coefficients a, b, and c of a quadratic equation and will determine whether the equation has two different real roots, one real root (a double root), or no real roots.
2. Modify the program that you wrote for Exercise 1 so that, if the equation has at least one real root, the program will output the values of the two different real roots or of the one double real root. If the equation has no real roots, the output should so state.

11–3 Using Quadratic Equations to Solve Problems

Many problems involve solving quadratic equations.

EXAMPLE 1 One number is ten more than a second number. If the product of the two numbers is 144, find the numbers.

SOLUTION

Step 1 The problem asks for two numbers whose difference is 10 and whose product is 144.

Step 2 Let x = the first number. Then:

$x - 10$ = the second number

Step 3 The product of the numbers is 144.

$$\underbrace{x(x-10)}_{\text{The product of the numbers}} \underset{\text{is}}{=} 144$$

Step 4

$$
\begin{aligned}
x(x-10) &= 144 \\
x^2 - 10x &= 144 \\
x^2 - 10x + (-5)^2 &= 144 + (-5)^2 \\
(x-5)^2 &= 169
\end{aligned}
$$

Since the coefficient of x is an even number, completing the square is a convenient method to use in solving the equation.

$$
\begin{array}{lcl}
x - 5 = \sqrt{169} & \text{or} & x - 5 = -\sqrt{169} \\
x - 5 = 13 & & x - 5 = -13 \\
x = 18 & \text{or} & x = -8 \\
x - 10 = 8 & & x - 10 = -18
\end{array}
$$

Step 5 Checking the results is left to you.

$\therefore$ the numbers are either 18 and 8 or −8 and −18.

EXAMPLE 2 A landscaper designed a rectangular grass plot with length 3 m less than four times its width. If the landscaper purchased 51 m^2 of sod to use for the plot, what is the length of the plot?

SOLUTION

Step 1 The problem asks for the length of a rectangular grass plot.

The length is 3 m less than four times the width.

The area is 51 m^2.

Step 2 Let w = the width of the plot. Then:

$4w - 3$ = the length of the plot

Step 3 The area of the plot is 51 m^2.

$$\underbrace{w(4w-3)}_{\text{The area of the plot}} \underset{\text{is}}{=} \underbrace{51}_{51\text{ m}^2}$$

Step 4

$$w(4w - 3) = 51$$

$$4w^2 - 3w - 51 = 0 \longleftarrow$$

Since no other method of solution seems easier, use the quadratic formula.

$$w = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-51)}}{2(4)}$$

$$= \frac{3 \pm \sqrt{9 + 816}}{8} = \frac{3 \pm \sqrt{825}}{8} = \frac{3 \pm 5\sqrt{33}}{8}$$

$$w = \frac{3 + 5\sqrt{33}}{8} \approx 3.97 \text{ or } w = \frac{3 - 5\sqrt{33}}{8} \approx -3.22$$

Since length cannot be negative, reject the second root.

$$w \approx 3.97$$

$$4w - 3 \approx 4(3.97) - 3 = 12.88$$

Step 5

Check: $(3.97)(12.88) \stackrel{?}{=} 51$

$$51.1336 \approx 51 \quad \checkmark$$

∴ the length of the grass plot is approximately 12.88 m.

EXAMPLE 3 The total cost of a skiing trip was to be \$960, to be shared equally by each of the members of the Outdoor Club. At the last minute, ten people decided not to go on the trip, thus raising the cost to each of the other members by \$16. How many members actually went on the trip?

SOLUTION

Step 1 The problem asks for the number of members who went on the ski trip.

Step 2 Let x = the number of members actually on the trip. Then:

$x + 10$ = the number of members in the club

Step 3 The actual cost to each member going on the trip is \$16 more than the originally planned cost.

$$\frac{960}{x} = \frac{960}{x + 10} + 16$$

Step 4

$$\frac{960}{x} = \frac{960}{x + 10} + 16$$

$$960(x + 10) = 960x + 16x(x + 10)$$

$$960x + 9600 = 960x + 16x^2 + 160x$$

$$0 = 16x^2 + 160x - 9600$$

$$0 = x^2 + 10x - 600 \longleftarrow$$

$$0 = (x + 30)(x - 20)$$

Since the factors of $x^2 + 10x - 600$ can be seen readily, factoring is a convenient method of solving the equation.

$x + 30 = 0$ or $x - 20 = 0$

$x = -30$ | $x = 20$

Since the number of people cannot be negative, reject the first root.

Step 5 Checking the results is left to you.

∴ twenty members of the club actually went on the ski trip.

Oral Exercises

Tell whether you would solve each equation by factoring, completing the square, or using the quadratic formula.

1. $2x^2 - 6x = 0$
2. $x^2 - 5x + 5 = 0$
3. $x^2 + 12x - 8 = 0$
4. $32x^2 = 64$
5. $5x^2 - 7x + 2 = 0$
6. $x^2 - 2x + 15 = 0$
7. $9x^2 - 180 = 0$
8. $2x^2 - 3x - 2 = 0$
9. $7x^2 + 2x - 8 = 0$
10. $x^2 + 18x - 24 = 0$
11. $x^2 - 10x + 3 = 0$
12. $15x^2 - 7x + 1 = 0$

Problems

Solve. Approximate irrational roots to the nearest hundredth. Reject impossible roots.

A **1.** A bulletin board is 30 cm longer than it is high. Its area is 2800 cm^2. Find the length and height of the bulletin board.

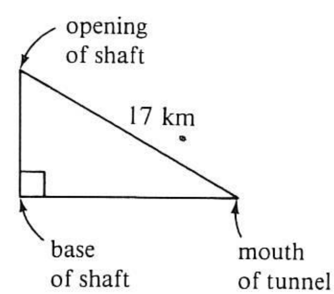
2. A painting is 6 cm longer than it is high. Its area is 691 cm^2. Find the length and height of the painting.

3. The sum of a number and its reciprocal is $\frac{25}{12}$. Find the number.

4. The difference when a number is decreased by twice its reciprocal is $\frac{1}{6}$. Find the number.

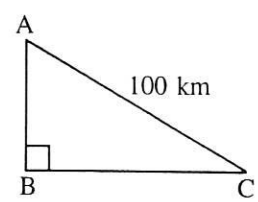
5. If the area of the floor of an office is 24 m^2 and the length of the floor is 2 m longer than four times its width, find the dimensions of the floor.

6. The area of the Johnsons' dining room floor is 36 m^2. If the length of the floor is 1 m longer than twice its width, find the dimensions of the floor.



Ex. 7

7. A tunnel through a mountain has a vertical ventilation shaft whose outside opening is 17 km up the slope of the mountain from the mouth of the tunnel. The distance from the base of the shaft to the mouth of the tunnel is 7 km more than the height of the ventilation shaft. Find the height of the shaft and its distance from the mouth of the tunnel.



Ex. 8

8. Town A is directly north of town B and town C is directly east of town B. The distance between towns A and C is 100 km. If the distance from town A to town B is 20 km less than the distance from town C to town B, find the distance between towns A and B and the distance between towns C and B.

B 9. Kedra earned $192 from the sale of flower arrangements that she had made. If she had charged $8 more apiece, she could have sold two fewer arrangements and still have earned the same amount. How many arrangements did Kedra sell?

10. Pamela and Anita can complete a job in 6 h if they work together. Working alone, Pamela takes 4 h longer to complete the job than Anita does when she works alone. How long would it take each woman to complete the job working alone?

11. Jose rowed his fishing boat 4 km up a stream and then returned the same distance. The current in the stream was flowing at a speed of 3 km/h. If the round trip took Jose 1 h 20 min, find the speed of his boat in still water.

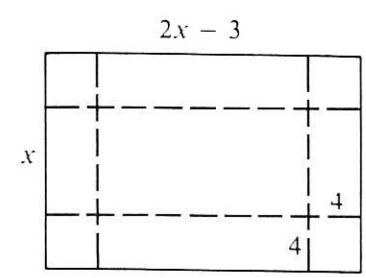
12. The members of the Historical Society sold a number of tickets to the Easton town picnic and raised $240 from the sale. If the Society had lowered its price by $2 a ticket, the members would have needed to sell 20 more tickets to earn the $240. How many tickets did the Society members actually sell?

13. Ron takes 2 h less than his brother Russ to cut enough firewood to fill his family's wood bin. Working together, they can fill the bin in $4\frac{4}{9}$ h. How long would it take each of them to cut the wood alone?

14. The average speed in still air of a certain plane is 500 km/h. With a constant wind blowing, the plane takes $4\frac{1}{6}$ h to fly 1040 km into the wind and then return the same distance. What is the speed of the wind?

C 15. Lauren bought a certain number of shares of stock in the XYZ Company for $405. As soon as the price of a share in the company increased by $10, she sold all but 16 shares and received $483. How many shares did she buy?

16. An open box was formed from a rectangular sheet of metal by cutting a square of side 4 cm from each of the corners of the rectangular sheet and then folding up the edges. The length of the original sheet of metal was 3 cm less than twice its width, and the volume of the box was 532 cm³. Find the dimensions of the sheet of metal.



Ex. 16

17. A car starts traveling due east along a road. At the same time and from the same point, a second car starts traveling due north at a speed that is 20 km/h faster than that of the first car. After 2 h, the cars are 200 km apart. At what speeds are they traveling?

Self-Test 1

VOCABULARY completing the square (p. 545)
quadratic formula (p. 548)
double root (p. 549)
discriminant of a quadratic equation (p. 549)

Solve by completing the square.

1. $x^2 + 4x - 3 = 0$ 2. $y^2 = 6y - 1$ *Obj. 1, p. 543*

Solve by using the quadratic formula.

3. $2z^2 + 5z + 1 = 0$ 4. $3w^2 - 2 = 4w$ *Obj. 2, p. 543*

5. The perimeter of a rectangle is 40 cm. If the area of the rectangle is 96 cm^2, find the width and length of the rectangle. *Obj. 3, p. 543*

Check your answers with those at the back of the book.

Chu Shih-chieh

13th–14th centuries

Chu Shih-chieh was one of the greatest Chinese mathematicians, yet little is known of his personal life. He is thought to have lived in Yen-shan, near Beijing, and to have traveled throughout the country as a wandering teacher for twenty years.

Chu is the author of *Suan-hsüeh ch'i-meng (Introduction to Mathematical Studies),* written in 1299. This work was used for centuries in East Asia as a textbook for beginners. Another of Chu's studies, *Ssu-yüan yü-chien (Precise Mirror of the Four Elements),* written in 1303, marked the height of the development of algebra in China. In this text Chu demonstrated a method for representing up to four unknowns in one algebraic equation. Also in this work Chu developed a method for finding rational square roots. Chu Shih-chieh made substantial contributions as well to the theory of series and the method of finite differences.

Graphing Quadratic and Other Polynomial Functions

OBJECTIVES for Sections 11-4 through 11-6:
1. *To graph quadratic functions.*
2. *To graph simple polynomial functions.*
3. *To graph quadratic inequalities.*

11–4 Quadratic Functions

Any function of the type

$$f\colon x \longrightarrow ax^2 + bx + c, \quad a \neq 0,$$

is called a **quadratic function.** If the domain of such a function is $\mathcal{R}$, then its graph on a coordinate plane is a smooth curve called a **parabola.**

The simplest quadratic function is $f\colon x \longrightarrow x^2$, whose graph is specified by the equation $y = x^2$. Figure 1 shows the curve, together with the table of values used to construct it.

x	y
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

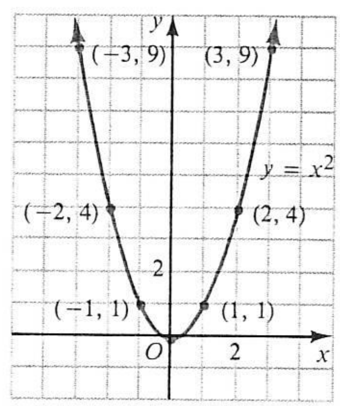


Figure 1

From Figure 1 you can see that the origin is the lowest point, or *minimum point*, on the graph of $y = x^2$. In general, a point on a curve is said to be a **minimum point** of the curve if its y-coordinate is less than or equal to the y-coordinate of every other point on the curve.

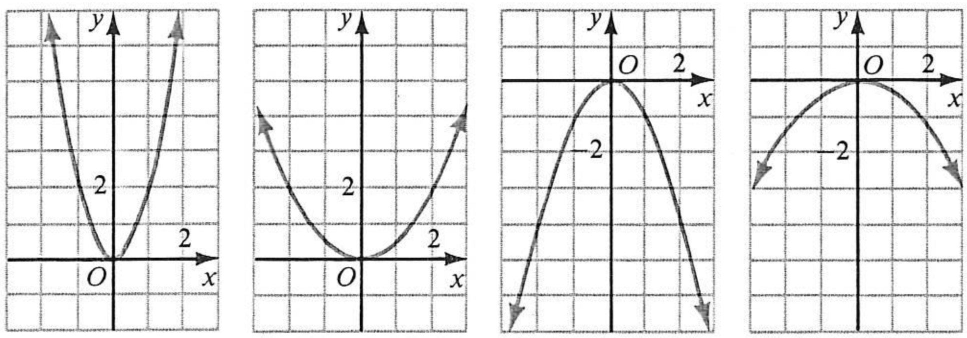
Recall from Chapter 5 that any member of the range of a function is called a *value* of the function. If the point (j, k) is a minimum point of the graph of a function, then k is called the **minimum value** of the function. The definitions for a **maximum point,** or highest point, and the **maximum value** are similar.

It can be proved that the graph of each quadratic function with domain $\mathcal{R}$ has one maximum point or one minimum point, but not both. This maximum or minimum point is called the **vertex** of the parabola. Correspondingly, it can be proved that each quadratic function with domain $\mathcal{R}$ has one maximum value or one minimum value, but not both.

Now imagine that the graph of $y = x^2$ that is shown in Figure 1 could be folded along the y-axis. Do you see that the part of the curve which lies in the first quadrant would then *coincide* with the part of the curve that lies in the second quadrant? This illustrates the fact that, given any point (u, t) that lies on the parabola, the point $(-u, t)$ also lies on the parabola. This is confirmed by the fact that, if $t = u^2$, then $t = (-u)^2$ also.

Since it divides the curve $y = x^2$ into two matching parts, the y-axis is called the **axis of symmetry** of the curve, and the curve is said to be *symmetric with respect to the y-axis.* It can be proved that the graph of any quadratic function with domain $\mathcal{R}$ has an axis of symmetry.

Several other quadratic functions are graphed in Figure 2.



$y = 2x^2$ $\quad$ $y = \frac{1}{2}x^2$ $\quad$ $y = -x^2$ $\quad$ $y = -\frac{1}{3}x^2$

Figure 2

As you can see, the equations specifying the functions graphed in Figure 2 are of the form

$$y = ax^2,$$

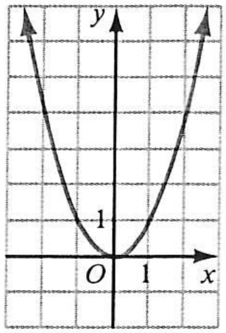
where a is 2, $\frac{1}{2}$, -1, and $-\frac{1}{3}$, respectively. Notice that

when $a > 0$, the parabola opens upward;
when $a < 0$, the parabola opens downward.

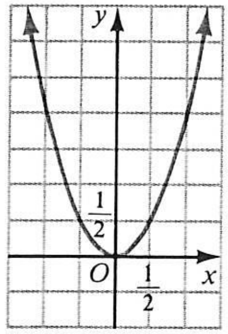
Notice also that the less $|a|$ is, the broader the parabola appears to be when the same scale is used. These remarks about a also apply to the general quadratic function

$$f\colon x \longrightarrow ax^2 + bx + c.$$

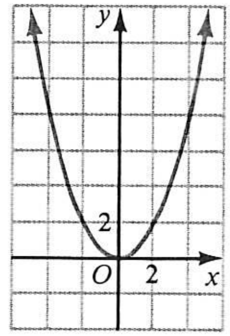
In Figure 2, the curve $y = -\frac{1}{3}x^2$ appears broader than the curve $y = 2x^2$. However, if the scale is changed suitably, all parabolas can be made to look alike, as suggested by Figure 3 on the following page.



$y = x^2$



$y = 2x^2$



$y = \frac{1}{2}x^2$

Figure 3

Any member of the domain of a function for which the value of the function is 0 is called a **zero of the function.** For example, the zeros of the function

$$f: x \longrightarrow x^2 - 4x - 5$$

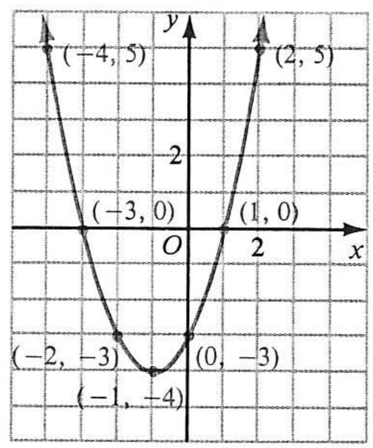
are 5 and -1 because, for these values of x,

$$x^2 - 4x - 5 = (x - 5)(x + 1) = 0.$$

EXAMPLE 1 Graph $f: x \longrightarrow x^2 + 2x - 3$ and give the zeros of the function, if any.

SOLUTION Set $y = x^2 + 2x - 3$ and compare this equation with $y = ax^2 + bx + c$. Since $a = 1$ and $1 > 0$, the parabola opens upward. Make a table of values for x and y, then sketch the graph.

x	y
−4	5
−3	0
−2	−3
−1	−4
0	−3
1	0
2	5



The graph intersects the x-axis at the points $(-3, 0)$ and $(1, 0)$.

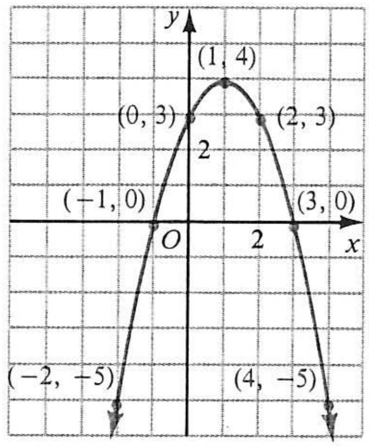
$\therefore$ the function has two zeros, -3 and 1.

Notice that the zeros of a function are the x-coordinates of those points at which the graph of the function intersects the x-axis.

EXAMPLE 2 Graph $f: x \longrightarrow 2x - x^2 + 3$ and give the zeros of the function, if any.

SOLUTION Set $y = 2x - x^2 + 3$ and compare this equation with $y = ax^2 + bx + c$. Since $a = -1$ and $-1 < 0$, the parabola opens downward. Make a table of values for x and y.

x	y
−2	−5
−1	0
0	3
1	4
2	3
3	0
4	−5



The graph intersects the x-axis at the points $(-1, 0)$ and $(3, 0)$.

$\therefore$ the zeros of the function are -1 and 3.

As you have seen, the zeros of a function are the values of x at the points where the graph of the function intersects the x-axis. Since $y = 0$ for every point on the x-axis, you can therefore find the zeros of the quadratic function specified by the equation

$$y = ax^2 + bx + c$$

by replacing y with 0 and solving the resulting quadratic equation.

Also, if you know two values of x, say x_1 and x_2, which are paired with the same value of y, then

$$x = \frac{x_1 + x_2}{2}$$

is the equation of the axis of symmetry, since the graphs of (x_1, y) and (x_2, y) are equidistant from this line. Thus, in Example 1 on the preceding page, note that $f(-2) = -3$ and $f(0) = -3$, and so the equation of the axis of symmetry is

$$x = \frac{-2 + 0}{2}, \quad \text{or} \quad x = -1,$$

as shown in Figure 4. The minimum point on this curve occurs then when $x = -1$. Thus, the vertex is at

$$(-1, -4).$$

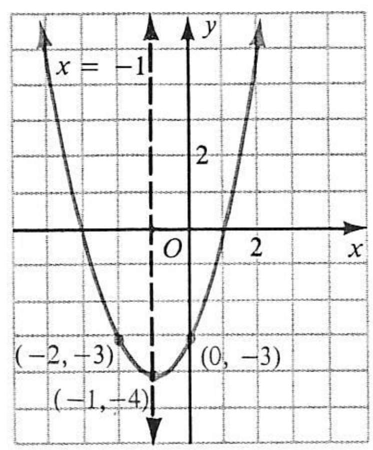


Figure 4

Notice that the equation of the axis of symmetry was found by taking the average of the x-coordinates of two points on the curve. Remember that the zeros of the function occur when $ax^2 + bx + c = 0$, that is, when

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If you take the average of these x values, you obtain the equation

$$x = \frac{x_1 + x_2}{2} = -\frac{b}{2a}.$$

This formula gives the x-coordinate of the vertex of a parabola and the equation of the axis of symmetry of the parabola.

> Given a parabola with equation $y = ax^2 + bx + c,\ a \neq 0$:
>
> **1.** The x-coordinate of the vertex of the parabola is $-\frac{b}{2a}$.
>
> **2.** The axis of symmetry of the parabola is the line $x = -\frac{b}{2a}$.

Once you have determined the x-coordinate of the vertex of a parabola, you can use this value to determine the maximum or minimum value of the quadratic function that is represented by the parabola.

EXAMPLE 3 Find the minimum value of $g: x \longrightarrow x^2 - 4x + 2$.

SOLUTION Find the x-coordinate of the vertex of the parabola that represents the function, using $a = 1$ and $b = -4$.

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = -(-2) = 2$$

Substitute this value of x into the equation $y = x^2 - 4x + 2$ in order to find the y-coordinate of the vertex.

$$y = x^2 - 4x + 2 = (2)^2 - 4(2) + 2 = -2$$

$\therefore$ the minimum value of the function is -2.

Oral Exercises

a. Tell whether the graph of each function opens upward or downward.

b. Tell whether the function has a maximum or a minimum value.

1. $f: x \longrightarrow 4x^2$

2. $g: x \longrightarrow -\frac{1}{2}x^2$

3. $h: x \longrightarrow 3x + x^2$

4. $f: x \longrightarrow 5x - x^2$

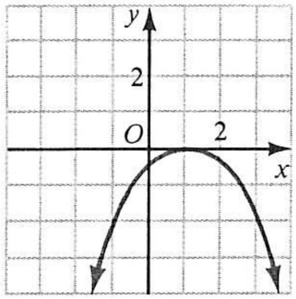
5. $g: x \longrightarrow -\frac{x^2}{4} + 2x$

6. $f: x \longrightarrow 3x^2 + 6x - 1$

7. $g: x \longrightarrow -(x - 1)^2$

8. $h: x \longrightarrow -(2 - x)^2$

Estimate the zeros, if any, of the functions whose graphs are sketched.

9.


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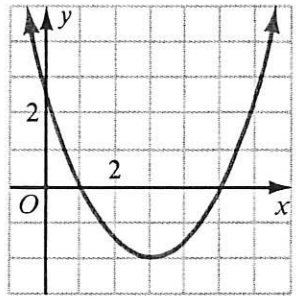

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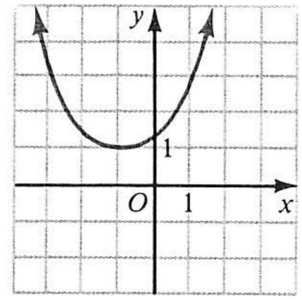

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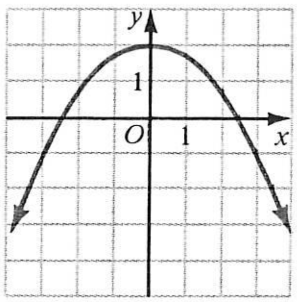

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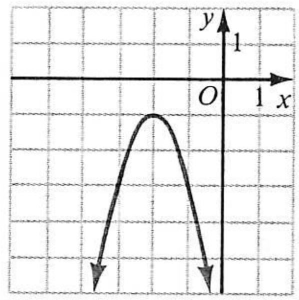

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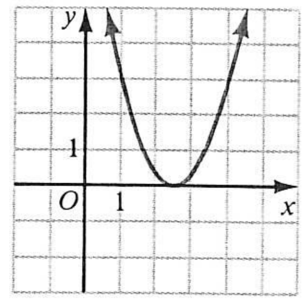


Written Exercises

Graph each function and determine its zeros, if any.

A

1. $f: x \longrightarrow 2x^2$
2. $g: x \longrightarrow -x^2$
3. $h: x \longrightarrow -\frac{1}{2}x^2$
4. $g: x \longrightarrow \frac{5}{2}x^2$
5. $F: x \longrightarrow -x^2 + 3$
6. $G: x \longrightarrow 2x^2 - 1$
7. $g: x \longrightarrow x^2 + 4x$
8. $h: x \longrightarrow -x^2 + 2x$
9. $H: x \longrightarrow x^2 + 2x + 1$
10. $G: x \longrightarrow -x^2 + 6x - 9$
11. $g: x \longrightarrow -x^2 - 6x - 7$
12. $f: x \longrightarrow x^2 - 2x - 3$

Find the minimum value of each function.

13. $f: x \longrightarrow x^2 + 3x$
14. $g: x \longrightarrow x^2 + 4$
15. $h: x \longrightarrow x^2 - x - 6$
16. $H: x \longrightarrow 3x^2 - 6x + 4$
17. $G: x \longrightarrow 4x^2 - 6$
18. $F: x \longrightarrow \frac{1}{2}x^2$

Find the maximum value of each function.

19. $f: x \longrightarrow -x^2 - 3x$
20. $F: x \longrightarrow -2x^2 + 4x$
21. $g: x \longrightarrow -x^2 - 6x - 10$
22. $G: x \longrightarrow -10x^2 + x + 5$
23. $h: x \longrightarrow -2x^2 + x$
24. $H: x \longrightarrow -\frac{1}{3}x^2$

In each of Exercises 25-28, graph the given equations on the same set of coordinate axes.

B

25. **a.** $y = x^2$ **b.** $y = x^2 + 1$ **c.** $y = x^2 - 3$

26. **a.** $y = \frac{1}{2}x^2$ **b.** $y = \frac{1}{2}x^2 + 1$ **c.** $y = \frac{1}{2}x^2 - 3$

27. **a.** $y = x^2$ **b.** $y = (x + 1)^2$ **c.** $y = (x - 3)^2$

28. **a.** $y = -2x^2$ **b.** $y = -2(x + 1)^2$ **c.** $y = -2(x - 3)^2$

29. For any real numbers a and k, describe how the graph of $y = ax^2 + k$ is related to the graph of $y = ax^2$.

30. For any real numbers a and h, describe how the graph of $y = a(x + h)^2$ is related to the graph of $y = ax^2$.

31. Graph $A = \{(x, y): x = y^2\}$.

a. Does the graph have a maximum point? a minimum point?

b. Is A a function?

32. Graph $B = \{(x, y): x = -y^2\}$.

a. Does B have a maximum point? a minimum point?

b. Is B a function?

33. Find and graph a linear function whose graph has no minimum points and no maximum points.

34. Find and graph a linear function whose graph has infinitely many minimum points and infinitely many maximum points. What is the minimum value of the function and what is the maximum value?

C

35. Find the coordinates of two points on the graph of the equation $y = x^2$ such that the x-coordinates, x_1 and x_2, differ by 1 and the y-coordinates, y_1 and y_2, differ by at least 1,000,000. What is the least possible positive integral value of x_1 that satisfies these conditions?

36. Find a value of k so that the line with equation $y = \frac{1}{1{,}000{,}000}x$ intersects the parabola with equation $y = kx^2$ at a point whose y-coordinate is 1.

37. If the point $P(u, v)$ lies on the graph of $y = ax^2 + bx + c$, $a \neq 0$, show that the point $Q\left(-\frac{b}{a} - u, v\right)$ also lies on the graph.

38. **a.** Is it possible to find the equation of a parabola passing through any two points whose x-coordinates are given? How many parabolas are there that pass through points with the given x-coordinates? (*Hint*: Refer to Figure 2 on page 558 and consider the points where $x = 1$ and $x = -1$.)

b. It can be established that two points lie on exactly one line whose equation can be determined. What is the least number n of arbitrary points needed to determine the equation of a parabola?

Computer Exercises For students with computer experience

1. Write a program that will allow you to input values for a, b, and c and will determine an equation of the axis of symmetry of the parabola whose equation is $y = ax^2 + bx + c$. The output should also indicate whether the parabola opens upward or downward. Be sure to account for the fact that the value of a must be nonzero.
2. Modify the program that you wrote for Exercise 1 so that it will also determine the coordinates of the vertex of the parabola.

11–5 Polynomial Functions

Any function of the form $x \longrightarrow P(x)$, where $P(x)$ represents a polynomial, is called a **polynomial function.** For example, the following are polynomial functions:

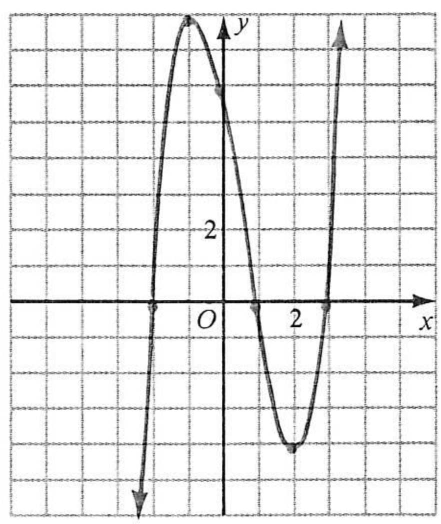
$f: x \longrightarrow \frac{2}{3}x - 4$ $\quad$ $h: x \longrightarrow 9x^3$ $\quad$ $g: x \longrightarrow 5 - 3x - 2x^2$

The graph of a polynomial function with domain $\mathcal{R}$ is a smooth curve, but many of these curves may be difficult to plot accurately. In more advanced courses you will learn many special plotting techniques. For the present, however, if you plot enough points, you will be able to make a rough sketch of the curve.

EXAMPLE 1 Sketch the graph of the function $f: x \longrightarrow x^3 - 2x^2 - 5x + 6$ and give the zeros of the function, if any.

SOLUTION Make a table of values for x and y.

x	y
−3	−24
−2	0
−1	8
0	6
1	0
2	−4
3	0
4	18

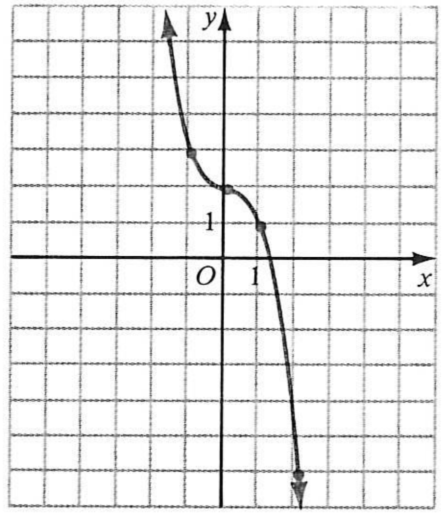


∴ the zeros of the function are −2, 1, and 3.

EXAMPLE 2 Make a rough sketch of the graph of the function $f: x \longrightarrow -x^3 + 2$ and estimate the zeros, if any.

SOLUTION

x	y
-2	10
$-\frac{3}{2}$	$5\frac{3}{8}$
-1	3
$-\frac{1}{2}$	$2\frac{1}{8}$
0	2
$\frac{1}{2}$	$1\frac{7}{8}$
1	1
$\frac{3}{2}$	$-1\frac{3}{8}$
2	-6



$\therefore$ there is one zero, and it lies between 1 and $1\frac{1}{2}$.
Estimate it from the graph as $1\frac{1}{4}$.

Oral Exercises

Name the zero(s) of each function.

1. $f: x \longrightarrow x^2 - 64$
2. $g: x \longrightarrow x^3 - 64$
3. $h: x \longrightarrow x^3 + 64$
4. $g: x \longrightarrow x(x - 2)(x + 1)$
5. $h: x \longrightarrow x(x^2 - 9)$
6. $f: x \longrightarrow x(x + 4)\left(x - \frac{1}{2}\right)$
7. $f: x \longrightarrow (x - 1)(x + 2)\left(x + \frac{3}{8}\right)$
8. $h: x \longrightarrow \left(x + \frac{1}{3}\right)\left(x - \frac{5}{8}\right)\left(x + \frac{1}{3}\right)$

Written Exercises

Sketch the graph of the given function and estimate the zeros, if any.

A

1. $f: x \longrightarrow x^3$
2. $g: x \longrightarrow -x^3$
3. $h: x \longrightarrow x^3 - 2$
4. $f: x \longrightarrow 2 - x^3$
5. $g: x \longrightarrow -x^3 + 3x$
6. $h: x \longrightarrow -x^3 - 3x$
7. $f: x \longrightarrow x^3 - 4x + 1$
8. $g: x \longrightarrow x^3 + 4x - 2$
9. $h: x \longrightarrow x^4 - 1$
10. $f: x \longrightarrow -x^4 + 1$
11. $g: x \longrightarrow x^4 + x^2$
12. $h: x \longrightarrow x^4 - x^2$

Find the required value of f when $f(x)$ is defined as follows.

a. $f(x) = x^3 - x^2$ b. $f(x) = -x^3 + x^2$

B 13. $f(-100)$ 14. $f(-1000)$ 15. $f(-10{,}000)$

16. $f(100)$ 17. $f(1000)$ 18. $f(10{,}000)$

19. a. What is true of the values of $f\colon x \longrightarrow x^3 - x^2$ as x takes on lesser and lesser values? greater and greater values?

b. What is true of the values of $f\colon x \longrightarrow -x^3 + x^2$ as x takes on lesser and lesser values? greater and greater values?

C 20. If P is a polynomial function such that $P(a) < 0$ and $P(b) > 0$ or such that $P(a) > 0$ and $P(b) < 0$, there is a theorem which states that P has *at least one* zero between a and b. Given $P\colon x \longrightarrow x^3 - 2x^2 + 12x - 10$, must P have at least one zero between -3 and 12?

21. Given $Q\colon x \longrightarrow (x - 1)^4 - 1$, $Q(-1) = 15$ and $Q(4) = 80$. On the basis of the theorem cited in Exercise 20, can you conclude that Q has no zeros between -1 and 4?

PROGRAMMING IN BASIC

The graphing program given on page 247 can be expanded as follows to provide a variation designed specifically for quadratic equations.

```
10   PRINT "TO GRAPH A QUADRATIC"
15   PRINT "EQUATION, Y = AX ↑ 2 + BX + C"
20   PRINT "(SENTENCE IS IN LINE 130):"
25   PRINT "INPUT A (<> 0), B, C";
30   INPUT A, B, C
40  }
 :  }  from program on page 247
70  }
73   LET D = B*B - 4*A*C
74   IF D < 0 THEN 79
75   LET D1 = SQR(D)/(2*A)
76   LET X1 = -B/(2*A)
77   PRINT "ZEROS ARE: "; X1 - D1;", ";X1 + D1
78   GOTO 80
79   PRINT "NO ZEROS"
80  }
 :  }  from program on page 247
120 }
130   IF Y = A*X*X + B*X + C THEN 250
140 }
 :  }  from program on page 247
340 }
```

Exercises

1. Type in and RUN the program as given. INPUT 9 for the extent of the graph and 0.25, 1, and −8 for the values of A, B, and C, respectively.

RUN the program for each of the following quadratic functions. INPUT 9 for the extent of the graph.

2. $y = x^2$

3. $y = 0.25x^2$

4. $y = 0.25x^2 - 4$

5. $y = 0.25x^2 - 6$

6. $y = 0.25x^2 - 9$

7. $y = -0.25x^2$

8. $y = -0.25x^2 + 4$

9. $y = -0.25x^2 + 9$

10. $y = 0.25x^2 - x - 3$

11–6 Quadratic Inequalities

Recall from Sections 5-6 and 6-8 the methods used in graphing linear inequalities and systems of linear inequalities. Similar methods may be used in graphing *quadratic inequalities* as well.

A **quadratic inequality** in two variables is an inequality whose *associated equation* is a quadratic equation. For example, consider the quadratic equation

$$y = x^2 - 1.$$

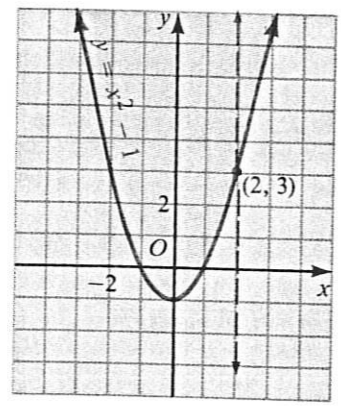


Figure 5

The graph of this equation is a parabola that separates a coordinate plane into two regions. One of these regions is *above* the parabola, as shown by the colored shading in Figure 5, and the other region is *below* the parabola, as shown by the gray shading. The parabola itself is the *boundary* of the two regions.

If you start at any point of the parabola, say (2, 3), and move vertically *upward*, the y-coordinates of the points on the plane increase. Thus, the region above the parabola is the graph of the quadratic inequality

$$y > x^2 - 1.$$

On the other hand, if you move vertically *downward* from (2, 3), the y-coordinates of the points *decrease*, and so the region below the parabola is the graph of the quadratic inequality

$$y < x^2 - 1.$$

As in graphing linear inequalities, you use a *solid* line to indicate that the boundary is to be included as part of a graph and a *dashed* line to indicate that it is not.

EXAMPLE 1 Graph $y < x^2 - 6x + 5$ on a coordinate plane.

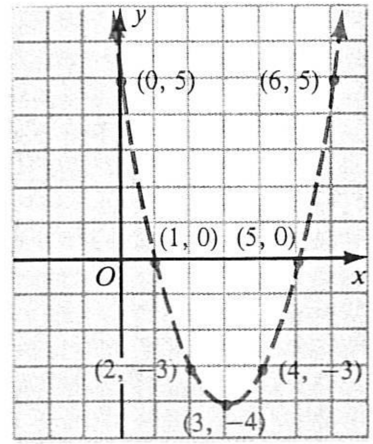
SOLUTION

1. Graph the associated equation
$$y = x^2 - 6x + 5$$
as a *dashed* curve.

2. Shade the region *below* the curve.

Check: Select any point in the shaded region and determine whether its coordinates satisfy the original inequality.

$$(0, 0):\ y < x^2 - 6x + 5$$
$$0 \overset{?}{<} 0^2 - 6(0) + 5$$
$$0 < 5 \quad \checkmark$$



Thus (0, 0) is in the solution set, and the correct region of the graph has been shaded.

The solution set of a *system of inequalities* includes all points on a coordinate plane that satisfy each inequality in the system. To solve a system of inequalities, you graph each inequality in the system on the same coordinate plane.

EXAMPLE 2 Graph the solution set of the following system of inequalities:

$$y > 1$$
$$y \geq x^2 - 2$$

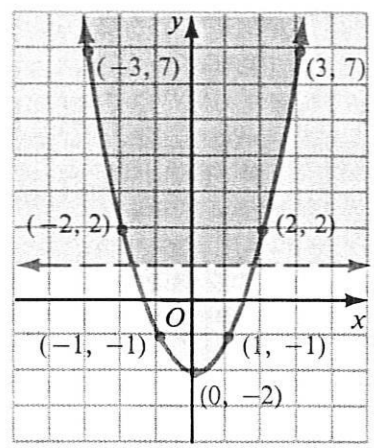
SOLUTION

1. Graph $y = 1$ as a *dashed* line. The graph of $y > 1$ is the open half-plane above this line.

2. Graph $y = x^2 - 2$ as a *solid* curve. The graph of $y > x^2 - 2$ is the region above this curve.

3. The intersection of the two shaded regions is the graph of the given system (double shading).

4. Check your work by selecting any point within the double-shaded region, such as (1, 2). This ordered pair should satisfy *each* of the original inequalities.



$$y > 1 \qquad\qquad y \geq x^2 - 2$$
$$2 > 1 \quad \checkmark \qquad 2 \overset{?}{\geq} (1)^2 - 2$$
$$2 \geq -1 \quad \checkmark$$

Thus (1, 2) is in the solution set, and the correct region of the graph has been shaded.

Oral Exercises

a. Tell whether the graph of the inequality is a solid or dashed curve.

b. Tell whether the graph contains the region above or below the curve.

1. $y \le x^2 + 1$	**2.** $y > x^2 - 2$	**3.** $y > 4 - x^2$
4. $y \le x^2 + 5$	**5.** $y \ge x^2 - x$	**6.** $y < x - x^2$
7. $y < 3x - x^2$	**8.** $y \le x^2 - 3x$	**9.** $y < x^2 + 6x + 9$
10. $y \ge x^2 - x - 12$	**11.** $y \ge 5 - 4x - x^2$	**12.** $y > 8 + 2x - x^2$

Written Exercises

A **1–12.** Graph each inequality in Oral Exercises 1–12.

Graph the solution set of each system of inequalities.

B

13. $y > x^2$, $y \le 5$	**14.** $y \ge x^2$, $x < 2$	**15.** $y \le -\frac{1}{2}x^2$, $x > -1$	**16.** $y < 3x^2$, $y \ge 2$
17. $y < x^2 - 2x$, $y \ge -1$	**18.** $y > x^2 + 2x$, $y < 2$	**19.** $y \ge x^2 + x - 6$, $x \le 1$	**20.** $y \le x^2 - 3x + 2$, $x \ge 1$
21. $y \ge x^2 - 1$, $y \le x$	**22.** $y < x^2 - 2$, $y \ge -x$	**23.** $y < x^2 + 2x - 3$, $y > x - 2$	**24.** $y \ge 4 - 3x - x^2$, $y < x + 3$

C

25. $y \ge x^2 - 1$, $y \le x^2 + 1$	**26.** $y \le x^2 - 2$, $y \ge 2 - x^2$	**27.** $y > x^2 - 2x - 8$, $y \le 8 - 2x - x^2$	**28.** $y < 3 - 2x - x^2$, $\lvert y \rvert \le 2$

Self-Test 2

VOCABULARY

quadratic function (p. 557)
parabola (p. 557)
minimum (or maximum) point of a curve (p. 557)
minimum (or maximum) value of a function (p. 557)
vertex of a parabola (p. 558)
axis of symmetry (p. 558)
zero of a function (p. 559)
polynomial function (p. 564)
quadratic inequality (p. 567)

Graph each of the following on a coordinate plane.

1. $f: x \longrightarrow -x^2$	**2.** $g: x \longrightarrow x^2 + 3x - 5$	*Obj. 1, p. 557*
3. $F: x \longrightarrow -x^3$	**4.** $G: x \longrightarrow x^4 + 1$	*Obj. 2, p. 557*
5. $y \le x^2$	**6.** $y > x^2 + 2x$	*Obj. 3, p. 557*

Check your answers with those at the back of the book.

EXTRA

Rational Exponents

When you simplify $2^3 \cdot 2^2$ by writing $2^3 \cdot 2^2 = 2^{3+2} = 2^5$, you apply the law of exponents for products of powers, that is,

$$a^m \cdot a^n = a^{m+n}.$$

Can you determine what value n must have if this law is to be true for

$$2^n \cdot 2^n = 2?$$

Since $$2^n \cdot 2^n = 2^{n+n} = 2^{2n},$$

you have $$2^{2n} = 2.$$

But if powers of the same base are equal, the exponents of the powers must be equal provided the base is not -1, 0, or 1. This means, in this case, that $2n = 1$, or $n = \frac{1}{2}$. If

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2,$$

what meaning can be given to the symbol $2^{\frac{1}{2}}$? Because you know that

$$\sqrt{2} \cdot \sqrt{2} = 2 \quad \text{and} \quad (-\sqrt{2})(-\sqrt{2}) = 2,$$

it makes sense to *define* the symbol $2^{\frac{1}{2}}$ to represent one of the square roots of 2. Selecting the positive square root, you say that

$$2^{\frac{1}{2}} = \sqrt{2}.$$

This example suggests the following definition.

> If a is a positive real number and n is a positive integer, or if a is a negative real number and n is a positive odd integer,
>
> $$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Note that, if a is a negative real number and n is a positive even integer, $\sqrt[n]{a}$ does not name a real number, and hence neither does $a^{\frac{1}{n}}$.

EXAMPLE 1 Simplify.

a. $49^{\frac{1}{2}}$ b. $81^{\frac{1}{4}}$ c. $(-27)^{\frac{1}{3}}$ d. $(-81)^{\frac{1}{4}}$

SOLUTION

a. $49^{\frac{1}{2}} = \sqrt{49} = 7$

b. $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

c. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

d. $(-81)^{\frac{1}{4}}$ does not name a real number.

Now, consider the expression $(\sqrt{9})^3$. By the definition of a power with a positive integral exponent,

$$(\sqrt{9})^3 = \sqrt{9} \cdot \sqrt{9} \cdot \sqrt{9}$$
$$= 3 \cdot 3 \cdot 3 = 27.$$

Also, if the law of exponents for powers of powers, $(a^m)^n = a^{mn}$, is to be true for $(9^{\frac{3}{2}})^2$, you must have

$$(9^{\frac{3}{2}})^2 = 9^3 = 729 = 27^2, \quad \text{or} \quad 9^{\frac{3}{2}} = 27.$$

Thus, a reasonable definition of $9^{\frac{3}{2}}$ would be

$$9^{\frac{3}{2}} = (\sqrt{9})^3.$$

This example suggests the following definition.

> If a is a positive real number, m is an integer, and n is a positive integer; or if a is a negative real number, m is an integer, and n is a positive odd integer; then
>
> $$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m.$$

As before, if a is a negative real number and n is a positive *even* integer, $a^{\frac{m}{n}}$ does not name a real number.

EXAMPLE 2 Simplify.

a. $4^{\frac{3}{2}}$ b. $(-27)^{-\frac{2}{3}}$ c. $(-9)^{\frac{5}{2}}$

SOLUTION

a. $4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = (2)^3 = 8$

b. $(-27)^{-\frac{2}{3}} = [(-27)^{\frac{1}{3}}]^{-2} = (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

c. $(-9)^{\frac{5}{2}} = [(-9)^{\frac{1}{2}}]^5$ is not defined because $(-9)^{\frac{1}{2}}$ is not a real number.

It can be shown that under the definitions given here for $a^{\frac{1}{n}}$ and $a^{\frac{m}{n}}$, all the laws of exponents for positive integral exponents may be applied to powers with *rational exponents*, as can the definition of a negative exponent.

EXAMPLE 3 Simplify.

a. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{5}{4}}$ b. $\frac{9^{\frac{3}{4}}}{9^{\frac{1}{4}}}$ c. $(3^{-\frac{2}{3}})^6$

SOLUTION

a. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{5}{4}} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{5}{4}} = 2^{\frac{8}{4}} = 2^2 = 4$

b. $\frac{9^{\frac{3}{4}}}{9^{\frac{1}{4}}} = 9^{\frac{3}{4}-\frac{1}{4}} = 9^{\frac{1}{2}} = 3$

c. $(3^{-\frac{2}{3}})^6 = 3^{(-\frac{2}{3})(6)} = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

When the restriction $a > 0$ is placed on a, it can also be shown that

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

if m is an integer and n is a natural number. For example, $(16^2)^{\frac{1}{4}}$ and $(16^{\frac{1}{4}})^2$ name the same number, since

$$(16^2)^{\frac{1}{4}} = (256)^{\frac{1}{4}} = 4 \qquad \text{and} \qquad (16^{\frac{1}{4}})^2 = (2)^2 = 4.$$

Note that the form $(16^{\frac{1}{4}})^2$ is much easier to simplify.

Exercises

Simplify. If the expression does not represent a real number, so state. Assume that the domain of every variable is the set of positive real numbers.

1. $4^{\frac{1}{2}}$ **2.** $16^{\frac{1}{2}}$ **3.** $(-8)^{\frac{1}{3}}$ **4.** $81^{\frac{1}{4}}$

5. $(-4)^{\frac{1}{2}}$ **6.** $(9)^{\frac{3}{2}}$ **7.** $(16)^{\frac{3}{4}}$ **8.** $(8)^{\frac{2}{3}}$

9. $(36)^{-\frac{1}{2}}$ **10.** $(-8)^{-\frac{1}{3}}$ **11.** $3^{\frac{1}{2}} \cdot 3^{\frac{3}{2}}$ **12.** $5^{\frac{8}{3}} \div 5^{\frac{2}{3}}$

13. $(2y^2)^{\frac{1}{2}}(2y^2)^{\frac{3}{2}}$ **14.** $(27x)^{\frac{1}{3}} \div (27x)^{-\frac{2}{3}}$ **15.** $[(2x)^{\frac{2}{3}}]^3$ **16.** $[(81y)^{-\frac{1}{4}}]^2$

Chapter Summary

1. Any quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, can be solved by *completing the square*. It can also be solved by using the *quadratic formula*, producing the roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

2. The *discriminant* $b^2 - 4ac$ can be used to determine the number of roots of a quadratic equation. If the discriminant is
   1. positive, there are two real roots;
   2. zero, there is one real root, called a *double root*;
   3. negative, there is no real root.
3. The graph of a quadratic function $f: x \longrightarrow ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, is a *parabola*. Every parabola has a *vertex* (*maximum* or *minimum point*) and an *axis of symmetry*.
4. The graph of every *polynomial function* is a smooth curve.
5. A parabola separates a coordinate plane into two regions, the set of points above the parabola and the set of points below the parabola. These regions are used in graphing quadratic inequalities.

Chapter Review

Write the letter of the correct answer.

1. Name the term that completes the square in the expression 11-1
$x^2 - 8x + \underline{\ ?\ }$.

a. 4 b. 16 c. 64 d. -64

2. If $(n - 2)^2 = 6$, $n = \underline{\ ?\ }$.

a. 38 b. -4 or 8
c. $2 + \sqrt{6}$ or $2 - \sqrt{6}$ d. $-2 + \sqrt{6}$ or $-2 - \sqrt{6}$

3. Solve $4z^2 = 4z + 12$ by completing the square.

a. $\left\{\frac{\sqrt{13}}{2}, -\frac{\sqrt{13}}{2}\right\}$ b. $\left\{\frac{\sqrt{13}}{4}, -\frac{\sqrt{13}}{4}\right\}$
c. $\left\{1 + \frac{\sqrt{13}}{2}, 1 - \frac{\sqrt{13}}{2}\right\}$ d. $\left\{\frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}\right\}$

4. Solve $2x^2 - 2x - 1 = 0$ using the quadratic formula. 11-2

a. $\left\{\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}\right\}$ b. $\left\{\frac{-1 + \sqrt{3}}{2}, \frac{-1 - \sqrt{3}}{2}\right\}$
c. $\left\{-\frac{5}{4}, \frac{7}{4}\right\}$ d. no real roots

5. Solve $9x^2 = 12x - 2$ using the quadratic formula.

a. $\left\{\frac{2 + \sqrt{6}}{3}, \frac{2 - \sqrt{6}}{3}\right\}$ b. $\left\{\frac{-2 + \sqrt{6}}{3}, \frac{-2 - \sqrt{6}}{3}\right\}$
c. $\left\{\frac{2 + \sqrt{2}}{3}, \frac{2 - \sqrt{2}}{3}\right\}$ d. no real roots

6. Use the discriminant to determine the number of real roots of the equation $5n^2 + 2n = 2$.

a. 1 b. 2 c. 3 d. none

7. One of the two shorter sides of a right triangle is 7 cm longer than the other, and the hypotenuse is 3 cm longer than twice the measure of the shortest side. Find the length of the hypotenuse. 11-3

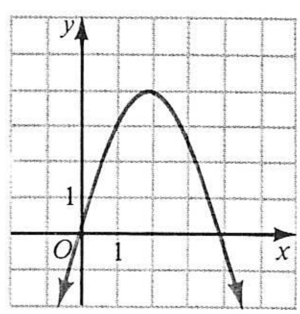
a. 5 cm b. 13 cm c. 24 cm d. 27 cm

8. Which of the following functions has a graph that opens downward? 11-4

a. $f: x \longrightarrow x^2 - 3x$ b. $f: x \longrightarrow 3x - x^2$
c. $f: x \longrightarrow (3 - x)^2$ d. $f: x \longrightarrow -(3x - x^2)$

9. Which of the following functions is graphed on the coordinate plane at the right?

a. $g: x \longrightarrow 4x - x^2$ b. $g: x \longrightarrow 4x + x^2$
c. $g: x \longrightarrow x^2 - 2x$ d. $g: x \longrightarrow 4x^2 - x$



10. Find the minimum value of $h: x \longrightarrow x^2 - 2$.

a. -2 b. -1 c. 0 d. 1

11. Which of the following is *not* a zero of $f: x \longrightarrow x^3 - x$? 11-5

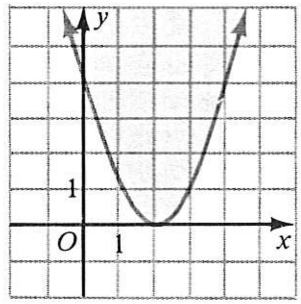
a. -2 b. -1 c. 0 d. 1

12. Find the zero(s) of $g: x \longrightarrow x^3 - 1$.

a. 1 b. 0, 1 c. -1, 1 d. -1, 0, 1

13. Which of the following inequalities is graphed on the coordinate plane at the right? 11-6

a. $y \leq x^2 - 4x + 4$ b. $y > x^2 - 4x + 4$
c. $y < x^2 - 4x + 4$ d. $y \geq x^2 - 4x + 4$



14. Describe the graph of the system $\begin{matrix} y > x^2 - 2 \\ y < 0 \end{matrix}$ on a coordinate plane.

a. a region bounded by the graph of $y = x^2 - 2$ and the y-axis
b. a region bounded by the graph of $y = x^2 - 2$ and the x-axis
c. the entire coordinate plane
d. none of the above

Chapter Test

1. Find a value of k so that $x^2 - 9x + k$ is a trinomial square. 11-1

Solve by completing the square. Express irrational solutions in simplest form.

2. $a^2 - 8a + 4 = 0$ 3. $3z^2 = 18z - 24$

Solve by using the quadratic formula. Express irrational solutions in simplest form.

4. $j^2 + 5j - 2 = 0$ 5. $4k^2 + 5 = 9k$ 11-2

6. Determine the number of real roots of the equation $4y^2 + 9 = 9y$.

Solve. Approximate irrational solutions to the nearest hundredth.

7. A rectangular swimming pool is 5 m wide and 10 m long. The pool is surrounded by a concrete walk of uniform width. The combined area of the pool and the walk is 50 m^2 more than the area of the swimming pool alone. Find the width of the walk. 11-3

8. The dimensions of a rectangle can be represented by consecutive odd integers, and its area is 255 cm^2. Find the dimensions.

Graph each function on a coordinate plane.

9. $f: x \longrightarrow x^2 + 1$

10. $g: x \longrightarrow -x^2 - 2x + 3$ 11-4

For each function, determine its zeros, if any, and its maximum or minimum value.

11. $G: x \longrightarrow -x^2 + 4x$

12. $H: x \longrightarrow x^2 + 2x + 1$

Sketch the graph of each function and estimate its zeros, if any.

13. $g: x \longrightarrow x^3 - x$

14. $h: x \longrightarrow 1 - x^3$ 11-5

15. Graph $y < x^2 + 2$ on a coordinate plane. 11-6

16. Graph the solution set of the system $\begin{array}{l} y \geq x^2 \\ x < 3 \end{array}$ on a coordinate plane.

Cumulative Review

Chapter 6

Solve each system of equations using any method.

1. $6x - y = 11$
   $2x + y = 5$

2. $8x - 3y = 3$
   $3x - 2y = -5$

3. $7r = 8 + 3s$
   $21r - 9s = 42$

4. $5s + t = 12$
   $s = t$

5. $3c + 5d = 10$
   $c = d - 2$

6. $3a = 8 + 5b$
   $12a - 20b = 24$

7. $2j - 3k = 4$
   $3j + k = 17$

8. $u + v = 6$
   $u - v = 10$

9. $9m + 5n = -6$
   $4m + 3n = 2$

10. $a + 3z = 10$
    $a + 9z = 22$

Solve.

11. There are 28 students in a computer programming class. The number of sophomores is 24 less than three times the number of juniors. How many sophomores and how many juniors are in the class?

12. Flying with the wind, an airplane can travel 1500 km in 2.5 h, but flying against the wind, the airplane requires 0.5 h more to make the return trip. Find the speed of the wind.

Chapter 7

Simplify.

13. $(u^2v - uv^2) - (u^2v + 2uv^2)$
14. $-8m^2n^2 - 3mn + 4m^2n + 6mn$
15. $-rs(r^2 - rs + s^2)$
16. $(-a)(a^2b)^3(ab^2)^2$
17. $(4x - 3)^2$
18. $(x - 3)(x^2 + 3x + 9)$

Factor completely.

19. $-12cd^2e - 48c^3d^3e$
20. $4m(2n + 9) - 3(2n + 9)$
21. $1 + 12w + 36w^2$
22. $3g^2 - 3gh + 2g - 2h$
23. $16x^2 - 50x + 25$
24. $24y^2 + 10y - 4$

Solve.

25. The sum of two integers is 36 and their product is 128. Find the integers.
26. A rectangular plot of land has an area of 1050 m^2. Its length exceeds its width by 5 m. Find the length of the plot.

Chapter 9

Solve.

27. $\frac{b + 12}{3} = \frac{b}{5}$
28. $\frac{2x + 3}{4} - \frac{4x + 1}{3} = \frac{5}{6}$
29. $\frac{a}{14} - \frac{2}{7} \leq \frac{a}{7}$
30. $\frac{2x - 1}{9} > \frac{1}{3}$
31. $\frac{w - 2}{w + 2} - 1 = \frac{12}{w^2 - 4}$
32. $\frac{x - 2}{x} - \frac{x}{x - 1} = \frac{5}{2}$
33. If y varies directly as x, and if $y = 42$ when $x = 14$, find y when $x = 8$.
34. If y varies inversely as x, and if $y = 18$ when $x = 24$, find y when $x = 12$.
35. If y varies jointly as x and z, and if $y = 4$ when $x = 6$ and $z = 2$, find y when $x = 12$ and $z = 10$.
36. If y varies directly as x and inversely as z, and if $y = 12$ when $x = 4$ and $z = 3$, find y when $x = 12$ and $z = 18$.
37. It takes Louise 6 h to do a certain job, and it takes Margaret 9 h to do the same job. How long would it take the two of them working together to complete the job?
38. When a car starts from rest and travels at a constant acceleration, the distance that it travels varies jointly as its acceleration and the square of the time that it has been traveling. If a car travels 320 m from rest in 4 s at an acceleration of 40 m/s^2, how far will the car travel from rest in 6 s at an acceleration of 25 m/s^2?

Simplify.

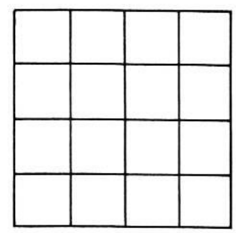
39. $\sqrt{225}$
40. $\sqrt{192}$
41. $(\sqrt{150})^2$
42. $\sqrt{(-24)^2}$
43. $4\sqrt{2} \cdot \sqrt{50}$
44. $\sqrt[4]{80}$
45. $\sqrt{\frac{7}{18}}$
46. $\frac{\sqrt{96}}{\sqrt{8}}$
47. $\sqrt[3]{\frac{5}{9}}$
48. $(2\sqrt{3})(8\sqrt{21})$
49. $\frac{\sqrt{3}-2}{\sqrt{3}+4}$
50. $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

Solve. Approximate irrational roots to the nearest hundredth.

51. $2x^2 - 72 = 0$
52. $3z^2 + 35 = 107$
53. $x - \sqrt{x} = 6$
54. $\sqrt{x^2 - 10} = 5$

Contest Problems

1. Determine the total number of distinct squares that can be outlined on the four-by-four "checkerboard" that is shown at the right.



2. Simplify the following expression.

$$\frac{2x-6}{2x-10} \cdot \left(\frac{x^3-27}{3x^2-75}\right)^{-1} \div \frac{3x+15}{2x^2+6x+18}$$

3. Find the least three-digit number that is itself a perfect square but the sum of whose digits is *not* a perfect square.

4. Solve $\sqrt{c^3 + c^3 + c^3 + c^3 + c^3} = 25$.

5. If $x + y = 11$ and $y = \frac{15}{x}$, find the value of $x^2 + y^2$.

6. Solve $x = \frac{x^x}{\sqrt[x]{x}}$.

7. Complete this statement: If n is an integer, the average of the numbers 8, n, and $8n + 1$ is always divisible by 3 and __?__.

8. Find all values of d for which the following expression is not defined.

$$\frac{\frac{d}{d-2} - \frac{d}{d+2}}{\frac{d}{d-2} + \frac{d}{d+2}}$$