Chapter 10

Irrational Numbers and Radicals

Rational and Irrational Numbers

OBJECTIVES for 10-1 through 10-4:
1. To apply basic properties of the rational numbers.
2. To express rational numbers as fractions or decimals.
3. To find the square roots of expressions that have rational square roots.
4. To simplify radicals.
5. To find approximations of irrational square roots.

10–1 Properties of Rational Numbers

Recall from Chapter 1 that a rational number is any number that can be expressed as the quotient of two integers, where the divisor is not zero. The following are examples of rational numbers.

$\frac{5}{8}$ $\qquad 3\frac{2}{9} = \frac{29}{9}$ $\qquad 0 = \frac{0}{2}$ $\qquad -4 = \frac{-4}{1}$ $\qquad 0.43 = \frac{43}{100}$

Any given rational number can be expressed as a quotient of integers in many different ways. For example,

$$\frac{5}{8} = \frac{10}{16} = \frac{15}{24} = \frac{-10}{-16},$$

and so on.

To decide which of a group of rational numbers is greatest, you can rewrite each fraction with the same positive denominator and then compare the numerators, as shown in Example 1 on the following page.

EXAMPLE 1 Compare $\frac{9}{5}$ and $\frac{11}{6}$.

SOLUTION Use the product of the denominators as a common denominator.

$$5 \times 6 = 30$$

Rewrite each fraction as an equivalent fraction with the common denominator.

$$\frac{9}{5} = \frac{9 \cdot 6}{5 \cdot 6} = \frac{54}{30} \qquad \frac{11}{6} = \frac{11 \cdot 5}{6 \cdot 5} = \frac{55}{30}$$

Compare the numerators. Since $54 < 55$, $\frac{54}{30} < \frac{55}{30}$.

$\therefore \frac{9}{5} < \frac{11}{6}$

The method used to compare rational numbers in Example 1 may be generalized as follows.

If a and b are integers and c and d are positive integers, then

$$\frac{a}{c} > \frac{b}{d} \text{ if and only if } ad > bc;$$

$$\frac{a}{c} < \frac{b}{d} \text{ if and only if } ad < bc.$$

EXAMPLE 2 Compare.

a. $\frac{2}{3}$ and $\frac{5}{6}$ **b.** $-\frac{3}{4}$ and $-\frac{4}{3}$ **c.** $\frac{80}{112}$ and $\frac{200}{125}$

SOLUTION **a.** $2 \cdot 6 < 5 \cdot 3$

$\therefore \frac{2}{3} < \frac{5}{6}$

b. Rewrite $-\frac{3}{4}$ and $-\frac{4}{3}$ as $\frac{-3}{4}$ and $\frac{-4}{3}$, respectively.

$(-3)(3) > (-4)(4)$

$\therefore \frac{-3}{4} > \frac{-4}{3}$, or $-\frac{3}{4} > -\frac{4}{3}$

c. First simplify each fraction.

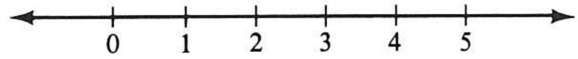
$\frac{80}{112} = \frac{5}{7} \qquad \frac{200}{125} = \frac{8}{5}$

Then compare $\frac{5}{7}$ and $\frac{8}{5}$.

$5 \cdot 5 < 8 \cdot 7$

$\therefore \frac{5}{7} < \frac{8}{5}$, or $\frac{80}{112} < \frac{200}{125}$

Rational numbers are different from integers in many ways. For example, on a number line there is not always another integer between any two given integers.

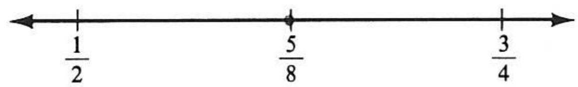


There are exactly four integers between 0 and 5: 1, 2, 3, and 4

There are *no* integers between 0 and 1.

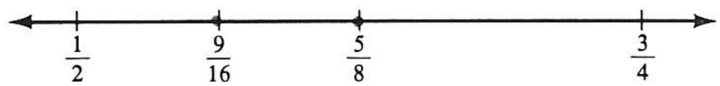
Between any two rational numbers, however, it is always possible to find another rational number. For example, consider $\frac{1}{2}$ and $\frac{3}{4}$. You can find the rational number that is one half of the way from $\frac{1}{2}$ to $\frac{3}{4}$ by adding one half the difference between $\frac{1}{2}$ and $\frac{3}{4}$ to $\frac{1}{2}$:

$$\frac{1}{2} + \frac{1}{2}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

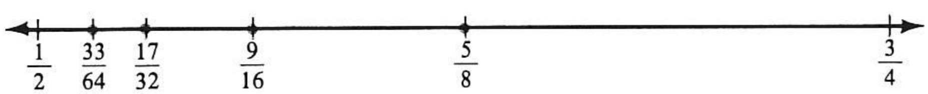


You can then use this method to find the number that is one half of the way from $\frac{1}{2}$ to $\frac{5}{8}$:

$$\frac{1}{2} + \frac{1}{2}\left(\frac{5}{8} - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{8}\right) = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$



Similarly, $\frac{17}{32}$ is one half of the way from $\frac{1}{2}$ to $\frac{9}{16}$, $\frac{33}{64}$ is one half of the way from $\frac{1}{2}$ to $\frac{17}{32}$, and so on.



Therefore, there exist at least four other rational numbers between $\frac{1}{2}$ and $\frac{3}{4}$. In fact, continuing with this process, you could find an infinite number of rational numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

In general, given two rational numbers a and b, $a < b$, the number that is one half of the way from a to b is

$$a + \frac{1}{2}(b - a);$$

the number one third of the way from a to b is

$$a + \frac{1}{3}(b - a);$$

and so on. These formulas suggest the *density property of rational numbers*, which is stated on the following page.

> ***Density Property of Rational Numbers***
>
> Between any two different rational numbers, there is another rational number.

EXAMPLE 3 Find a rational number between $\frac{1}{5}$ and $\frac{7}{10}$.

SOLUTION Choose, for example, the number one fourth of the way from $\frac{1}{5}$ to $\frac{7}{10}$.

$$\frac{1}{5} + \frac{1}{4}\left(\frac{7}{10} - \frac{1}{5}\right) = \frac{1}{5} + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{5} + \frac{1}{8} = \frac{13}{40}$$

Check: Is $\frac{13}{40}$ between $\frac{1}{5}$ and $\frac{7}{10}$?

$$\frac{1}{5} \stackrel{?}{<} \frac{13}{40} \stackrel{?}{<} \frac{7}{10}$$

$$(1)(40) \stackrel{?}{<} (13)(5) \quad \text{and} \quad (13)(10) \stackrel{?}{<} (7)(40)$$

$$40 < 65 \ \checkmark \qquad\qquad 130 < 280 \ \checkmark$$

$\therefore$ one rational number between $\frac{1}{5}$ and $\frac{7}{10}$ is $\frac{13}{40}$.

Oral Exercises

Show that each number is a rational number by expressing it as a quotient of integers.

1. 5 **2.** $16\frac{1}{3}$ **3.** 0.05 **4.** -2.6

5. 75% **6.** 36% **7.** 0 **8.** $2 + \frac{1}{9}$

Written Exercises

Replace each __?__ with one of the symbols <, =, or > to make a true statement.

A **1.** $\frac{2}{5}$ __?__ $\frac{7}{12}$ **2.** $\frac{1}{8}$ __?__ $\frac{7}{63}$ **3.** $-\frac{5}{3}$ __?__ $-\frac{7}{5}$

4. $\frac{3}{7}$ __?__ $\frac{13}{19}$ **5.** $\frac{2}{9}$ __?__ $\frac{18}{81}$ **6.** $-\frac{16}{56}$ __?__ $-\frac{24}{84}$

7. $\frac{32}{60}$ __?__ $\frac{325}{625}$

8. $\frac{96}{112}$ __?__ $\frac{140}{154}$

9. $-3\frac{7}{8}$ __?__ $-\frac{96}{27}$

10. $5\frac{4}{9}$ __?__ $\frac{138}{27}$

11. $\frac{147}{63}$ __?__ $\frac{315}{135}$

12. $\frac{90}{216}$ __?__ $\frac{108}{225}$

Write the given numbers in order from least to greatest.

13. $\frac{3}{16}, \frac{5}{14}, \frac{8}{29}$

14. $\frac{30}{18}, \frac{54}{36}, \frac{60}{75}$

15. $-\frac{15}{40}, -\frac{14}{63}, -\frac{27}{81}$

16. $\frac{32}{5}, 6\frac{3}{7}, \frac{56}{9}$

17. $\frac{5}{16}, \frac{13}{65}, \frac{4}{24}, \frac{7}{43}$

18. $2\frac{3}{4}, \frac{84}{40}, \frac{46}{16}, \frac{49}{20}$

Find the rational number that is one half of the way from *a* to *b*.

19. $a = \frac{1}{5}, b = \frac{1}{4}$

20. $a = \frac{1}{8}, b = \frac{7}{12}$

21. $a = -\frac{5}{6}, b = -\frac{2}{5}$

22. $a = -\frac{4}{9}, b = \frac{11}{6}$

23. $a = 2\frac{1}{4}, b = 3\frac{1}{2}$

24. $a = -4\frac{1}{3}, b = -4\frac{1}{6}$

B

25. Find the rational number that is one third of the way from $\frac{3}{5}$ to $\frac{8}{3}$.

26. Find the rational number that is one fourth of the way from $-\frac{1}{12}$ to $\frac{4}{9}$.

27. Find the rational number that is one sixth of the way from $-\frac{5}{4}$ to $-\frac{1}{6}$.

28. Find the rational number that is one seventh of the way from $1\frac{3}{4}$ to $4\frac{3}{8}$.

29. Find the rational number that is three sevenths of the way from $5\frac{2}{3}$ to $13\frac{4}{9}$.

30. Find the rational number that is two fifths of the way from $-1\frac{2}{7}$ to $\frac{9}{2}$.

31. Find an expression for the rational number that is one fifth of the way from $\frac{2a}{3}$ to $\frac{9a}{4}$, given that $a \neq 0$.

32. Find an expression for the rational number that is one half of the way from $-3b$ to $-\frac{5b}{2}$, given that $b \neq 0$.

C

33. Find the rational number that is one half of the way from a to b if a and b are the numbers that are, respectively, one third and two thirds of the way from $\frac{3}{7}$ to $\frac{12}{5}$.

34. Find two rational numbers g and h such that g is one fifth of the way from $\frac{5}{2}$ to h and h is one half of the way from g to $\frac{15}{4}$.

10–2 Decimals and Fractions

To find *decimal representations* of rational numbers, you can use the division process as follows.

$$\frac{5}{8} = 5 \div 8 \qquad\qquad \frac{7}{11} = 7 \div 11$$

$$\begin{array}{r} 0.625 \\ 8\overline{)\,5.000} \\ \underline{4\,8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array} \qquad\qquad \begin{array}{r} 0.6363 \\ 11\overline{)\,7.0000} \\ \underline{6\,6} \\ 40 \\ \underline{33} \\ 70 \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

$$\therefore \frac{5}{8} = 0.625 \qquad\qquad \therefore \frac{7}{11} = 0.6363\ldots$$

In the division of 5 by 8, the quotient 0.625 is called a **terminating decimal** because the division process *terminates* when a final remainder of 0 is reached.

In the division of 7 by 11, on the other hand, the quotient 0.6363 . . . is a *nonterminating* decimal. The division process never terminates because the remainders 4 and 7 (printed in red) keep appearing, and a remainder of 0 is never reached. The quotient is called a **repeating decimal** because it is a nonterminating decimal in which the same block of digits *repeats* without end. In a repeating decimal, a bar is often used to indicate the block of digits that repeats. Therefore,

$$0.6363\ldots = 0.\overline{63}.$$

In general, when an integer p is divided by a positive integer q, the decimal quotient will either terminate or repeat. The remainder at each step in the division process will be $0, 1, 2, \ldots,$ or $q - 1$. Therefore, after only zeros are left in the dividend, within at most q steps either 0 occurs as a remainder and the division process ends, or one of $1, 2, \ldots, q - 1$ recurs, producing a repeating sequence of remainders and hence a repeating block of digits in the quotient. Thus, if an integer is divided by 11, for example, the only remainders that can appear are the integers 0 through 10 and the repeating block of digits in the quotient will contain *at most* 10 digits.

Just as rational numbers can be represented by decimals, many decimals may be expressed as fractions. You already know that a *terminating* decimal is equivalent to a fraction that has a power of 10 as its denominator.

Thus,

$$3.17 = \frac{317}{100}, \qquad 0.051 = \frac{51}{1000}, \qquad \text{and} \qquad 1.0009 = \frac{10{,}009}{10{,}000}.$$

To find a fraction equivalent to a *repeating* decimal, you can use the procedure shown in the following example.

EXAMPLE Express each decimal as a fraction in simplest form.

a. $0.\overline{14}$

b. $3.2\overline{8}$

c. $0.\overline{675}$

SOLUTION **a.** Let $N = 0.\overline{14}$.

$$\begin{array}{lrl} \text{Multiply:} & 100N &= 14.\overline{14} \\ & N &= 0.\overline{14} \\ \hline \text{Subtract:} & 99N &= 14 \\ \text{Divide:} & N &= \frac{14}{99} \end{array}$$

$\therefore\ 0.\overline{14} = \frac{14}{99}$

b. Let $N = 3.2\overline{8}$.

$$\begin{array}{lrl} \text{Multiply:} & 10N &= 32.8\overline{8} \\ & N &= 3.2\overline{8} \\ \hline \text{Subtract:} & 9N &= 29.6 \\ \text{Divide:} & N &= \frac{29.6}{9} = \frac{296}{90} = \frac{148}{45} \end{array}$$

$\therefore\ 3.2\overline{8} = \frac{148}{45}$

c. Let $N = 0.\overline{675}$.

$$\begin{array}{lrl} \text{Multiply:} & 1000N &= 675.\overline{675} \\ & N &= 0.\overline{675} \\ \hline \text{Subtract:} & 999N &= 675 \\ \text{Divide:} & N &= \frac{675}{999} = \frac{225}{333} = \frac{75}{111} = \frac{25}{37} \end{array}$$

$\therefore\ 0.\overline{675} = \frac{25}{37}$

Notice that in part (a) of Example 1 the multiplication of N by 100 shifts the repeating block 2 places to the left. The subtraction then produces a terminating decimal. A similar procedure is used in parts (b) and (c). In general, if a repeating block of digits contains n digits, you multiply by 10^n.

The discussion on the preceding pages suggests the following facts about rational numbers.

1. Each rational number can be named by a terminating or a repeating decimal.
2. Each repeating or terminating decimal names a unique rational number.

It is possible to construct a decimal numeral, however, that is nonterminating and nonrepeating. For example, the decimal numerals

$$0.37\underbrace{33}7\underbrace{333}7\underbrace{3333}7\ldots$$

and

$$0.2468\underbrace{10}\underbrace{12}\underbrace{14}\underbrace{16}\underbrace{18}\ldots$$

both follow patterns and could be carried out to any number of decimal places, but no block of digits would keep repeating. Since every rational number can be expressed as a terminating or a repeating decimal, these numerals cannot be equivalent to rational numbers. These nonrepeating, nonterminating decimals are called *irrational numbers.* Recall from Chapter 1 that an irrational number is any real number that is not a rational number.

Ordinarily, the decimal numerals for irrational numbers do not have systematic patterns for their digits as do the preceding examples. You may already know one such irrational number,

$$\pi = 3.14159\ldots.$$

Often, you may need to use π in a computation, such as finding the circumference or area of a circle. In a computation, you usually use a *rational approximation* for π, such as

$$\pi \approx 3.14 \text{ or } \pi \approx 3.1416,$$

where the symbol $\approx$ is read *equals approximately* (or *is approximately equal to*).

It is often convenient to *round* a lengthy or infinite decimal in order to use an approximation of the original decimal in a computation. For example, you may approximate $2\frac{6}{13}$, or $2.\overline{461538}$, as follows.

$$\begin{aligned} 2.\overline{461538} &\approx 2.4615 && \text{to the nearest ten-thousandth} \\ &\approx 2.462 && \text{to the nearest thousandth} \\ &\approx 2.46 && \text{to the nearest hundredth} \\ &\approx 2.5 && \text{to the nearest tenth} \\ &\approx 2 && \text{to the nearest unit} \end{aligned}$$

In rounding you use the rules stated at the top of the following page.

1. If the first digit dropped is 5 or more, add 1 to the last digit retained.
2. If the first digit dropped is less than 5, leave the retained digits unchanged.

The decimals studied in this section have all represented rational or irrational numbers. Since these two sets of numbers make up the set of real numbers, the following property is true.

Property of Completeness

Each decimal represents a real number, and every real number can be represented as a decimal.

Oral Exercises

Replace each _?_ with one of the words *rational* or *irrational* to make a true statement.

1. Each terminating decimal numeral represents a(n) _?_ number.
2. 0.554555445555444 . . . names a(n) _?_ number.
3. Nonterminating, nonrepeating decimal numerals represent _?_ numbers.
4. Each repeating or terminating decimal names a unique _?_ number.
5. 0.979797 . . . names a(n) _?_ number.
6. Let a = 2.232332333233332 . . . and b = 0.101001000100001 . . . .
   a. $a + b$ is a(n) _?_ number.
   b. $a - b$ is a(n) _?_ number.

Replace each _?_ with the number that makes a true statement.

7. The decimal equivalent of the rational number $\frac{n}{13}$, where n is a nonzero integer, will start repeating or will terminate after at most _?_ digits beyond the decimal point.
8. The decimal equivalent of the rational number $\frac{n}{18}$, where n is a nonzero integer, will start repeating or will terminate after at most _?_ digits beyond the decimal point.
9. The repeating block of digits in the quotient $\frac{6}{29}$ will contain at most _?_ digits.
10. The repeating block of digits in the quotient $\frac{9}{47}$ will contain at most _?_ digits.

Round each decimal as indicated.

a. to the nearest thousandth **b.** to the nearest hundredth **c.** to the nearest tenth

11. 0.43792 **12.** $0.\overline{25}$ **13.** 12.1357 **14.** $-6.3\overline{84}$

Express each rational number as a terminating or repeating decimal.

15. $\frac{2}{3}$ **16.** $\frac{8}{5}$ **17.** $\frac{7}{2}$ **18.** $\frac{5}{6}$ **19.** $\frac{1}{4}$ **20.** $\frac{1}{12}$

21. **a.** Which denominators in Exercises 15–20 have prime factors other than 2 and 5?

b. Which fractions in Exercises 15–20 are equivalent to terminating decimals?

22. Construct a decimal numeral that names an irrational number.

Written Exercises

Express each rational number as a terminating or repeating decimal.

A **1.** $\frac{1}{4}$ **2.** $\frac{3}{8}$ **3.** $\frac{5}{9}$ **4.** $\frac{4}{9}$ **5.** $\frac{5}{12}$ **6.** $\frac{7}{12}$

7. $-\frac{21}{40}$ **8.** $\frac{35}{16}$ **9.** $-\frac{56}{9}$ **10.** $2\frac{4}{11}$ **11.** $4\frac{3}{7}$ **12.** $-\frac{88}{7}$

Express each rational number as a fraction in simplest form.

13. 0.84 **14.** 0.015 **15.** 52.54 **16.** 129.6

17. $0.\overline{4}$ **18.** $0.\overline{7}$ **19.** $0.\overline{25}$ **20.** $0.\overline{43}$

21. $2.\overline{31}$ **22.** $12.\overline{5}$ **23.** $5.\overline{160}$ **24.** $-7.\overline{038}$

25. $-1.0\overline{3}$ **26.** $2.7\overline{3}$ **27.** $-11.\overline{6655}$ **28.** $10.0\overline{27}$

Multiply. Express the product as a fraction in simplest form. (Hint: First convert each decimal to fraction form.)

B **29.** $(0.5)(0.\overline{4})$ **30.** $(0.\overline{6})(0.75)$ **31.** $(3.\overline{7})(0.6)$ **32.** $(1.5)(0.\overline{2})$

33. $(0.\overline{3})(0.\overline{5})$ **34.** $(0.\overline{18})(0.\overline{8})$ **35.** $(-0.\overline{45})(0.\overline{36})$ **36.** $(0.\overline{81})(0.\overline{77})$

Add. Express the sum as a fraction in simplest form. (Hint: First convert each decimal to fraction form.)

37. $0.375 + \frac{3}{4}$ **38.** $0.625 + \frac{2}{3}$ **39.** $0.2 + 0.\overline{2}$ **40.** $0.4 + 0.\overline{4}$

41. $0.\overline{7} + \frac{1}{3}$ **42.** $0.2\overline{6} + \frac{3}{10}$ **43.** $0.\overline{25} + 0.\overline{62}$ **44.** $0.8\overline{3} + 0.\overline{54}$

45. $1.0\overline{4} + 2.0\overline{5}$ **46.** $-6.\overline{1} + 3.\overline{3}$ **47.** $0.\overline{18} + 5.\overline{81}$ **48.** $9.\overline{753} + 1.\overline{246}$

C **49.** **a.** Express $3.\overline{9}$, $7.\overline{9}$, and $15.\overline{9}$ in fraction form.

b. Use the result of part (a) to determine the value of any repeating decimal of the form $\_.\overline{9}$.

50. Show that if a fraction in lowest terms can be represented by a terminating decimal, the only numbers that can be prime factors of the denominator are 2 and 5.

PROGRAMMING IN BASIC

When you use a computer's built-in division operation, the quotient that the computer outputs will contain a fixed number of digits. If you wish to have a greater number of digits in the quotient, however, the following program will instruct the computer to perform the division using the same steps that you might use. When you input any positive numbers N and D, N < D, the program will compute the quotient N/D digit-by-digit. If the remainder R is zero at any step, the division is complete and the word "TERMINATES" is printed. Otherwise the division is carried out for D + 3 steps.

```
10  PRINT "TO COMPUTE N/D"
20  PRINT "INPUT N, D (0 < N < D)";
30  INPUT N, D
40  PRINT N;"/";D;" = 0.";
50  LET R = N
60  FOR I = 1 TO D + 3
70  LET R = R*10
80  LET Q = INT(R/D)
90  PRINT Q;
100 LET R = R - Q*D
110 IF R = 0 THEN 150
120 NEXT I
130 PRINT " . . . "
140 GOTO 160
150 PRINT " TERMINATES "
160 END
```

Exercises

1. Type in the program as given. Then RUN the program to find a decimal equivalent for each of the following fractions:

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{21}, \frac{1}{25}, \frac{1}{37}$$

2. If you can obtain a printout of the program, draw a red box around the loop in lines 60–120.

In the program given on the preceding page, note that a nonterminating division is carried out for D + 3 steps regardless of the point at which the digits of the quotient begin to repeat. If you wish to stop the program when the first such repeat occurs, however, you can store the remainders in an array and compare each new remainder with those previously found. In order to do this, make the following changes in the program.

```
5   DIM R(200)
20  PRINT "INPUT N, D (0 < N < D < 100)";
```

Change the loop:

```
60   LET I = 1
65   LET R(I) = R
120  LET I = I + 1
125  GOTO 65
```

Insert the following lines:

```
112  REM *SEARCH FOR FIRST REPEAT
113  FOR J = 1 to I
114  IF R = R(J) THEN 130
115  NEXT J
```

3. Type in the changes, then RUN the revised program to find a decimal equivalent of the same fractions listed in Exercise 1.
4. If you can obtain a printout of the program, draw a red box around the loop in lines 65–125 and a blue box around lines 112–115.

The following changes in the program will insert apostrophes to mark off the block of repeating digits in the quotient. The program uses a flag, F, and sets F = 0 until the first repeat is found. The program then sets F = 1 while it searches for the next repeat.

```
44   REM *SET FLAG TO ZERO
45   LET F = 0
111  IF F > 0 THEN 118
116  GOTO 120
117  REM *SEARCH FOR SECOND REPEAT
118  IF R = X THEN 138
123  IF F > 0 THEN 70
128  REM *FIRST REPEAT MARKED
130  PRINT "'";
131  REM *STORE REPEATING REMAINDER
132  LET X = R
133  REM *SET FLAG TO ONE
134  LET F = 1
135  GOTO 120
137  REM *SECOND REPEAT MARKED
138  PRINT "'. . ."
```

5. Type in the changes, then RUN the revised program to find a decimal equivalent of the same fractions listed in Exercise 1.
6. If you can obtain a printout of the program, draw a red box around the loop in lines 65–125. Draw a blue box around lines 112–115, and a second blue box around lines 117–118.

10–3 Rational Square Roots

Just as the inverse of addition is subtraction and the inverse of multiplication is division, the inverse operation of squaring a number is finding a *square root*. In general, a number b is called a **square root** of a positive real number a if

$$b^2 = a.$$

Since $b^2 = (-b)^2$, a positive real number a has two square roots, a positive square root and a negative square root.

EXAMPLE 1 Solve $x^2 = 36$.

SOLUTION The problem asks for all numbers whose square is 36. Since $(6)^2 = 36$ and $(-6)^2 = 36$, $x = 6$ or $x = -6$.

$\therefore$ the solution set is $\{6, -6\}$.

In equations of the type $x^2 = k$, it may be convenient to use the symbol $\pm$ (read "plus-or-minus") in the solution. Therefore, in Example 1,

$$x = 6 \text{ or } x = -6 \text{ may be written as } x = \pm 6.$$

The positive square root of a number a is called the **principal square root** and is represented by the symbol

$$\sqrt{a}.$$

The negative square root is therefore represented by the symbol

$$-\sqrt{a}.$$

Thus, if $a = 49$, then $\sqrt{a} = 7$ and $-\sqrt{a} = -7$. Of course, if $a = 0$, then both $\sqrt{a}$ and $-\sqrt{a}$ represent 0.

In an expression like $\sqrt{a}$, the symbol $\sqrt{\ }$ is called a **radical sign.** Any numeral or expression under a radical sign is called the **radicand.** When a mathematical expression appears under a radical sign, the entire expression is called a **radical.**

Because the square of each real number is a nonnegative real number, a negative real number has no real square root. Thus, expressions such as $\sqrt{-2}$, $\sqrt{-3}$, and so on do not name real numbers.

Since squaring a number and taking the square root of a number are inverse operations, you may see that if a number has a real square root, squaring that square root results in the number itself. Thus,

$$(\sqrt{16})^2 = 16 \quad \text{and} \quad (\sqrt{3})^2 = 3.$$

By the definition of principal square root, the expression $\sqrt{x^2}$ denotes a nonnegative number whether x represents a nonnegative or a negative number. Thus,

$$\sqrt{(5)^2} = 5 \quad \text{and} \quad \sqrt{(-5)^2} = 5.$$

Since $|a|$ is nonnegative for all real numbers a, the following is true.

For any real number a,

$$\sqrt{a^2} = |a|.$$

EXAMPLE 2 Simplify.

a. $\sqrt{(n-2)^2}$ b. $\sqrt{m^2n^2}$ c. $\sqrt{x^4}$

SOLUTION

a. $\sqrt{(n-2)^2} = |n-2|$

b. $\sqrt{m^2n^2} = |mn|$

c. $\sqrt{x^4} = |x^2|$

Since x^2 is nonnegative for any real number x, $|x^2| = x^2$.

$\therefore \sqrt{x^4} = x^2.$

When a radicand does not involve a variable, however, it is not necessary to use absolute value notation.

EXAMPLE 3 Simplify.

a. $\sqrt{64}$ b. $\sqrt{\left(\frac{2}{5}\right)^2}$ c. $\sqrt{(-4)^2}$

SOLUTION

a. $\sqrt{64} = 8$

b. $\sqrt{\left(\frac{2}{5}\right)^2} = \frac{2}{5}$

c. $\sqrt{(-4)^2} = 4$

Any number that can be expressed as the square of a rational number is called a **perfect square.** For example:

64 is a perfect square because $64 = 8^2$.

$\frac{9}{16}$ is a perfect square because $\frac{9}{16} = \left(\frac{3}{4}\right)^2$.

Equivalently, any number with a rational square root is a perfect square. The square roots of real numbers other than perfect squares are all irrational numbers. For example,

$$\sqrt{2},\ \sqrt{15},\ \text{and}\ \sqrt{120}$$

all represent irrational numbers.

Oral Exercises

Tell whether or not the given symbol represents a real number. If the number represented is real, tell whether it is rational or irrational.

1. $\sqrt{11}$
2. $\sqrt{-4}$
3. $-\sqrt{49}$
4. $\sqrt{(-3)^2}$
5. $\sqrt{\frac{1}{5}}$
6. $\sqrt{\frac{16}{49}}$
7. $\sqrt{0.81}$
8. $\sqrt{-0.25}$
9. $\sqrt{(-1)^3}$
10. $-\sqrt{72}$
11. $\sqrt{0.1}$
12. $\frac{2}{\sqrt{3}}$

Simplify.

13. $\sqrt{36}$
14. $-\sqrt{9}$
15. $\sqrt{121}$
16. $\sqrt{\frac{1}{4}}$
17. $(\sqrt{64})^2$
18. $\sqrt{(-5)^2}$
19. $-\sqrt{0.01}$
20. $(\sqrt{81})^2$
21. $\sqrt{(-2)^4}$
22. $\sqrt{10{,}000}$
23. $\sqrt{\frac{4}{225}}$
24. $-\sqrt{1.44}$
25. Is the sentence $\sqrt{a^2} = a$ true for every nonnegative real value of a? for every negative real value of a?
26. Is the sentence $\sqrt{a^2} = |a|$ true for every nonnegative real value of a? for every negative real value of a?

Written Exercises

Simplify.

A

1. $\sqrt{81}$
2. $\sqrt{144}$
3. $-\sqrt{9}$
4. $-\sqrt{0.04}$
5. $\sqrt{(17)^2}$
6. $-\sqrt{(21)^2}$
7. $\sqrt{(-4)^2}$
8. $\sqrt{(-10)^2}$
9. $(\sqrt{25})^2$
10. $(\sqrt{49})^2$
11. $-\sqrt{\frac{16}{81}}$
12. $\sqrt{\frac{9}{169}}$

Solve.

13. $x^2 = 25$
14. $y^2 = 64$
15. $z^2 - 15 = 66$
16. $x^2 + \frac{7}{12} = \frac{5}{6}$
17. $100y^2 = 81$
18. $3x^2 - 8 = 19$
19. $5 - 4z^2 = -59$
20. $2x^2 + 150 = 1400$

Simplify.

B 21. $\sqrt{64c^2}$

22. $\sqrt{144m^2}$

23. $\sqrt{9m^2n^2}$

24. $\sqrt{400x^2y^2}$

25. $\sqrt{16u^4}$

26. $-\sqrt{4f^4}$

27. $\sqrt{\frac{c^2}{e^2}}$

28. $\sqrt{\frac{16z^2}{b^2}}$

29. $\sqrt{25a^4b^2}$

30. $\sqrt{144c^6d^4}$

31. $\sqrt{\frac{x^4}{9y^2}}$

32. $\sqrt{\frac{49m^2}{36n^6}}$

33. $-\sqrt{1.21g^4h^4}$

34. $\sqrt{6.25a^4b^4}$

35. $\sqrt{(5ef)^2}$

36. $\sqrt{(-3jk)^2}$

37. $\sqrt{(r+s)^2}$

38. $\sqrt{(p+q)^4}$

39. $\sqrt{(f+g)^8}$

40. $\sqrt{(1-m)^6}$

41. $\sqrt{196x^4y^{16}}$

42. $\sqrt{2500r^{10}s^{18}}$

43. $\sqrt{\frac{169j^{30}}{64k^{20}}}$

44. $\sqrt{\frac{36d^{48}}{441m^{54}}}$

45. Show that $\sqrt{25a^2 + 30ab + 9b^2} = |5a + 3b|$ is true for all real values of a and b.

46. Show that $\sqrt{16x^2 - 72xy + 81y^2} = |4x - 9y|$ is true for all real values of x and y.

Simplify. Assume that variable expressions appearing as exponents denote positive integers.

C 47. $\sqrt{c^{4m}}$

48. $\sqrt{(e^m f^{2n})^2}$

49. $\sqrt{(q^{3m})(q^{7m})}$

50. $\sqrt{\frac{q^{2+m}}{q^{8-m}}}$

51. Does $\sqrt{a^2 - 1}$ represent a real number for all real values of a?

52. Does $\sqrt{a^2 + 1}$ represent a real number for all real values of a?

53. Does $f: x \longrightarrow \sqrt{x}$ define a function? If so, what is the domain? the range?

54. Does $f: x \longrightarrow -\sqrt{x}$ define a function? If so, what is the domain? the range?

55. Find the fallacy in the following argument that every real number is equal to its opposite.

If a is a real number, then $a = \sqrt{a^2} = \sqrt{(-a)^2} = -a$.

56. Prove that each positive real number has at most one positive square root. That is, if a, b, and c are any positive real numbers such that $b^2 = a$ and $c^2 = a$, then $b = c$.

10–4 Irrational Square Roots

Since the decimal representations of irrational numbers such as $\sqrt{2}$ or $\sqrt{15}$ are nonterminating and nonrepeating, you cannot name an exact decimal numeral for these numbers. However, you can find a rational approximation for an irrational number. To find a rational approximation for an irrational number, begin by locating the irrational number between two consecutive integers or between two rational numbers. For example, since

$$9 < 15 < 16,$$

you know that

$$\sqrt{9} < \sqrt{15} < \sqrt{16},$$

or

$$3 < \sqrt{15} < 4.$$

To obtain a more precise idea of the value of $\sqrt{15}$, find numbers whose squares are close to 15. Thus, since $(3.8)^2 = 14.44$, and since $(3.9)^2 = 15.21$, you know that

$$3.8 < \sqrt{15} < 3.9.$$

Continuing to find numbers whose squares approach 15, you may find that $(3.87)^2 = 14.9769$ and $(3.88)^2 = 15.0544$, and so

$$3.87 < \sqrt{15} < 3.88.$$

Similarly, you may find that

$$3.872 < \sqrt{15} < 3.873,$$

and

$$3.8729 < \sqrt{15} < 3.8730.$$

Therefore,

$$\sqrt{15} \approx 3.8730.$$

Rational approximations of all irrational square roots can be determined by the method outlined for $\sqrt{15}$. The *Table of Square Roots* on page 683 gives these approximations for the integers from 1 to 100.

EXAMPLE 1 Using the Table of Square Roots, give a rational approximation of each of the following.

a. $\sqrt{51}$ **b.** $3 + \sqrt{5}$ **c.** $6\sqrt{13}$

SOLUTION **a.** $\sqrt{51} \approx 7.141$

b. $3 + \sqrt{5} \approx 3 + 2.236 = 5.236$

c. $6\sqrt{13} \approx 6(3.606) = 21.636$

When a radicand is an integer greater than 100, its square root will not be listed in the table. To determine a method for finding these square roots, first consider the following.

$$\sqrt{16 \cdot 25} = \sqrt{400} = 20$$

$$\sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$$

Then, by the transitive axiom of equality,

$$\sqrt{16 \cdot 25} = \sqrt{16} \cdot \sqrt{25}.$$

This relationship suggests the following fact about square roots.

Product Property of Square Roots

For any nonnegative real numbers a and b,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

This property can be used to simplify irrational square roots as in the following example.

EXAMPLE 2 Simplify. **a.** $\sqrt{120}$ **b.** $\sqrt{180}$

SOLUTION

a. $\sqrt{120} = \sqrt{4 \cdot 30}$
$= \sqrt{4} \cdot \sqrt{30}$
$= 2\sqrt{30}$

b. $\sqrt{180} = \sqrt{9 \cdot 20}$
$= \sqrt{9 \cdot 4 \cdot 5}$
$= \sqrt{9} \cdot \sqrt{4} \cdot \sqrt{5}$
$= 3 \cdot 2 \cdot \sqrt{5}$
$= 6\sqrt{5}$

Simplifying the radicals $\sqrt{120}$ and $\sqrt{180}$ now enables you to use the table to approximate these square roots. Thus,

$$\sqrt{120} = 2\sqrt{30} \approx 2(5.477) = 10.954$$

and

$$\sqrt{180} = 6\sqrt{5} \approx 6(2.236) = 13.416.$$

To approximate irrational square roots that cannot be found in the table or that cannot be simplified using the product property, you may wish to use the *divide-and-average method* discussed in Appendix A on page 685.

The product property of square roots is helpful in simplifying some expressions in which the radicand is a fraction. For example:

$$\sqrt{\frac{2}{25}} = \sqrt{\frac{1}{25} \cdot 2} = \sqrt{\frac{1}{25}} \cdot \sqrt{2} = \frac{1}{5}\sqrt{2}, \text{ or } \frac{\sqrt{2}}{5}$$

The product property of square roots may also be helpful in simplifying *rational* square roots.

EXAMPLE 3 Simplify $\sqrt{1024}$.

SOLUTION

$$\begin{aligned}\sqrt{1024} &= \sqrt{4 \cdot 256}\\ &= \sqrt{4 \cdot 4 \cdot 64}\\ &= \sqrt{4} \cdot \sqrt{4} \cdot \sqrt{64}\\ &= 2 \cdot 2 \cdot 8\\ &= 32\end{aligned}$$

You may use the product property in solving equations with irrational roots.

EXAMPLE 4 Solve. Simplify irrational solutions.

a. $x^2 - 18 = 90$ **b.** $10x^2 - 2430 = 570$

SOLUTION

a.

$$\begin{aligned}x^2 - 18 &= 90\\ x^2 &= 108\\ x &= \pm\sqrt{108}\\ &= \pm\sqrt{4 \cdot 9 \cdot 3}\\ &= \pm\sqrt{4} \cdot \sqrt{9} \cdot \sqrt{3}\\ &= \pm 2 \cdot 3 \cdot \sqrt{3}\\ x &= \pm 6\sqrt{3}\end{aligned}$$

$\therefore$ the solution set is $\{6\sqrt{3}, -6\sqrt{3}\}$.

b.

$$\begin{aligned}10x^2 - 2430 &= 570\\ 10x^2 &= 3000\\ x^2 &= 300\\ x &= \pm\sqrt{300}\\ &= \pm\sqrt{100 \cdot 3}\\ &= \pm\sqrt{100} \cdot \sqrt{3}\\ x &= \pm 10\sqrt{3}\end{aligned}$$

$\therefore$ the solution set is $\{10\sqrt{3}, -10\sqrt{3}\}$.

Oral Exercises

Name the two consecutive integers between which each square root lies.

1. $\sqrt{17}$ **2.** $\sqrt{50}$ **3.** $\sqrt{38.5}$ **4.** $\sqrt{92.8}$ **5.** $\sqrt{\frac{1}{2}}$ **6.** $\sqrt{\frac{32}{5}}$

7. $\sqrt{175}$ **8.** $\sqrt{255}$ **9.** $-\sqrt{402}$ **10.** $-\sqrt{628}$ **11.** $3\sqrt{15}$ **12.** $4\sqrt{70}$

Simplify.

13. $\sqrt{60}$ **14.** $\sqrt{84}$ **15.** $\sqrt{240}$ **16.** $\sqrt{726}$ **17.** $(\sqrt{5})^2$ **18.** $(\sqrt{48})^2$

Approximate each square root to the nearest thousandth. Use the Table of Square Roots as necessary.

19. $\sqrt{85}$ **20.** $\sqrt{56}$ **21.** $\sqrt{160}$ **22.** $\sqrt{384}$ **23.** $\sqrt{8200}$ **24.** $\sqrt{1728}$

Written Exercises

Simplify.

A

1. $\sqrt{48}$
2. $\sqrt{18}$
3. $\sqrt{75}$
4. $\sqrt{128}$
5. $\sqrt{320}$
6. $\sqrt{1200}$
7. $2\sqrt{96}$
8. $3\sqrt{108}$
9. $\sqrt{\frac{5}{16}}$
10. $\sqrt{\frac{27}{25}}$
11. $\sqrt{0.72}$
12. $\sqrt{0.32}$

Solve. Simplify irrational solutions.

13. $a^2 = 11$
14. $b^2 = 28$
15. $c^2 - 7 = 0$
16. $z^2 - 10 = 0$
17. $d^2 + 5 = 25$
18. $m^2 - 6 = 34$
19. $3e^2 + 7 = 97$
20. $4x^2 - 12 = 48$
21. $12x^2 - 50 = 670$
22. $10y^2 + 63 = 513$
23. $8x^2 + 12 = 102$
24. $50z^2 - 26 = 190$

Simplify. Approximate the answer to the nearest thousandth. Use the Table of Square Roots as necessary.

25. $14 - \sqrt{91}$
26. $\sqrt{20} - \sqrt{10}$
27. $7\sqrt{67}$
28. $4\sqrt{83}$

B

29. $5 + 2\sqrt{34}$
30. $9\sqrt{24} - 16$
31. $\frac{\sqrt{43} - 5}{2}$
32. $\frac{2\sqrt{76} - 4}{3}$
33. $\sqrt{180} + \sqrt{250}$
34. $\frac{6}{5}\sqrt{567}$
35. $\frac{\sqrt{432} - 2}{3}$
36. $\frac{8 - 3\sqrt{648}}{2}$

Determine which of the two numbers is greater.

37. $5 + \sqrt{10}$, $20 - \sqrt{85}$
38. $12 - 3\sqrt{5}$, $3 + 2\sqrt{6}$
39. $2 + \sqrt{8}$, $8 - \sqrt{11}$
40. $\sqrt{75} - 5$, $\sqrt{5} + 3$
41. $1 + \sqrt{148}$, $\sqrt{204} - 1$
42. $2\sqrt{260}$, $\sqrt{600} + 10$
43. Show that $(3 - \sqrt{5})^2 < 3 - \sqrt{5}$.
44. Show that $(8 - \sqrt{54})^2 < 8 - \sqrt{54}$.
45. Solve $2r + \sqrt{2} = 5$. Approximate the value of r to the nearest hundredth.
46. Solve $\sqrt{12} - 3x = \sqrt{3}$. Approximate the value of x to the nearest hundredth.
47. Find two nonzero rational numbers x and y such that $x\sqrt{5} + y\sqrt{5}$ is rational.
48. Find two nonzero rational numbers a and b such that $a\sqrt{34} + b\sqrt{34}$ is rational.

Evaluate $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ for the given values of a and b.

C 49. $a = 4, b = 9$ 50. $a = 9, b = 4$ 51. $a = 16, b = 25$

52. $a = 36, b = 4$ 53. $a = 64, b = 49$ 54. $a = 100, b = 100$

55. Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ when a and b are nonnegative numbers.

Let r and s be irrational numbers. Determine whether or not the expression may represent a rational number. Give examples to illustrate your hypothesis.

56. $2r$ 57. $r - 1$ 58. $r \cdot s$ 59. $r \div s$

60. If p is a rational number and $\sqrt{q}$ is an irrational number, show that $(p + \sqrt{q})$ is an irrational number.

61. On a number line, graph the solution set of $\sqrt{x} > x$.

Computer Exercises

For students with computer experience

1. Write a program that will allow you to input a positive integer less than 1000 and will find the greatest square factor of the integer. A sample output would be

   THE GREATEST SQUARE FACTOR OF 162 IS 81.

   (*Hint*: Begin with $31^2 = 961$ as a trial factor and continue down to $1^2 = 1$. The first such square that is a factor of the given number is the greatest square factor.)

2. Incorporate the program that you wrote for Exercise 1 into a program that will simplify the square root of any integer less than 1000. A sample output would be

   SQR(162) = 9 * SQR(2).

3. Write a program that will *locate* the square root of a given positive integer *without using the computer's square-root function.* (*Hint:* Start with 1 as a *trial* square root, square it, then check whether the square is greater than or equal to the given integer. If it is, print it. If it is not, add 1 to the trial root and check again. Continue adding 1 if the square is *less than* the given integer, and print the first trial root whose square is *greater than or equal to* it.)

4. Modify the program that you wrote for Exercise 3 so that it *estimates* the square root in successive steps. That is, the program should first print out the greatest *integer* less than or equal to the square root, then the greatest *tenth*, the greatest *hundredth*, and finally the greatest *thousandth*. For comparison, the program should also print out the value obtained by using the computer's square-root function.

Self-Test 1

VOCABULARY

density property of rational numbers (p. 484)
terminating decimal (p. 486)
repeating decimal (p. 486)
property of completeness (p. 489)
square root (p. 493)
principal square root (p. 493)
radical sign (p. 493)
radicand (p. 493)
radical (p. 493)
perfect square (p. 494)

Replace each $\underline{?}$ with one of the symbols <, =, or > to make a true statement. *Obj. 1, p. 481*

1. $\frac{6}{5} \underline{?} \frac{7}{6}$
2. $\frac{96}{72} \underline{?} \frac{39}{26}$
3. Find the rational number that is one half of the way from $\frac{2}{3}$ to 5.

Express each rational number as a terminating or repeating decimal.

4. $\frac{5}{8}$
5. $\frac{11}{24}$

Obj. 2, p. 481

Express each rational number as a fraction in simplest form.

6. 0.384
7. $0.4\overline{36}$

Simplify.

8. $\sqrt{64}$
9. $(\sqrt{36})^2$
10. $\sqrt{(-18)^2}$

Obj. 3, p. 481

11. $\sqrt{24}$
12. $2\sqrt{84}$
13. $\sqrt{320}$

Obj. 4, p. 481

Approximate each square root to the nearest thousandth. Use the Table of Square Roots as necessary.

14. $\sqrt{88}$
15. $\sqrt{69}$
16. $\sqrt{245}$

Obj. 5, p. 481

Check your answers with those at the back of the book.

ON THE CALCULATOR

Most calculators have a square root key that can be used to find the square root of a nonnegative number. Use a calculator to approximate each square root to the nearest hundredth.

1. $\sqrt{7}$
2. $\sqrt{19}$
3. $\sqrt{125}$
4. $\sqrt{4000}$
5. $\sqrt{2.5}$
6. $\sqrt{3.15}$
7. $\sqrt{0.625}$
8. $\sqrt{0.004}$

Careers

Transportation Engineering

Transportation engineers are responsible for designing and developing surface transportation systems that are safe, efficient, and economical. Their work may involve the planning of new systems or the improvement of existing ones.

At the beginning of a project, transportation engineers draft plans that specify the details of the proposed construction or repair. They must then prepare cost estimates for required materials and labor, and they may also need to report on the effect of the proposed changes on the environment. Once a project is underway, they often supervise on-site construction and equipment maintenance.

Transportation engineers continue to be involved in a project even after the actual construction or repair is finished. For example, some engineers work to prepare accurate maps of the completed system, while others are involved in ongoing analysis of traffic flow.

EXAMPLE Roads expand with a large increase in temperature. It is the job of the transportation engineer to design a road so that it can withstand such expansion without warping and breaking up. One formula that is used to determine the amount of expansion, I, is

$$I = \frac{9}{5}kl(T - t),$$

where T is the actual temperature in degrees Celsius, t is the temperature in degrees Celsius at which the road was built, l is the length of the road, and k is a constant. How many feet will a two-mile stretch of road expand at 38°C if $k = 0.000012$ and the road was built at 16°C?

SOLUTION Substitute the given information into the formula.

$$I = \frac{9}{5}(0.000012)(2)(38 - 16)$$
$$\approx 0.00095$$

Since the length of the road was given in *miles*, multiply this result by 5280 to determine the number of *feet* of expansion.

$$0.00095 \times 5280 = 5.016$$

$\therefore$ the road will expand *approximately* 5 ft.

READING ALGEBRA Problem Solving

The first time that you read a problem, you may know how to solve it immediately. If you do, solve it carefully. As discussed on page 143, be sure that you have answered the question that was asked and that your answer is reasonable.

More often, however, you will not be able to solve a problem after just one reading. In such cases, reread the problem until you understand exactly what it is about. Questions like these may be helpful.

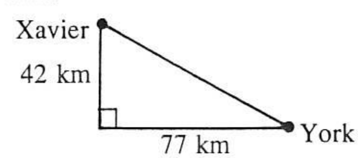
1. What am I asked to find?
2. What am I told?
3. Have I done problems like this before? How? Can my experience help in solving this problem?
4. Can I draw a diagram or make a chart using the given information?
5. What do I expect the answer to be?

Consider the following problem.

Suppose you plan to fly your airplane from Xavier to York. Looking at a road map, you see that York is 42 km south and 77 km east of Xavier. What is the distance if you fly there directly?

Your answers to the questions should be specific, like the following.

1. I am looking for the straight-line distance from Xavier to York.
2. York is 42 km south and 77 km east of Xavier.
3. I have done problems with distances in the same and opposite directions, by adding or subtracting. The distance should not be more than $42 + 77 = 119$ km or less than $77 - 42 = 35$ km.
4. I can draw a diagram like the one at the right and find the approximate distance by measuring the third side of the triangle.
5. I expect the answer to be a distance in kilometers.



Although you may not have seen a problem like this before, the answers to the questions show one way that you might approach it. In the following section you will learn a method of finding a more exact answer.

Exercises

Read the following problem and answer the five questions above.

A door is 95 cm wide and 220 cm high. You need to attach a diagonal brace to the door. How long must the brace be?

Using the Pythagorean Theorem

OBJECTIVES for Sections 10-5 and 10-6:
1. *To use the Pythagorean Theorem and its converse.*
2. *To use the distance formula to find the distance between two points on a coordinate plane.*

10–5 The Pythagorean Theorem

While the irrational number $\sqrt{2}$ cannot be represented as a terminating decimal, it is nevertheless possible to construct a picture of a line segment whose length is $\sqrt{2}$ units, as you will see later in this section.

First, look at the tile pattern shown in Figure 1. You can verify by counting the small triangles, each of which has the same area, that the area of the large square is equal to the sum of the areas of the two small red squares. This figure illustrates a special case of the *Pythagorean Theorem*. The development of this theorem is credited to Pythagoras, a famous Greek philosopher and mathematician who lived from about 580–500 B.C. It is believed that his method of proof was based on comparison of areas.

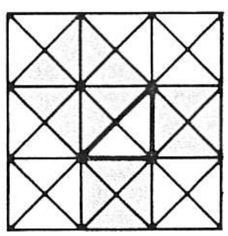


Figure 1

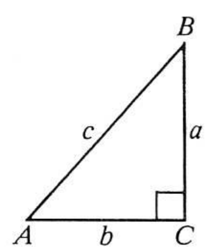


Figure 2

In stating the general theorem, it is convenient to use the customary labeling of a right triangle that is shown in Figure 2. The vertex of the right angle is labeled C. The length of the side opposite the right angle, called the **hypotenuse**, is represented by c. The lengths of the other two sides are represented by a and b.

> ***Pythagorean Theorem***
>
> In any right triangle, the square of the length c of the hypotenuse is equal to the sum of the squares of the lengths a and b of the other two sides; that is:
>
> $$c^2 = a^2 + b^2$$

The diagrams in Figure 3 suggest a proof of this theorem. Each diagram shows a square, with sides of length $(a + b)$ units, separated into a series of smaller squares and triangles. Recall that the area of a square is the square of the length of one of its sides and use this fact to determine two expressions for the area of a square with side of $(a + b)$ units. If you equate these two expressions, you will obtain the equation stated in the theorem.

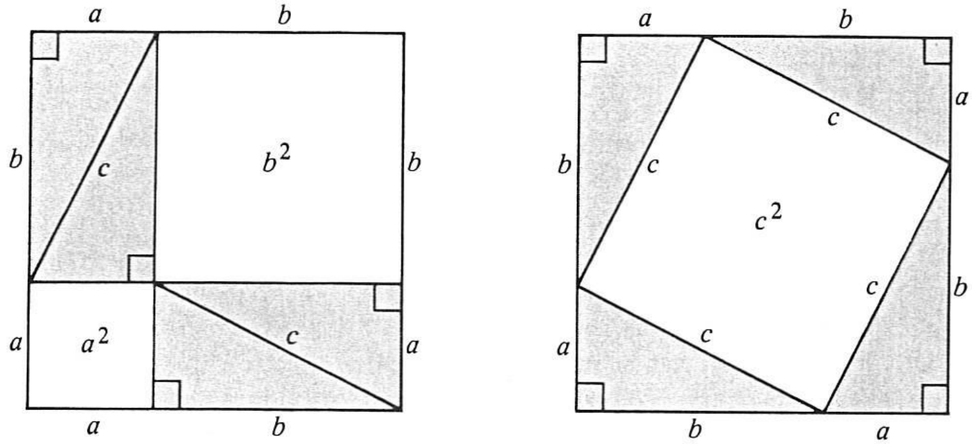


Figure 3

$$(a + b)^2 = a^2 + b^2 + 4\left(\frac{1}{2}ab\right) \qquad (a + b)^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

$$a^2 + b^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 4\left(\frac{1}{2}ab\right)$$

$$a^2 + b^2 = c^2$$

Thus, by the symmetric axiom of equality,

$$c^2 = a^2 + b^2.$$

Using the Pythagorean Theorem, it is now possible to construct a line segment of length $\sqrt{2}$ units. First, draw a right triangle whose shorter sides are 1 unit long, as shown in Figure 4. Then the length of the hypotenuse, c, can be found by using the Pythagorean Theorem, with $a = 1$ and $b = 1$, as follows.

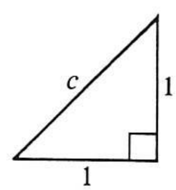


Figure 4

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 1^2 + 1^2 \\ &= 1 + 1 \\ c^2 &= 2 \\ c &= \pm\sqrt{2} \end{aligned}$$

Therefore, the length of the hypotenuse is $\sqrt{2}$ units, as shown in Figure 5. (Notice that the negative root, $-\sqrt{2}$, is rejected because lengths cannot be negative.)

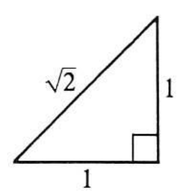


Figure 5

The triangles in Figure 6 illustrate the fact that the Pythagorean Theorem may be used to construct line segments of length $\sqrt{3}$ units, $\sqrt{4}$ units, $\sqrt{5}$ units, $\sqrt{6}$ units, and so on.

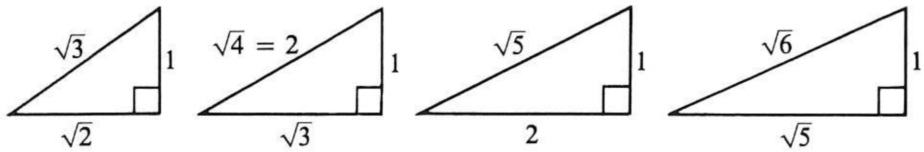


Figure 6

The triangles constructed in Figure 6 may now be used to locate the graphs of irrational square roots such as $\sqrt{2}$, $-\sqrt{2}$, and $\sqrt{3}$ on a number line, as shown in Figure 7. Here arcs are drawn to transfer the lengths of the hypotenuse of each triangle to a number line. Note that $-\sqrt{2}$ is located $\sqrt{2}$ units to the left of zero, $-\sqrt{3}$ is located $\sqrt{3}$ units to the left of zero, and so on.

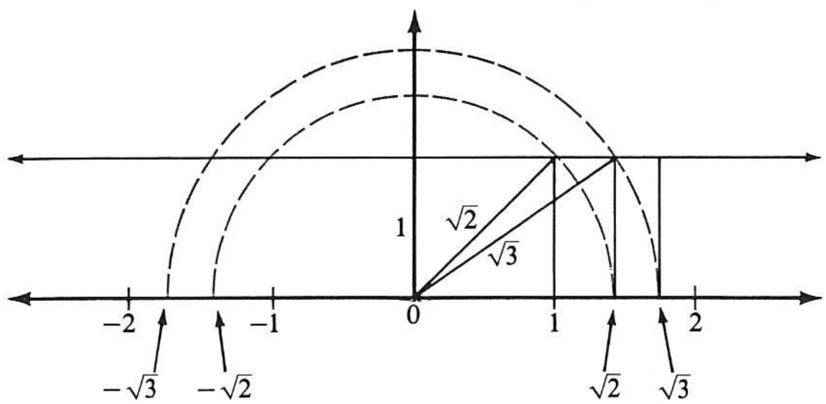


Figure 7

If you constructed a series of such triangles and arcs, you would be able to locate the graphs of $\sqrt{5}$, $\sqrt{6}$, and so on, on the number line.

Note that the *converse* of the Pythagorean Theorem is also true.

Converse of the Pythagorean Theorem

If the lengths of the sides of a triangle are such that the sum of the squares of the lengths of the two shorter sides is equal to the square of the length of the longest side, the triangle is a right triangle. The right angle will be opposite the longest side.

EXAMPLE 1 Determine whether or not each set of numbers could represent the lengths of the sides of a right triangle.

a. {12, 16, 20} b. {8, 10, 12}

SOLUTION a. $12^2 + 16^2 \stackrel{?}{=} 20^2$

$144 + 256 \stackrel{?}{=} 400$

$400 = 400$ √

∴ a triangle with sides of length 12, 16, and 20 units is a right triangle.

b. $8^2 + 10^2 \stackrel{?}{=} 12^2$

$64 + 100 \stackrel{?}{=} 144$

$164 \neq 144$

∴ a triangle with sides of length 8, 10, and 12 units is not a right triangle.

Any set of positive integers that satisfies the equation

$$c^2 = a^2 + b^2$$

is called a set of **Pythagorean numbers,** or a **Pythagorean triple.**

EXAMPLE 2 Show that {15, 20, 25} is a Pythagorean triple.

SOLUTION $15^2 + 20^2 \stackrel{?}{=} 25^2$

$225 + 400 \stackrel{?}{=} 625$

$625 = 625$ √

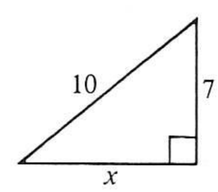
The Pythagorean Theorem and its converse can be used in solving problems.

EXAMPLE 3 A support wire 10 m long is attached to the top of a utility pole 7 m tall and is then stretched taut. How far from the base of the pole will the wire be attached to the ground?

SOLUTION

Step 1 The problem asks for the distance from the base of the pole to the point at which the wire touches the ground. Assume that the pole forms a right angle with the ground. Then the positions of the pole, the wire, and the ground can be represented by a right triangle.

Step 2 Let x = the distance in meters from the base of the pole to the ground end of the wire.



Step 3 Apply the Pythagorean Theorem, using $a = x$, $b = 7$, and $c = 10$.

$$a^2 + b^2 = c^2$$

$$x^2 + 7^2 = 10^2$$

Step 4

$$x^2 + 7^2 = 10^2$$
$$x^2 + 49 = 100$$
$$x^2 = 51$$
$$x = \pm\sqrt{51}$$

Since a distance cannot be negative, the positive square root is selected. Thus,

$$x = \sqrt{51} \approx 7.14 \text{ m.}$$

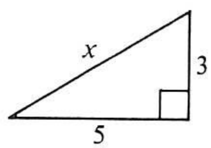
Step 5 Checking the results is left to you.

$\therefore$ the distance from the base of the pole to the ground end of the wire is $\sqrt{51}$ m, or approximately 7.14 m.

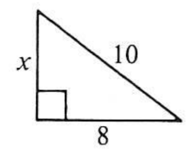
Oral Exercises

State an equation that expresses the relationship between the lengths of the sides of each triangle.

1.



2.



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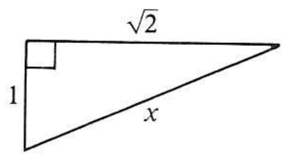

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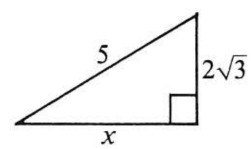
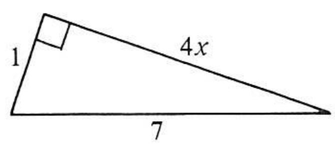
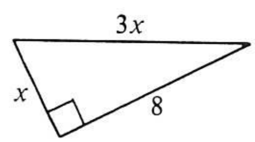


5.



6.



Tell whether or not each set of numbers is a Pythagorean triple.

7. $\{15, 12, 9\}$ **8.** $\{2, 3, 4\}$ **9.** $\{5, 12, 13\}$ **10.** $\{10, 24, 26\}$

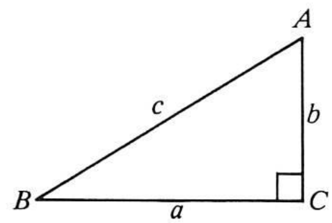
11. $\left\{\frac{3}{4}, 1, \frac{5}{4}\right\}$ **12.** $\{7, 24, 25\}$ **13.** $\{6, 10, 11\}$ **14.** $\{2, \sqrt{5}, 3\}$

15. Explain how to find the length of a diagonal of a square whose sides are 4 cm long.

16. Explain how to find the length of a diagonal of a rectangle with dimensions 5 cm and 8 cm.

Written Exercises

Exercises 1–12 refer to the right triangle pictured below. Find the required length. Approximate your answer to the nearest hundredth, using a calculator or the Table of Square Roots as necessary.



A

1. $a = 2,\ b = 5,\ c = \underline{\ ?\ }$
2. $a = 3,\ b = 7,\ c = \underline{\ ?\ }$
3. $b = 5,\ c = 9,\ a = \underline{\ ?\ }$
4. $b = 11,\ c = 13,\ a = \underline{\ ?\ }$
5. $a = 10,\ c = 15,\ b = \underline{\ ?\ }$
6. $a = 12,\ c = 16,\ b = \underline{\ ?\ }$
7. $a = \sqrt{10},\ b = 4,\ c = \underline{\ ?\ }$
8. $a = 6,\ b = \sqrt{23},\ c = \underline{\ ?\ }$
9. $a = \sqrt{19},\ c = \sqrt{87},\ b = \underline{\ ?\ }$
10. $a = \sqrt{59},\ c = \sqrt{95},\ b = \underline{\ ?\ }$
11. $b = \sqrt{42},\ c = \sqrt{74},\ a = \underline{\ ?\ }$
12. $b = \sqrt{37},\ c = \sqrt{109},\ a = \underline{\ ?\ }$

Determine whether or not each set of numbers could represent the lengths of the sides of a right triangle.

13. $\{6, 8, 10\}$
14. $\{8, 15, 17\}$
15. $\{2, 4, 6\}$
16. $\{9, 40, 41\}$
17. $\{7, 10, 11\}$
18. $\{6, 14, 15\}$
19. $\{24, 25, 7\}$
20. $\{20, 10, 15\}$
21. $\{9, 21, 30\}$
22. $\{14, 48, 50\}$
23. $\{12, 12, 25\}$
24. $\{35, 35, 35\}$

B

25. $\{5a, 12a, 13a\}$
26. $\{9q, 12q, 15q\}$
27. $\{\sqrt{29}, 8, \sqrt{93}\}$
28. $\{\sqrt{65}, \sqrt{15}, 9\}$
29. $\{\sqrt{2}, \sqrt{2}, 2\}$
30. $\{\sqrt{3}, \sqrt{3}, \sqrt{6}\}$
31. $\{1.2, 1.6, 2\}$
32. $\{1.8, \sqrt{7}, 3.2\}$

In Exercises 33–42, if *c* is the length of the hypotenuse and *a* and *b* the lengths of the other two sides of a right triangle, find the required length(s). Approximate your answer to the nearest hundredth, using a calculator or the Table of Square Roots as necessary.

33. $a = \frac{3}{8},\ b = \frac{1}{2},\ c = \underline{\ ?\ }$
34. $a = \frac{1}{3},\ b = \frac{4}{9},\ c = \underline{\ ?\ }$
35. $a = \frac{1}{2}b,\ b = 8,\ c = \underline{\ ?\ }$
36. $a = \frac{2}{5}b,\ b = 25,\ c = \underline{\ ?\ }$
37. $a = 32,\ b = \frac{3}{4}a,\ c = \underline{\ ?\ }$
38. $a = 27,\ b = \frac{4}{3}a,\ c = \underline{\ ?\ }$

C

39. $a = 2b,\ c = 10,\ a = \underline{\ ?\ },\ b = \underline{\ ?\ }$
40. $b = \frac{1}{2}a,\ c = \sqrt{35},\ a = \underline{\ ?\ },\ b = \underline{\ ?\ }$
41. $a = \frac{2}{3}c,\ b = 5,\ a = \underline{\ ?\ },\ c = \underline{\ ?\ }$
42. $c = \frac{6}{5}a,\ b = \sqrt{33},\ a = \underline{\ ?\ },\ c = \underline{\ ?\ }$

Problems

Make a sketch for each problem. Approximate each square root to the nearest hundredth. Use a calculator or the Table of Square Roots as necessary.

A

1. Find the length of a diagonal of a rectangle whose dimensions are 12 cm by 16 cm.
2. Find the length of a diagonal of a rectangle whose dimensions are 10 cm by 24 cm.
3. Find the length of a diagonal of a square whose sides are each 10 m.
4. Find the length of a diagonal of a square whose sides are each $\sqrt{7}$ cm.
5. Starting at the airfield, Ben flew his glider 8 km due east to point A and then 15 km due south to point B. If he flew directly back to the airfield from point B, how far did he travel in this third part of his trip?
6. A ship navigated a course due west for 18 km and then navigated a course due north for 80 km. At that point, what was the distance of the ship from its starting point?
7. A cable from the top of a circus tent pole is attached to the ground at a point 6 m from the base of the pole. If the cable is 44 m long, how high is the pole?
8. The foot of a 2.6 m ramp is 2.4 m from the base of a loading platform. Find the height of the platform.

B

9. A diagonal of a square is 12 cm long. Find the length of a side of the square.
10. A diagonal of a square is 8 cm long. Find the length of a side of the square.
11. The hypotenuse of a certain right triangle is twice as long as the shortest side, and the length of the third side is 15 m. Find the length of the shortest side.
12. Find the dimensions of a rectangle whose length is three times its width if one of its diagonals has length 30 cm.

C

13. Two sides of a right triangle have lengths 5 m and 8 m. Find the two possible lengths for the third side.
14. Two sides of a right triangle have lengths 6 cm and 10 cm. Find the two possible lengths for the third side.

15. Find the length of a diagonal of a rectangular box of length 10 cm, width 4 cm, and height 5 cm.

16. Find the altitude of an equilateral triangle in which each side has length 8 cm.

17. Develop a formula for finding the length d of the diagonal of a cube whose edge has length k units.

18. Show that if $\{r, s, t\}$ is a Pythagorean triple such that $r^2 + s^2 = t^2$, then for any positive integer k, $\{kr, ks, kt\}$ is also a Pythagorean triple.

Computer Exercises For students with computer experience

1. Write a program that will allow you to input three positive numbers *in any order* and will determine whether or not the numbers can represent the sides of a right triangle. (*Note*: Do *not* use the computer's exponentiation operation to square the numbers.)

2. If p and q are whole numbers such that $p > q$, the three numbers

$$a = p^2 - q^2, \qquad b = 2pq, \qquad \text{and} \qquad c = p^2 + q^2$$

always form a Pythagorean triple. Use this fact to write a program that lists all Pythagorean triples for which p and q are both less than 7. (*Hint*: Use two nested loops, one in which p ranges from 2 to 6 and the other in which q ranges from 1 to $p - 1$.)

10–6 The Distance Formula

Recall from Section 4-5 that the distance between two points on a number line is equal to the absolute value of the difference between their coordinates. To denote the distance between points P and Q, write PQ. Thus, in Figure 8,

$$AB = |1 - (-2)| = 3,$$
$$BC = |3 - 1| = 2, \text{ and}$$
$$CA = |-2 - 3| = 5.$$

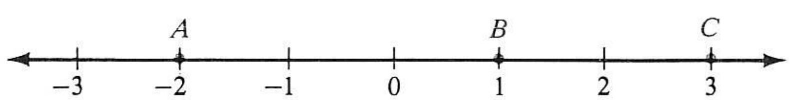


Figure 8

Since $|a - b| = |b - a|$, you can see that the order in which the coordinates are subtracted does not matter.

You can use a similar method to find the distance between two points on a coordinate plane. In Figure 9, notice that $\overline{LM}$ (read "line segment L,M") is parallel to the y-axis and $\overline{NP}$ is parallel to the x-axis.

Since the x-coordinates of points L and M are both 2, the length of $\overline{LM}$ will be the difference of the y-coordinates of the points:

$$LM = |-1 - 4| = 5$$

Similarly, since the y-coordinates of points N and P are both -3, take the difference of the x-coordinates of these points to find the length of $\overline{NP}$:

$$NP = |1 - (-4)| = 5$$

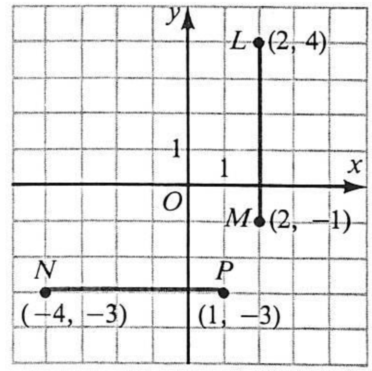


Figure 9

Suppose, now, that you wish to find the distance between two points that lie on a line which is *not* parallel to either axis, such as the points $A(3, 2)$ and $B(-2, 6)$ shown in Figure 10. By drawing the horizontal and vertical segments intersecting at $C(-2, 2)$, as shown, you can form a right triangle having $\overline{AB}$ as its hypotenuse. You can then find the length of each horizontal and vertical side and apply the Pythagorean Theorem.

$$AC = |-2 - 3| = 5$$
$$CB = |6 - 2| = 4$$
$$(AB)^2 = 5^2 + 4^2 = 25 + 16 = 41$$
$$AB = \sqrt{41}$$

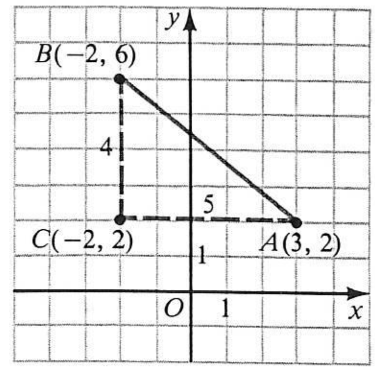


Figure 10

A formula for the distance between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a coordinate plane can be derived in a similar way. In Figure 11, notice that

$$P_1C = |x_2 - x_1| \quad \text{and} \quad CP_2 = |y_2 - y_1|.$$

Since triangle P_1P_2C is a right triangle:

$$(P_1P_2)^2 = (P_1C)^2 + (CP_2)^2$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2$$

However,

$$|x_2 - x_1|^2 = (x_2 - x_1)^2$$

and

$$|y_2 - y_1|^2 = (y_2 - y_1)^2.$$

Thus,

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and, therefore,

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

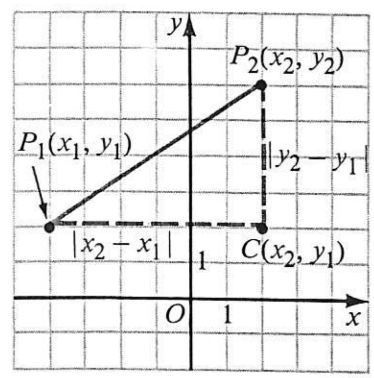


Figure 11

The method of finding the distance between two points that was discussed on the preceding page can be generalized as the following *distance formula*.

> ***Distance Formula***
>
> Given any points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:
>
> $$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE Find the distance between $P(-2, -3)$ and $Q(4, -1)$.

SOLUTION 1
$$PQ = \sqrt{[4 - (-2)]^2 + [-1 - (-3)]^2}$$
$$= \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

SOLUTION 2
$$QP = \sqrt{[-2 - 4]^2 + [-3 - (-1)]^2}$$
$$= \sqrt{(-6)^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$$

The two solutions in the Example show that it does not matter which point is considered (x_1, y_1) and which point (x_2, y_2).

Oral Exercises

Find the distance between the two points having the given coordinates.

1. $(2, -1), (5, 3)$
2. $(1, -6), (6, 6)$
3. $(-3, 20), (5, 5)$
4. $(8, 10), (2, 2)$
5. $(6, -18), (-4, 6)$
6. $(10, 10), (1, -30)$

Written Exercises

Find the distance between the two points having the given coordinates. Simplify irrational distances.

A

1. $(1, 1), (3, 2)$
2. $(2, 2), (-2, 1)$
3. $(2, 1), (5, 5)$
4. $(-7, -3), (5, 2)$
5. $(3, -2), (-3, 1)$
6. $(4, -1), (-1, 4)$
7. $(5, -1), (2, 2)$
8. $(-2, -1), (-1, 3)$
9. $(3, -6), (11, 9)$
10. $(8, 20), (-1, 8)$
11. $(24, -18), (6, 0)$
12. $(-4, 40), (-20, 8)$

B 13. $(a, b), (-a, -b)$

14. $(4a, b), (2a, -3b)$

15. $(a + b, 2b), (2b, a + b)$

16. $(a + b, a - b), (-b, -a)$

17. $(a, b), (b, a)$

18. $(2a, b), (-b, -2a)$

Using the distance formula and the converse of the Pythagorean Theorem, determine whether or not the triangle with the given vertices is a right triangle.

19. (1, 0), (4, 4), (4, 0)

20. (3, 1), (6, 1), (6, −5)

21. (2, −2), (8, 0), (4, −4)

22. (4, 2), (6, −3), (−1, 5)

23. (−1, 1), (1, 5), (6, −1)

24. (−4, −2), (1, −3), (−2, −5)

25. (−3, −2), (0, 4), (7, 3)

26. (10, 6), (−1, 2), (13, −1)

27. Given that (1, −2), (0, 5), and (−3, 1) are the vertices of a right triangle, determine an equation of the line that passes through the endpoints of the hypotenuse.

28. Given that (−1, −1), (5, 3), and (−3, 2) are the vertices of a right triangle, determine an equation of the line that passes through the endpoints of the shortest side.

29. Show that the points (18, 4), (12, 12), and (8, 4) are the vertices of an isosceles triangle.

30. Show that the points (6, 2), (2, −1), and (−1, 3) are the vertices of an isosceles right triangle.

C 31. Show that the distance d between the origin and a point whose coordinates are (x, y) is given by the formula $d = \sqrt{x^2 + y^2}$.

32. Find two values of x such that the point $(x, 5)$ is 5 units from the point (−1, 2).

33. Find two values of m such that the point $(m, -1)$ is 6.5 units from the point (−4, −7).

Computer Exercises

For students with computer experience

1. Write a program that will allow you to input the coordinates of any two points on a coordinate plane, (x_1, y_1) and (x_2, y_2), and will compute the distance between them.
2. Modify the program that you wrote for Exercise 1 so that you can input the coordinates of a third point, (x_3, y_3), and the program will determine whether or not the distance between (x_1, y_1) and (x_3, y_3) is equal to the distance between (x_2, y_2) and (x_3, y_3).
3. Modify the program that you wrote for Exercise 2 so that it will allow you to input the coordinates of any three noncollinear points and it will determine whether the triangle that has these three points as vertices is equilateral, isosceles, or scalene.

Self-Test 2

VOCABULARY hypotenuse (p. 505)
Pythagorean Theorem (p. 505)
Pythagorean numbers or Pythagorean triple (p. 508)
distance formula (p. 514)

Determine whether or not each set of numbers could represent the lengths of the sides of a right triangle.

Obj. 1, p. 505

1. {16, 30, 34} **2.** {12, 16, 25}

3. $\{\sqrt{2}, 5, 7\}$ **4.** {21, 72, 75}

If c is the length of the hypotenuse and a and b are the lengths of the other two sides of a right triangle, find the required length. Approximate your answer to the nearest hundredth, using a calculator or the Table of Square Roots as necessary.

5. $a = 18, b = 24, c = \underline{\ ?\ }$ **6.** $b = \sqrt{42}, c = 9, a = \underline{\ ?\ }$

7. $b = \sqrt{91}, c = 17, a = \underline{\ ?\ }$ **8.** $a = 13, c = \sqrt{281}, b = \underline{\ ?\ }$

Find the distance between the two points having the given coordinates. Simplify irrational distances.

Obj. 2, p. 505

9. (4, −2), (8, 3) **10.** (2, −8), (−3, 4)

11. (20, 15), (8, 45) **12.** (32, 4), (26, −12)

Check your answers with those at the back of the book.

Working with Radicals

OBJECTIVES for Sections 10-7 through 10-10:
1. To simplify expressions that contain radicals.
2. To simplify n*th roots.*
3. To solve equations involving radicals.

10–7 Products and Quotients of Square Roots

Recall that, in Section 10-4, you used the *product property of square roots* to simplify radicals whose radicands contain factors that are perfect squares. The following example shows how the same property is used to simplify *products* of square roots.

EXAMPLE 1 Simplify. Assume that all variables denote positive real numbers.

a. $\sqrt{2} \cdot \sqrt{6}$ b. $\sqrt{15yz} \cdot \sqrt{3yz}$ c. $12\sqrt{10} \cdot 4\sqrt{2}$

SOLUTION a.
$$\begin{aligned}\sqrt{2} \cdot \sqrt{6} &= \sqrt{2 \cdot 6} \\ &= \sqrt{12} \\ &= \sqrt{4} \cdot \sqrt{3} \\ &= 2\sqrt{3}\end{aligned}$$

b.
$$\begin{aligned}\sqrt{15yz} \cdot \sqrt{3yz} &= \sqrt{45y^2z^2} \\ &= \sqrt{9y^2z^2} \cdot \sqrt{5} \\ &= 3yz\sqrt{5}\end{aligned}$$

c.
$$\begin{aligned}12\sqrt{10} \cdot 4\sqrt{2} &= 48\sqrt{20} \\ &= 48 \cdot \sqrt{4} \cdot \sqrt{5} \\ &= 48 \cdot 2 \cdot \sqrt{5} \\ &= 96\sqrt{5}\end{aligned}$$

To determine a property that you may use to simplify *quotients* of square roots or square roots of quotients, first consider the following. Since

$$\sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

and

$$\frac{\sqrt{36}}{\sqrt{4}} = \frac{6}{2} = 3,$$

by the transitive axiom of equality,

$$\sqrt{\frac{36}{4}} = \frac{\sqrt{36}}{\sqrt{4}}.$$

This relationship suggests the following property of square roots.

Quotient Property of Square Roots

For any nonnegative real number a and positive real number b,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

This property, together with the *basic property of quotients*, provides a means of expressing a fraction with an irrational denominator or a radical containing a fraction as an equivalent fraction with a rational denominator, as shown on the following page.

EXAMPLE 2 Simplify. Assume that all variables represent positive real numbers.

a. $\sqrt{\frac{5}{6}}$ **b.** $\frac{3}{\sqrt{12}}$ **c.** $\sqrt{\frac{2}{5x}}$

SOLUTION

a. $\sqrt{\frac{5}{6}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5} \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{\sqrt{30}}{\sqrt{36}} = \frac{\sqrt{30}}{6}$, or $\frac{1}{6}\sqrt{30}$

b. $\frac{3}{\sqrt{12}} = \frac{3 \cdot \sqrt{3}}{\sqrt{12} \cdot \sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{36}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$, or $\frac{1}{2}\sqrt{3}$

c. $\sqrt{\frac{2}{5x}} = \frac{\sqrt{2}}{\sqrt{5x}} = \frac{\sqrt{2} \cdot \sqrt{5x}}{\sqrt{5x} \cdot \sqrt{5x}} = \frac{\sqrt{10x}}{\sqrt{25x^2}} = \frac{\sqrt{10x}}{5x}$, or $\frac{1}{5x}\sqrt{10x}$

The process of expressing a fraction with an irrational denominator as an equivalent fraction with a rational denominator is called **rationalizing the denominator.** Although sometimes it is helpful to use an expression with an irrational denominator, such as $\frac{1}{\sqrt{3}}$, in the process of computation, it is customary to express a final answer in a form in which the denominator has been rationalized.

The product and quotient properties of square roots can be used to express radicals in *simplest form* as follows.

An expression that contains a radical is in **simplest form** when:

1. no integral radicand has a perfect square factor other than 1,
2. no fractions are under a radical, and
3. no radicals are in a denominator.

You may apply this rule to expressions containing radicals that have polynomials as radicands when such polynomials represent non-negative real numbers. Of course, if the polynomial is a denominator, the polynomial must represent a *positive* real number.

Rationalizing the denominator may simplify the calculations involved in finding an approximation of an expression containing a radical. For example, contrast the following computations.

$$\frac{1}{\sqrt{2}} \approx \frac{1}{1.414} \approx 0.707$$

and

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707$$

You would probably find that the second of the two computations is easier to complete. Therefore, simplify any expression containing radicals before evaluating the expression.

Oral Exercises

Simplify. Assume that all variables represent positive real numbers.

1. $\sqrt{2} \cdot \sqrt{5}$
2. $\sqrt{2} \cdot \sqrt{10}$
3. $\sqrt{8} \cdot \sqrt{6}$
4. $\sqrt{12} \cdot \sqrt{6}$
5. $\frac{1}{\sqrt{5}}$
6. $\frac{1}{\sqrt{7}}$
7. $\sqrt{\frac{1}{3}}$
8. $\sqrt{\frac{2}{5}}$
9. $6\sqrt{2} \cdot 4\sqrt{30}$
10. $5\sqrt{15} \cdot 8\sqrt{10}$
11. $\sqrt{2x} \cdot \sqrt{xy}$
12. $\sqrt{3r} \cdot \sqrt{6r}$
13. $\sqrt{\frac{1}{z}}$
14. $\sqrt{\frac{5}{e}}$
15. $\frac{b}{\sqrt{b^3}}$
16. $\frac{v^2}{\sqrt{v^5}}$

Tell whether each statement is true or false for all positive real numbers *a* and *b*.

17. $5\sqrt{9a} = 8\sqrt{a}$
18. $\sqrt{a^{36}} = a^6$
19. $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$
20. $\sqrt{3b} \cdot \sqrt{7b} = \sqrt{21b}$
21. $\frac{\sqrt{b}}{a} = \sqrt{\frac{b}{a}}$
22. $\sqrt{\frac{a}{b}} = \frac{1}{b}\sqrt{ab}$
23. $\sqrt{a} \cdot \sqrt{a + ab} = a\sqrt{1 + b}$
24. $(\sqrt{a^2b^3})^2 = a^2b^3$

Written Exercises

Simplify.

A

1. $\sqrt{5} \cdot \sqrt{10}$
2. $\sqrt{2} \cdot \sqrt{32}$
3. $2\sqrt{3} \cdot \sqrt{6}$
4. $4\sqrt{5} \cdot \sqrt{18}$
5. $\sqrt{12} \cdot \sqrt{15}$
6. $\sqrt{24} \cdot \sqrt{10}$
7. $\frac{6}{\sqrt{2}}$
8. $\frac{8}{\sqrt{6}}$
9. $\sqrt{\frac{10}{3}}$
10. $\sqrt{\frac{5}{12}}$
11. $\frac{\sqrt{32}}{\sqrt{8}}$
12. $\frac{\sqrt{75}}{\sqrt{10}}$
13. $\sqrt{\frac{8}{3} \cdot \frac{3}{4}}$
14. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{15}{8}}$
15. $\frac{2\sqrt{800}}{\sqrt{20}}$
16. $\frac{5\sqrt{600}}{\sqrt{75}}$

Simplify. Approximate your answer to the nearest hundredth. Use a calculator or the Table of Square Roots as necessary.

17. $\sqrt{2} \cdot \sqrt{10}$
18. $\sqrt{18} \cdot \sqrt{12}$
19. $\sqrt{48} \cdot \sqrt{6}$
20. $\sqrt{32} \cdot \sqrt{24}$
21. $\sqrt{\frac{1}{5}}$
22. $\frac{1}{\sqrt{7}}$
23. $\frac{2}{\sqrt{7}}$
24. $\sqrt{\frac{5}{8}}$

Simplify. Assume that all variables represent positive real numbers and that the value of any expression under a radical sign is a positive real number.

B 25. $\sqrt{2c} \cdot \sqrt{3c}$ 26. $\sqrt{20u} \cdot \sqrt{8u}$ 27. $\sqrt{10e} \cdot \sqrt{10e^2}$ 28. $\sqrt{3q^2} \cdot \sqrt{48q}$

29. $\sqrt{\frac{3}{t}}$ 30. $\sqrt{\frac{5a}{b}}$ 31. $\sqrt{8r^3} \cdot \sqrt{32r^5}$ 32. $\sqrt{18j^6} \cdot \sqrt{12j^3}$

33. $\frac{5\sqrt{2e}}{\sqrt{e^3}}$ 34. $\frac{\sqrt{6h}}{\sqrt{2h^3}}$ 35. $\sqrt{\frac{45}{8c}}$ 36. $\sqrt{\frac{75}{28c}}$

37. $n\sqrt{\frac{m^3}{mn}}$ 38. $\frac{2c\sqrt{2c}}{\sqrt{c^3}}$ 39. $\frac{k}{i}\sqrt{\frac{5i^2}{3k}}$ 40. $\frac{5i}{6t}\sqrt{\frac{9t^3}{10i}}$

41. $\sqrt{g^5} \cdot \sqrt{\frac{3}{g^2h}}$ 42. $\sqrt{8c^3} \cdot \sqrt{\frac{3d^2}{c^5d^3}}$ 43. $\frac{2t\sqrt{12t^5}}{\sqrt{6t^7}}$ 44. $\frac{e}{f}\sqrt{\frac{27f^3}{32e^3}}$

C 45. $\frac{1}{\sqrt{a+b}}$ 46. $\frac{m+n}{\sqrt{m+n}}$ 47. $\sqrt{\frac{2x+1}{(2x+1)^2}}$ 48. $\sqrt{\frac{y+2}{y^2+4y+4}}$

49. $\sqrt{\frac{x-1}{x^2+2x-3}}$ 50. $\sqrt{\frac{x-2y}{x^2+xy-6y^2}}$ 51. $\frac{r^2-s^2}{\sqrt{(r-s)^3}}$ 52. $\frac{2a^2+8ab+8b^2}{\sqrt{(a+2b)^5}}$

10–8 Sums and Products of Expressions Containing Radicals

To simplify the sum

$$4\sqrt{5} + 9\sqrt{5},$$

note that the two terms have a common radical factor, $\sqrt{5}$, and apply the distributive axiom. Thus:

$$4\sqrt{5} + 9\sqrt{5} = (4+9)\sqrt{5} = 13\sqrt{5}$$

Terms that do *not* have a common radical factor cannot be combined. Thus:

$$2\sqrt{3} - 8\sqrt{2} + 12\sqrt{3} = 14\sqrt{3} - 8\sqrt{2}$$

To determine whether or not two or more terms have a common radical factor, write each radical in simplest form.

EXAMPLE 1 Simplify.

a. $\sqrt{20} + \sqrt{80} - \sqrt{45}$ b. $(5 + 2\sqrt{12}) - (3 + \sqrt{48})$

SOLUTION a. $\sqrt{20} + \sqrt{80} - \sqrt{45} = 2\sqrt{5} + 4\sqrt{5} - 3\sqrt{5} = 3\sqrt{5}$

b. $(5 + 2\sqrt{12}) - (3 + \sqrt{48}) = (5 + 4\sqrt{3}) - (3 + 4\sqrt{3}) = 2$

Part (b) of Example 1 illustrates the fact that a *sum* of irrational numbers may be a rational number.

> To simplify a sum of radicals:
>
> 1. Express each radical in simplest form.
> 2. Use the distributive axiom to combine radical terms having the same radicand.

The distributive axiom also enables you to simplify products of expressions containing radicals.

EXAMPLE 2 Simplify.

a. $\sqrt{2}(3\sqrt{2} + \sqrt{3})$ **b.** $(2 + \sqrt{3})(3 - \sqrt{3})$

c. $(\sqrt{6} + \sqrt{5})(\sqrt{6} - \sqrt{5})$ **d.** $(x\sqrt{3} + \sqrt{2})(x\sqrt{3} - \sqrt{2})$

SOLUTION

a.
$$\begin{aligned}\sqrt{2}(3\sqrt{2} + \sqrt{3}) &= (\sqrt{2})(3\sqrt{2}) + (\sqrt{2})(\sqrt{3})\\ &= 3 \cdot 2 + \sqrt{6}\\ &= 6 + \sqrt{6}\end{aligned}$$

b.
$$\begin{aligned}(2 + \sqrt{3})(3 - \sqrt{3}) &= 6 - 2\sqrt{3} + 3\sqrt{3} - 3\\ &= 3 + \sqrt{3}\end{aligned}$$

c.
$$\begin{aligned}&(\sqrt{6} + \sqrt{5})(\sqrt{6} - \sqrt{5})\\ &\quad = (\sqrt{6})(\sqrt{6}) - (\sqrt{6})(\sqrt{5}) + (\sqrt{5})(\sqrt{6}) - (\sqrt{5})(\sqrt{5})\\ &\quad = 6 - \sqrt{30} + \sqrt{30} - 5\\ &\quad = 6 - 5 = 1\end{aligned}$$

d.
$$\begin{aligned}(x\sqrt{3} + \sqrt{2})(x\sqrt{3} - \sqrt{2}) &= (x\sqrt{3})^2 - (\sqrt{2})^2\\ &= 3x^2 - 2\end{aligned}$$

Part (c) of Example 2 illustrates the fact that a *product* of irrational numbers may be a rational number. In particular, the product of two irrational numbers, such as the expressions $\sqrt{6} + \sqrt{5}$ and $\sqrt{6} - \sqrt{5}$, may be an integer.

If b and d are nonnegative real numbers, then the binomials

$$a\sqrt{b} + c\sqrt{d} \text{ and } a\sqrt{b} - c\sqrt{d}$$

are called **conjugates.** If a and c are integers and b and d are nonnegative integers, the product

$$(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d})$$

will be an integer.

You can use conjugates to rationalize some binomial denominators that contain radicals.

EXAMPLE 3 Simplify. a. $\dfrac{\sqrt{2}}{4 - \sqrt{3}}$ b. $\dfrac{\sqrt{6}}{\sqrt{2} + \sqrt{5}}$

SOLUTION a. $\dfrac{\sqrt{2}}{4 - \sqrt{3}} = \dfrac{\sqrt{2}}{4 - \sqrt{3}} \cdot \dfrac{4 + \sqrt{3}}{4 + \sqrt{3}}$

$= \dfrac{4\sqrt{2} + \sqrt{6}}{16 - 3}$

$= \dfrac{4\sqrt{2} + \sqrt{6}}{13}$

b. $\dfrac{\sqrt{6}}{\sqrt{2} + \sqrt{5}} = \dfrac{\sqrt{6}}{\sqrt{2} + \sqrt{5}} \cdot \dfrac{\sqrt{2} - \sqrt{5}}{\sqrt{2} - \sqrt{5}}$

$= \dfrac{\sqrt{12} - \sqrt{30}}{2 - 5}$

$= \dfrac{2\sqrt{3} - \sqrt{30}}{-3}$

$= \dfrac{\sqrt{30} - 2\sqrt{3}}{3}$

In Chapter 7, you factored polynomials over the set of polynomials with integral coefficients. If the factor set is extended to include polynomials with irrational coefficients, you may now factor additional polynomials. For example, in part (d) of Example 2,

$$3x^2 - 2 = (x\sqrt{3} + \sqrt{2})(x\sqrt{3} - \sqrt{2}).$$

Oral Exercises

Simplify.

1. $2\sqrt{2} + 4\sqrt{2}$
2. $12\sqrt{3} - 5\sqrt{3}$
3. $\sqrt{2}(5 + \sqrt{3})$
4. $\sqrt{3}(\sqrt{2} - \sqrt{5})$
5. $(3\sqrt{2})(2\sqrt{2})$
6. $(2\sqrt{5})^2$
7. $(4\sqrt{3})(3\sqrt{6})$
8. $(3\sqrt{8})(2\sqrt{12})$
9. $\sqrt{2} + \sqrt{8}$
10. $\sqrt{5} - \sqrt{20}$
11. $2\sqrt{3} - \sqrt{75}$
12. $7\sqrt{2} + \sqrt{128}$
13. $\sqrt{28} - \sqrt{63}$
14. $\sqrt{72} + \sqrt{200}$
15. $(2 + \sqrt{3})(2 - \sqrt{3})$
16. $(3 + \sqrt{5})(3 + \sqrt{5})$
17. $(\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3})$
18. $(\sqrt{10} - \sqrt{7})(\sqrt{10} - \sqrt{7})$

State the conjugate of each binomial.

19. $2 + \sqrt{3}$
20. $4 - \sqrt{5}$
21. $\sqrt{2} + \sqrt{7}$
22. $\sqrt{5} - \sqrt{15}$
23. $2\sqrt{2} - \sqrt{6}$
24. $4\sqrt{2} + 3\sqrt{3}$

Written Exercises

Simplify. Assume that all variables represent positive real numbers.

A

1. $5\sqrt{6} + \sqrt{6} - 2\sqrt{6}$
2. $8\sqrt{3} - 2\sqrt{3} + \sqrt{3}$
3. $7\sqrt{2} - \sqrt{18} + 3\sqrt{8}$
4. $\sqrt{75} - 2\sqrt{3} + 4\sqrt{27}$
5. $8\sqrt{2} - 4\sqrt{8} - 4\sqrt{50}$
6. $6\sqrt{5} + \sqrt{500} - 2\sqrt{125}$
7. $\sqrt{16c} + \sqrt{c} - \sqrt{81c}$
8. $t\sqrt{st^2} - t^2\sqrt{s} - \sqrt{49st^2}$
9. $(5\sqrt{5})(3\sqrt{6})$
10. $(6\sqrt{15})(2\sqrt{3})$
11. $\sqrt{3}(\sqrt{6} - \sqrt{12})$
12. $\sqrt{2}(\sqrt{32} + \sqrt{10})$
13. $(\sqrt{3} + 1)(\sqrt{3} - 1)$
14. $(3 + \sqrt{5})(3 - \sqrt{5})$
15. $(2 + \sqrt{13})(4 - \sqrt{13})$
16. $(\sqrt{6} + 1)(\sqrt{6} - 4)$
17. $(2\sqrt{2} + 3)(\sqrt{2} - 1)$
18. $(3\sqrt{3} - 2)(\sqrt{3} + 4)$
19. $(4\sqrt{3} + 5)(3\sqrt{3} - 2)$
20. $(8 - 2\sqrt{2})(3 - 4\sqrt{2})$
21. $(\sqrt{6} - \sqrt{11})(\sqrt{6} + \sqrt{11})$
22. $(\sqrt{15} + \sqrt{8})(\sqrt{15} - \sqrt{8})$
23. $(2\sqrt{5} - \sqrt{3})(2\sqrt{5} + \sqrt{3})$
24. $(\sqrt{2} + 3\sqrt{6})(\sqrt{2} - 3\sqrt{6})$

B

25. $(\sqrt{a} - \sqrt{c})^2$
26. $(\sqrt{b} + \sqrt{d})^2$
27. $(\sqrt{3} + \sqrt{6})^2$
28. $(\sqrt{10} - \sqrt{2})^2$
29. $(2\sqrt{5} + \sqrt{3})^2$
30. $(3\sqrt{r} - 2\sqrt{s})^2$
31. $\frac{2}{\sqrt{3} - 1}$
32. $\frac{5}{\sqrt{2} + 3}$
33. $\frac{4\sqrt{6}}{2 + \sqrt{2}}$
34. $\frac{5\sqrt{15}}{5 - \sqrt{3}}$
35. $\frac{\sqrt{3} + 2}{\sqrt{3} - 1}$
36. $\frac{\sqrt{5} - 1}{\sqrt{2} + 5}$
37. $\frac{\sqrt{6} + 2\sqrt{2}}{\sqrt{2} - \sqrt{3}}$
38. $\frac{\sqrt{3} - \sqrt{5}}{3\sqrt{3} + \sqrt{5}}$
39. $\frac{\sqrt{2} + \sqrt{5}}{2\sqrt{5} - \sqrt{3}}$
40. $\frac{\sqrt{10} + \sqrt{3}}{\sqrt{2} + 3\sqrt{3}}$
41. $\frac{\sqrt{5} - 2\sqrt{3}}{\sqrt{5} + 3\sqrt{2}}$
42. $\frac{2\sqrt{6} - \sqrt{3}}{\sqrt{6} + \sqrt{5}}$
43. $15\sqrt{\frac{2}{3}} + \sqrt{150} - 8\sqrt{1\frac{1}{2}}$
44. $3\sqrt{2\frac{2}{5}} - \frac{2}{5}\sqrt{240} + \sqrt{540}$
45. $3j\sqrt{98k} - 5k\sqrt{\frac{242j^2}{k}}$
46. $3\sqrt{a^5b} + a^2b\sqrt{\frac{25a}{b}}$
47. $\sqrt{\frac{35u^2}{72} + \frac{5u^2}{24}}$
48. $\sqrt{\frac{5}{2c} - \frac{2}{9c} - \frac{1}{36c}}$
49. Determine an expression in simplest form that represents the area of a square whose perimeter is $(16\sqrt{5} + 4\sqrt{2})$ m.
50. Determine an expression in simplest form that represents the area of a square whose perimeter is $(12\sqrt{2} + 20\sqrt{3})$ m.

Let $f(x) = 3x^2 - 3x + 1$. Compute each of the following.

C **51.** $f(\sqrt{3})$ **52.** $f(-\sqrt{2})$ **53.** $f(\sqrt{2} - 1)$ **54.** $f(\sqrt{3} + 2)$

55. Show that $(2 + \sqrt{5})$ and $(2 - \sqrt{5})$ are solutions of $x^2 - 4x - 1 = 0$.

56. Show that $(-3 + 2\sqrt{5})$ and $(-3 - 2\sqrt{5})$ are solutions of $x^2 + 6x - 11 = 0$.

10–9 *n*th Roots

Recall from Section 10-3 that if $b^2 = a$, then b is called a *square root* of a. Similarly, if

$$b^n = a,$$

then b is called an *nth root* of a. In general, if n is a positive integer and a is a real number, any real number whose nth power equals a is called an *n*th root of a.

If $b^3 = a$, then b is called the **cube root** of a. There is only *one* real number, 4, whose cube is 64. Therefore, 4 is the cube root of 64. The number -4 is the cube root of -64, since $(-4)^3 = -64$. In general:

> *If a is a real number and n is a positive odd integer*, then there is only *one* real nth root of a, denoted by
>
> $$\sqrt[n]{a}.$$

For example, $\sqrt[3]{64} = 4$ and $\sqrt[3]{-64} = -4$.

On the other hand, if you want to find the sixth root of 64, there are *two* values, 2 and -2, since $2^6 = 64$ and $(-2)^6 = 64$. In general:

> *If a is a positive real number and n is a positive even integer*, then there are *two* real nth roots of a. The *positive* nth root of a is denoted by
>
> $$\sqrt[n]{a},$$
>
> and the *negative* nth root of a is denoted by
>
> $$-\sqrt[n]{a}.$$

The positive nth root is referred to as the **principal *n*th root.** For example, $\sqrt[6]{64} = 2$ and $-\sqrt[6]{64} = -2$. The *principal* sixth root of 64 is 2.

Although 64 has two real sixth roots, -64 has *no* real sixth roots, since there are no real numbers that satisfy $x^6 = -64$. In general:

> *If a is a negative real number and n is a positive even integer*, then there are *no* real nth roots of a.

When n is even and a is negative, the symbols $\sqrt[n]{a}$ and $-\sqrt[n]{a}$ do not represent real numbers.

If $a = 0$ and n is a positive integer, then 0 is the only nth root of a, no matter whether n is odd or even, since $0^n = 0$ for any positive integer n.

Thus, for any positive integer n,

$$\sqrt[n]{0} = 0.$$

In the symbol $\sqrt[n]{a}$, the positive integer n is called the **index.** In a radical such as $\sqrt{10}$ the index is understood to be 2, but it is usually not written.

The properties of nth roots are similar to the properties of square roots.

> ***Properties of nth Roots***
>
> If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then:
>
> **1.** $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ **2.** If $b \neq 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

If a number has a real nth root, raising the nth root to the nth power results in the number itself. Thus,

$$(\sqrt[3]{8})^3 = 8, \qquad (\sqrt[4]{81})^4 = 81, \qquad \text{and} \qquad (\sqrt[n]{a})^n = a.$$

If a real number is not the nth power of some rational number, any nth root of the number will be irrational. The following example shows how the properties of nth roots can be used to express these irrational nth roots in *simplest form.*

EXAMPLE Simplify. a. $\sqrt[3]{16}$ b. $\sqrt[4]{162} + \sqrt[4]{32} - \sqrt[3]{54}$ c. $\frac{3}{\sqrt[3]{25}}$

SOLUTION a. $\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$

b. $\sqrt[4]{162} + \sqrt[4]{32} - \sqrt[3]{54} = (\sqrt[4]{81})(\sqrt[4]{2}) + (\sqrt[4]{16})(\sqrt[4]{2}) - (\sqrt[3]{27})(\sqrt[3]{2})$

$= 3\sqrt[4]{2} + 2\sqrt[4]{2} - 3\sqrt[3]{2}$

$= 5\sqrt[4]{2} - 3\sqrt[3]{2}$

c. $\frac{3}{\sqrt[3]{25}} = \frac{3}{\sqrt[3]{25}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}} = \frac{3\sqrt[3]{5}}{\sqrt[3]{125}} = \frac{3\sqrt[3]{5}}{5}$

Oral Exercises

Tell whether or not the given symbol represents a real number. If the number represented is real, tell whether it is rational or irrational.

1. $\sqrt[3]{8}$
2. $\sqrt[4]{625}$
3. $\sqrt[8]{-1}$
4. $\sqrt[3]{0}$
5. $\sqrt{-\frac{3}{64}}$
6. $-\sqrt[4]{-16}$
7. $\sqrt{\frac{16}{125}}$
8. $\sqrt[4]{-\frac{1}{32}}$
9. $\sqrt[3]{-\frac{81}{64}}$
10. $\sqrt[5]{\frac{12}{25}}$
11. $\sqrt[12]{-\frac{56}{75}}$
12. $\sqrt[6]{\frac{64}{729}}$

Written Exercises

Simplify.

A

1. $\sqrt[3]{27}$
2. $\sqrt[3]{125}$
3. $\sqrt[4]{81}$
4. $\sqrt[5]{-32}$
5. $(\sqrt[7]{-128})^7$
6. $(\sqrt[6]{64})^6$
7. $\sqrt[4]{80}$
8. $\sqrt[3]{40}$
9. $\sqrt[3]{24} + \sqrt[3]{81}$
10. $\sqrt[4]{48} - \sqrt[4]{243}$
11. $\frac{3}{\sqrt[3]{2}}$
12. $\frac{9}{\sqrt[4]{3}}$
13. $\frac{6}{\sqrt[4]{8}}$
14. $\frac{12}{\sqrt[5]{4}}$
15. $\sqrt[3]{\frac{5}{8}}$
16. $\sqrt[6]{\frac{7}{64}}$

B

17. $\sqrt[5]{\frac{9}{16}}$
18. $\sqrt[4]{\frac{16}{27}}$
19. $\sqrt[3]{\frac{5}{32}}$
20. $\sqrt[3]{-\frac{1}{81}}$

Simplify. Assume that all variables represent positive real numbers and that the value of any expression under a radical sign is positive.

21. $\sqrt[3]{x^9}$
22. $\sqrt[4]{y^{12}}$
23. $\sqrt[10]{x^{20}y^{30}z^{10}}$
24. $\sqrt[15]{a^{45}b^{30}c^{60}}$
25. $\sqrt[3]{x^7y^{11}z^8}$
26. $\sqrt[6]{m^{13}p^{21}t^{48}}$
27. $\sqrt[4]{\frac{y^9}{x^2}}$
28. $\sqrt[5]{\frac{b^6}{a^4y^2}}$

C

29. $\sqrt[3]{\frac{x}{x^2 - 2x + 1}}$
30. $\sqrt[4]{\frac{y + 1}{(y - 3)^3}}$
31. $\sqrt[6]{\frac{x - 3}{x^2 - 2x - 3}}$
32. $\sqrt[5]{\frac{a + 4}{(a + 1)(a^2 + 5a + 4)}}$

ON THE CALCULATOR

The square root key of a calculator can be used to find the nth root of a nonnegative real number provided that n is a power of two. For example, since $2^2 = 4$, you can find the *fourth* root by pressing the square root key *twice*. Similarly, since $2^3 = 8$, you can find the *eighth* root by pressing the square root key *three times*. Following this pattern, you can use the square root key to find the *sixteenth* root, the *thirty-second* root, and so on.

Use a calculator to find the indicated roots. Approximate the answer to the nearest hundredth.

1. $\sqrt{3021}$
2. $\sqrt[4]{72}$
3. $\sqrt[32]{468}$
4. $\sqrt{0.0142}$
5. $\sqrt[16]{12}$
6. $\sqrt[64]{100}$
7. $\sqrt[8]{0.379}$
8. $\sqrt[8]{\frac{13}{55}}$

10–10 Equations Involving Radicals

Sometimes you will need to solve an equation in which a term has a variable in a radicand. In such cases, you transform the equation into an equivalent equation in which the term with the variable in the radicand is alone as one side of the equation. Then raise both sides of the equation to the power equal to the index of the radical and solve the resulting equation.

EXAMPLE 1 Solve.

a. $\sqrt{d} - 3 = 4$

b. $\sqrt[3]{x + 6} = 2$

c. $2 + \sqrt{x - 5} = 3$

SOLUTION

a.
$$\begin{aligned} \sqrt{d} - 3 &= 4 \\ \sqrt{d} &= 7 \\ (\sqrt{d})^2 &= 7^2 \\ d &= 49 \end{aligned}$$

Check:
$$\begin{aligned} \sqrt{d} - 3 &= 4 \\ \sqrt{49} - 3 &\stackrel{?}{=} 4 \\ 7 - 3 &\stackrel{?}{=} 4 \\ 4 &= 4 \ \checkmark \end{aligned}$$

$\therefore$ the solution set is $\{49\}$.

b.
$$\begin{aligned} \sqrt[3]{x + 6} &= 2 \\ (\sqrt[3]{x + 6})^3 &= 2^3 \\ x + 6 &= 8 \\ x &= 2 \end{aligned}$$

Check:
$$\begin{aligned} \sqrt[3]{x + 6} &= 2 \\ \sqrt[3]{2 + 6} &\stackrel{?}{=} 2 \\ \sqrt[3]{8} &\stackrel{?}{=} 2 \\ 2 &= 2 \ \checkmark \end{aligned}$$

$\therefore$ the solution set is $\{2\}$.

c.
$$\begin{aligned} 2 + \sqrt{x - 5} &= 3 \\ \sqrt{x - 5} &= 1 \\ (\sqrt{x - 5})^2 &= 1^2 \\ x - 5 &= 1 \\ x &= 6 \end{aligned}$$

Check:
$$\begin{aligned} 2 + \sqrt{x - 5} &= 3 \\ 2 + \sqrt{6 - 5} &\stackrel{?}{=} 3 \\ 2 + \sqrt{1} &\stackrel{?}{=} 3 \\ 2 + 1 &\stackrel{?}{=} 3 \\ 3 &= 3 \ \checkmark \end{aligned}$$

$\therefore$ the solution set is $\{6\}$.

The equation you obtain by raising both sides of an equation to a power is not necessarily equivalent to the original equation. Therefore, the solutions of the new equation may not all be solutions of the original equation. However, every solution of the original equation will always be a solution of the new equation. It is important to check each solution of the new equation in the original equation to discover any *extraneous* solutions.

EXAMPLE 2 Solve $\sqrt{5x-4} - x = -2$

SOLUTION

$$\sqrt{5x-4} - x = -2$$
$$\sqrt{5x-4} = x - 2$$
$$(\sqrt{5x-4})^2 = (x-2)^2$$
$$5x - 4 = x^2 - 4x + 4$$
$$0 = x^2 - 9x + 8$$
$$0 = (x-8)(x-1)$$

$x = 8$ or $x = 1$

Check:

$$\sqrt{5x-4} - x = -2 \qquad \sqrt{5x-4} - x = -2$$
$$\sqrt{5(8)-4} - 8 \stackrel{?}{=} -2 \qquad \sqrt{5(1)-4} - 1 \stackrel{?}{=} -2$$
$$\sqrt{40-4} - 8 \stackrel{?}{=} -2 \qquad \sqrt{5-4} - 1 \stackrel{?}{=} -2$$
$$\sqrt{36} - 8 \stackrel{?}{=} -2 \qquad \sqrt{1} - 1 \stackrel{?}{=} -2$$
$$6 - 8 \stackrel{?}{=} -2 \qquad 1 - 1 \stackrel{?}{=} -2$$
$$-2 = -2 \ \checkmark \qquad 0 \neq -2$$

$\therefore$ the solution set is $\{8\}$.

Not all equations involving radicals have a real-number solution. For example, consider the equation

$$\sqrt{2x-3} + 5 = 2.$$

Transforming the equation, you will see that

$$\sqrt{2x-3} = -3.$$

The equation $\sqrt{2x-3} = -3$, however, will never have a real-number solution since a square root radical always denotes a nonnegative number. Thus, the solution set of the equation is $\emptyset$.

Oral Exercises

Tell whether or not each equation has a real-number solution.

1. $\sqrt{x} = 5$
2. $\sqrt{y} = -2$
3. $-\sqrt{x} = -1$
4. $-\sqrt{y} = 3$
5. $\sqrt[4]{p} = -3$
6. $\sqrt[3]{q} = -5$
7. $\sqrt{z} - 4 = 0$
8. $\sqrt{x} + 6 = 9$
9. $\sqrt[3]{t} + 12 = 16$
10. $\sqrt[4]{m} + 2 = 4$
11. $\sqrt{z} - 4 = -12$
12. $6 - \sqrt{2x} = 4$

Describe the first step you would take in solving each equation.

13. $\sqrt{4x - 3} = 3$

14. $\sqrt[3]{2m + 1} = 3$

15. $\sqrt{2 - x} + 4 = 6$

16. $\sqrt{1 + 3x} - 10 = -3$

17. $\sqrt{5x} + 10 = x$

18. $4\sqrt{x} - 2x = 1$

Written Exercises

Solve.

A

1. $\sqrt{x} - 2 = 6$
2. $\sqrt{x} + 4 = 9$
3. $3\sqrt{y} + 2 = 8$
4. $4\sqrt{x} - 3 = 1$
5. $2\sqrt{n} = -12$
6. $-5\sqrt{z} = 10$
7. $3\sqrt{y} = 1$
8. $4\sqrt{y} = 3$
9. $\frac{\sqrt{m}}{2} + 2 = 7$
10. $2 - \frac{\sqrt{h}}{3} = 1$
11. $\sqrt{x - 4} = 3$
12. $\sqrt[3]{2 - x} = 2$
13. $\sqrt[4]{2x + 4} = 1$
14. $\sqrt{4 - 5y} = 8$
15. $\sqrt[3]{\frac{4n}{3}} + 1 = 4$
16. $\sqrt[3]{\frac{2}{5}x} - 3 = 1$
17. $3\sqrt{n} - \frac{1}{5} = \frac{8}{5}$
18. $\frac{2}{3} - 4\sqrt{z} = 2$

B

19. $\sqrt{\frac{2x - 5}{2}} = 1$
20. $\sqrt{\frac{4 - 3x}{7}} = 2$
21. $\sqrt{x} = 2\sqrt{3}$
22. $3\sqrt{y} = 6\sqrt{2}$
23. $\sqrt{x^2 - 2} = 4$
24. $\sqrt{5 - y^2} = 1$
25. $x + \sqrt{x} = 6$
26. $y - \sqrt{y} = 12$
27. $\sqrt{x + 2} - x = -10$
28. $\sqrt{3x} - x = -6$
29. $\sqrt{3x + 13} - 2x = 8$
30. $\sqrt{33 - 4x} + x = 3$

C

31. $\sqrt{a^2 + 5} = 2a - 1$
32. $\sqrt{2a^2 - 7a + 10} = 2a - 5$
33. $\sqrt[3]{a^3 + 3a^2} = a + 1$
34. $\sqrt[4]{2b^2 - 1} = b$
35. $\sqrt{3x + 4} - \sqrt{x} = 2$
36. $\sqrt{x - 5} - \sqrt{2x + 7} = -3$

Solve each system of equations.

37. $3\sqrt{a} - 2\sqrt{b} = -2$
    $4\sqrt{a} + 5\sqrt{b} = 28$

38. $5\sqrt{a} + 6\sqrt{b} = 45$
    $15\sqrt{a} - 8\sqrt{b} = 5$

Self-Test 3

VOCABULARY

rationalize a denominator (p. 518)
simplest form of a radical (p. 518)
conjugates (p. 521)
*n*th root (p. 524)
cube root (p. 524)
principal *n*th root (p. 524)
index (p. 525)

Simplify.

1. $\sqrt{18} \cdot 2\sqrt{27}$
2. $\sqrt{\dfrac{32}{45}}$
3. $\sqrt{48} - \sqrt{72} + \sqrt{75}$ *Obj. 1, p. 516*
4. $\sqrt{6}(\sqrt{3} + \sqrt{2})$
5. $\dfrac{\sqrt{5} - 2}{\sqrt{7} + 3}$
6. $(4 - \sqrt{2})(5 + \sqrt{2})$
7. $\sqrt[5]{243}$
8. $\sqrt[3]{\dfrac{5}{54}}$
9. $\sqrt[4]{32} + \sqrt[4]{1250}$ *Obj. 2, p. 516*

Solve.

10. $\sqrt{x + 1} - 6 = 2$
11. $\sqrt[3]{2x + 5} = 3$
12. $\sqrt{3x} + x = 6$ *Obj. 3, p. 516*

Check your answers with those at the back of the book.

EXTRA

Imaginary Numbers

In Sections 10-3 and 10-4, you learned to solve equations such as

$$x^2 = 16 \qquad \text{and} \qquad x^2 + 4 = 22.$$

You found, however, that you were unable to solve equations such as

$$x^2 = -4$$

over the set of real numbers because the square of a real number is always a nonnegative real number.

The solutions of this equation do exist, however, if the replacement set of the variable is extended to include *imaginary numbers.* **Imaginary numbers** involve the **imaginary unit,** i, which has the property that

$$i^2 + 1 = 0, \qquad \text{or} \qquad i^2 = -1.$$

Thus, i is called a "square root of -1," and you write

$$i = \sqrt{-1}.$$

The imaginary unit can be used in solving equations as follows.

$$\begin{aligned} x^2 &= -4 \\ x &= \pm\sqrt{-4} \\ &= \pm\sqrt{-1} \cdot \sqrt{4} \\ &= \pm i \cdot 2 \\ x &= \pm 2i \end{aligned}$$

In fact, for every positive real number r,

$$\sqrt{-r} = i\sqrt{r} \qquad \text{and} \qquad -\sqrt{-r} = -i\sqrt{r}.$$

Thus,

$$(i\sqrt{r})^2 = -r \qquad \text{and} \qquad (-i\sqrt{r})^2 = -r.$$

These relationships can be used in simplifying expressions involving radicals whose radicands are negative numbers.

EXAMPLE 1 Simplify.

a. $\sqrt{-48}$ **b.** $\sqrt{-20} + \sqrt{-45}$

c. $\sqrt{-8} \cdot \sqrt{-18}$ **d.** $\sqrt{-\frac{5}{7}}$

SOLUTION **a.** $\sqrt{-48} = \sqrt{-1} \cdot \sqrt{48} = i \cdot 4\sqrt{3} = 4i\sqrt{3}$

b. $$\begin{aligned} \sqrt{-20} + \sqrt{-45} &= i\sqrt{20} + i\sqrt{45} \\ &= 2i\sqrt{5} + 3i\sqrt{5} \\ &= 5i\sqrt{5} \end{aligned}$$

c. $$\begin{aligned} \sqrt{-8} \cdot \sqrt{-18} = i\sqrt{8} \cdot i\sqrt{18} &= 2i\sqrt{2} \cdot 3i\sqrt{2} \\ &= 6i^2 \cdot 2 \\ &= 6(-1) \cdot 2 = -12 \end{aligned}$$

d. $$\sqrt{-\frac{5}{7}} = \frac{\sqrt{-5}}{\sqrt{7}} = \frac{i\sqrt{5}}{\sqrt{7}} = \frac{i\sqrt{5}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{i\sqrt{35}}{7}$$

Notice that in part (c) of Example 1 the product property of square roots could not be used until the radicals were expressed as real square roots.

When you simplify increasing powers of i, you find the values repeating in cycles of four according to the pattern i, -1, $-i$, 1.

$i^1 = i$

$i^2 = i \cdot i = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^5 \cdot i = i \cdot i = -1$

$i^7 = i^6 \cdot i = -1 \cdot i = -i$

$i^8 = i^7 \cdot i = -i \cdot i = 1$

Thus, $i^9 = i$, $i^{10} = -1$, $i^{11} = -i$, $i^{12} = 1$, and so on.

Since $i(-i) = (-i)(i) = -i^2 = -(-1) = 1$, i and $-i$ are reciprocals. That is,

$$\frac{1}{i} = -i \quad \text{and} \quad \frac{1}{-i} = i.$$

These facts about reciprocals are used in simplifying fractions whose denominators are imaginary numbers.

EXAMPLE 2 Simplify.

a. $\frac{18}{30i}$ b. $\frac{2}{\sqrt{-40}}$ c. $\frac{\sqrt{3}}{-\sqrt{-24}}$

SOLUTION

a. $\frac{18}{30i} = \frac{18}{30} \cdot \frac{1}{i}$

$= \frac{3}{5}(-i)$

$= -\frac{3}{5}i$

b. $\frac{2}{\sqrt{-40}} = \frac{2}{2i\sqrt{10}}$

$= \frac{1}{\sqrt{10}} \cdot \frac{1}{i}$

$= \frac{-i}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$

$= \frac{-i\sqrt{10}}{10}$

c. $\frac{\sqrt{3}}{-\sqrt{-24}} = \frac{\sqrt{3}}{-2i\sqrt{6}}$

$= \frac{1}{-i} \cdot \frac{\sqrt{3}}{2\sqrt{6}}$

$= \frac{i\sqrt{3}}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{i\sqrt{18}}{2 \cdot 6}$

$= \frac{3i\sqrt{2}}{12} = \frac{i\sqrt{2}}{4}$

Exercises

Express in simplest form as *i*, −1, −*i*, or 1.

1. i^{16}
2. i^{29}
3. i^{62}
4. i^{81}
5. $\frac{1}{i^3}$
6. $\frac{1}{-i^4}$

Simplify.

7. $\sqrt{-28}$
8. $\sqrt{-32}$
9. $\sqrt{-60}$
10. $\sqrt{-72}$
11. $\sqrt{-36} + \sqrt{-81}$
12. $\sqrt{-54} - \sqrt{-150}$
13. $\sqrt{-12} \cdot \sqrt{-96}$
14. $\sqrt{-108} \cdot \sqrt{-75}$
15. $\frac{16}{56i}$
16. $\frac{24}{-32i}$
17. $\frac{8}{\sqrt{-180}}$
18. $\frac{\sqrt{6}}{2\sqrt{-240}}$

Solve.

19. $x^2 + 49 = 0$
20. $x^2 + 13 = 1$
21. $x^2 + 120 = 40$
22. $70 - x^2 = 95$
23. $3x^2 + 50 = 2$
24. $2x^2 - 5 = -95$

Chapter Summary

1. The *density property of rational numbers* asserts that between any two rational numbers there is always another rational number.

2. A number can be represented by a *terminating* or *repeating* decimal if and only if it is a *rational number*. Numbers that can only be expressed as nonterminating, nonrepeating decimals are *irrational numbers*.

3. If $b^2 = a$, then b is called a *square root* of a. The nonnegative, or *principal*, square root is denoted by the symbol $\sqrt{a}$. The negative square root is denoted by $-\sqrt{a}$.

4. For any real number a, $\sqrt{a^2} = |a|$.

5. If the square root of a given rational number is rational, then the given number is a *perfect square*. Square roots of numbers that are not perfect squares may be approximated using a table of square roots.

6. The *Pythagorean Theorem* is a statement of an important relationship among the sides of a right triangle: If the length of the hypotenuse is c and the lengths of the other sides are a and b, then
$$c^2 = a^2 + b^2.$$

7. The *distance formula* is a special application of the Pythagorean Theorem: The distance d between two points (x_1, y_1) and (x_2, y_2) is
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

8. The following two properties are used to simplify radicals.
   a. *Product property of square roots*: For any nonnegative real numbers a and b,
   $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$
   b. *Quotient property of square roots*: For any nonnegative real number a and any positive real number b,
   $$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

9. For every positive integer n, a solution of $b^n = a$ is an *nth root* of a. The symbol $\sqrt[n]{a}$ denotes the *principal nth root* of a and $-\sqrt[n]{a}$ denotes the negative nth root of a. The positive integer n is called the *index*.

10. To solve an equation in which a term has a variable in a radicand, transform the equation into an equivalent equation in which the term with the variable in the radicand is alone as one side of the equation. Then raise both sides of the equation to the power equal to the index of the radical.

Chapter Review

Write the letter of the correct answer.

1. Which of the following statements is false? 10-1

   a. $\frac{126}{81} = \frac{84}{54}$ b. $\frac{28}{42} > \frac{32}{56}$ c. $4\frac{2}{9} > \frac{47}{11}$ d. $-\frac{8}{9} < -\frac{5}{12}$

2. Find the rational number that is one fifth of the way from $-\frac{3}{2}$ to 6.

   a. $\frac{3}{2}$ b. -1 c. 0 d. $\frac{9}{2}$

3. Express $2\frac{5}{18}$ as a decimal. 10-2

   a. $0.2\overline{7}$ b. $0.22\overline{7}$ c. $2.\overline{27}$ d. $2.2\overline{7}$

4. Which of the following is an irrational number?

   a. 2.01 b. $2.\overline{01}$ c. $2.\overline{01011}$ d. $2.010110111\ldots$

5. Express $(0.\overline{4})(1.8)$ as a fraction in simplest form.

   a. $\frac{18}{25}$ b. $\frac{4}{5}$ c. $\frac{36}{5}$ d. $\frac{8}{25}$

6. Which statement is true for all real values of x? 10-3

   a. $\sqrt{x^2} = x$ b. $\sqrt{(x + 1)^2} = x + 1$
   c. $\sqrt{x^2} = |x|$ d. $\sqrt{x}$ is a real number.

7. Simplify $-\sqrt{(-9)^2}$.

   a. 3 b. 9 c. -9 d. -3

8. Simplify $\sqrt{81a^2b^2}$.

   a. $9ab$ b. $-9ab$ c. $9|ab|$ d. $|9|ab$

9. Name the two consecutive integers between which $\sqrt{13.4}$ lies. 10-4

   a. 3 and 4 b. 4 and 5 c. 10 and 11 d. 11 and 12

10. Simplify $\sqrt{45}$.

    a. $9\sqrt{5}$ b. $3\sqrt{5}$ c. $5\sqrt{3}$ d. $5\sqrt{9}$

11. Solve $4y^2 + 3 = 6$.

    a. $\left\{\frac{3}{2}\right\}$ b. $\left\{-\frac{3}{2}, \frac{3}{2}\right\}$ c. $\left\{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$ d. $\left\{-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right\}$

12. In a right triangle, the shorter sides measure 10 cm and 24 cm. Find the length of the hypotenuse. 10-5

    a. 26 cm b. 34 cm c. $2\sqrt{119}$ cm d. 20 cm

13. Which of the following sets of numbers is *not* a Pythagorean triple?

a. $\{3, 4, 5\}$ b. $\{5, 12, 13\}$ c. $\{8, 15, 17\}$ d. $\{1, 2, \sqrt{5}\}$

14. A ladder that is 10 m long leans against a wall. If the ladder meets the wall at a point that is 8 m above the ground, how far from the base of the wall is the foot of the ladder?

a. 2 m b. 6 m c. 18 m d. $\sqrt{18}$ m

15. Find the distance between the points $A(2, -3)$ and $B(-1, 4)$. *10-6*

a. $\sqrt{10}$ b. $\sqrt{58}$ c. $2\sqrt{119}$ d. 20

Simplify. Assume that all variables represent positive real numbers.

16. $\frac{2}{\sqrt{2}}$ *10-7*

a. $\frac{\sqrt{2}}{2}$ b. $\sqrt{2}$ c. $2\sqrt{2}$ d. 2

17. $\sqrt{6x^2} \cdot \sqrt{8x}$

a. $16x\sqrt{3x}$ b. $3x\sqrt{4x}$ c. $4x\sqrt{3x}$ d. $4x^2\sqrt{3x}$

18. $3\sqrt{50} + 2\sqrt{18}$ *10-8*

a. $5\sqrt{68}$ b. $21\sqrt{2}$ c. $13\sqrt{2}$ d. $6\sqrt{5} + 4\sqrt{3}$

19. $(\sqrt{5} - 1)(\sqrt{5} + 2)$

a. 3 b. 23 c. $\sqrt{5} + 3$ d. $\sqrt{5} + 23$

20. What is the conjugate of $2\sqrt{3} - 6\sqrt{5}$?

a. $\sqrt{3} + \sqrt{5}$ b. $6\sqrt{5} - 2\sqrt{3}$
c. $3\sqrt{2} - 5\sqrt{6}$ d. $2\sqrt{3} + 6\sqrt{5}$

21. Simplify $\sqrt[3]{-27}$. *10-9*

a. -9 b. -3 c. 9 d. none of these

22. Simplify $\sqrt[3]{\frac{3}{4}}$.

a. $\sqrt[3]{3}$ b. $\frac{\sqrt[3]{3}}{2}$ c. $\frac{\sqrt[3]{6}}{2}$ d. $\frac{\sqrt[3]{12}}{4}$

Solve.

23. $\sqrt[3]{4 + x} = 5$ *10-10*

a. $\{1\}$ b. $\{11\}$ c. $\{21\}$ d. $\{121\}$

24. $6 - \sqrt{x} = x$

a. $\{4\}$ b. $\{4, 9\}$ c. $\{9\}$ d. $\{3\}$

Chapter Test

1. Find the rational number that is one third of the way from $-\frac{3}{4}$ to $\frac{1}{3}$. *10-1*

2. Write $\frac{36}{54}$, $\frac{60}{144}$, and $\frac{42}{96}$ in order from least to greatest.

3. Express $\frac{14}{11}$ as a decimal. *10-2*

4. Express $-0.41\overline{6}$ as a fraction in simplest form.

5. Express $0.8\overline{3} + \frac{1}{3}$ as a fraction in simplest form.

Simplify.

6. $\sqrt{(-36)^2}$ 7. $-\sqrt{\frac{64}{81}}$ 8. $\sqrt{(x-3)^2}$ *10-3*

9. $(\sqrt{24})^2$ 10. $\sqrt{450}$ 11. $\sqrt{\frac{20}{9}}$ *10-4*

12. Solve $16z^2 + 7 = 25$.

13. Given that the length of the hypotenuse of a right triangle is $\sqrt{34}$ cm and that the length of a second side is 3 cm, find the length of the third side. *10-5*

14. Roger rode his bike from his home due east for 10 km and then due north for 6 km. How far was he from his home? Approximate the answer to the nearest hundredth.

15. Find the distance between the points $A(3, 2)$ and $B(9, 5)$. *10-6*

16. Determine whether or not the triangle with vertices $(-1, -1)$, $(1, 2)$, and $(5, -5)$ is a right triangle.

Simplify. Assume that all variables represent positive real numbers.

17. $\sqrt{32} \cdot \sqrt{6}$ 18. $\sqrt{\frac{2}{3}}$ 19. $\sqrt{5y^3} \cdot \sqrt{20y^5}$ *10-7*

20. $\sqrt{5} + \sqrt{12} - \sqrt{20}$ 21. $\sqrt{15}(\sqrt{5} - \sqrt{6})$ 22. $\frac{3}{4 - \sqrt{2}}$ *10-8*

23. $\sqrt[5]{-32}$ 24. $\sqrt[3]{270}$ 25. $\sqrt[4]{\frac{7}{8}}$ *10-9*

Solve.

26. $\sqrt{x} = -3$ 27. $\sqrt{y-3} = 7$ 28. $\sqrt{z} + 2 = z$ *10-10*

Mixed Review

Simplify.

1. $(n)^2(2n) + (n)(n^2)$
2. $(7w^3 - w^2 + 9) - (2w^3 - 10 - 8w^2)$
3. $(2x - 5)(2x + 5)$
4. $(4t - 3)(3t - 2)$
5. $(x - 5)^3$
6. $4a^2b(5a^3 + 3ab - 2b^2)$
7. $\dfrac{2^{-2}x^2z^0}{4^{-1}x^{-2}z^{-3}}$
8. $\dfrac{a^2 - ab - 12b^2}{a^2 - 3ab - 4b^2}$
9. $\dfrac{m^2 - 4}{2m - 6} \div \dfrac{2 - m}{m - 3}$
10. $\dfrac{4}{x^2 + 2xy + y^2} - \dfrac{3}{x + y}$
11. $\dfrac{42a^3b^2 - 27a^2b^3 + 3a^2b}{-3a^2b}$
12. $\dfrac{v^2 + 2v - 15}{v^2 + 3v - 10} \cdot \dfrac{v^2 - 4}{6 + v - v^2}$
13. $\dfrac{d}{c + 3d} + \dfrac{c}{3c + d}$
14. $\dfrac{a - 3}{a^2 - 5a + 4} - \dfrac{a - 1}{12 + a - a^2}$
15. $\dfrac{\frac{5x}{4} - \frac{5z}{24}}{\frac{z}{12} - \frac{x}{2}}$
16. $\dfrac{1 - \frac{36}{w^2}}{1 + \frac{6}{w}}$

Solve.

17. $2d^2 + d = 15$
18. $y^3 = 7y^2 + 18y$
19. $(3 - 6a^2) - (5a - 6a^2) = -2$
20. $(2t + 5)(2t - 3) = (4t - 1)(t - 1)$
21. $\dfrac{3k}{8} - \dfrac{k}{4} < \dfrac{3}{2}$
22. $\dfrac{5m - 2}{6} - \dfrac{2m + 1}{3} > \dfrac{1}{2}$
23. $\dfrac{x - 16}{x} = \dfrac{3}{5}$
24. $\dfrac{-5}{2 - 3z} = \dfrac{2}{4 + z}$
25. $\dfrac{5}{2y + 3} = \dfrac{4}{y} - 3$
26. $\dfrac{2t + 3}{t + 1} - \dfrac{4t + 2}{t^2 - 1} = \dfrac{t}{t - 1}$
27. $\dfrac{r^2 - 7}{r^2 - 4r + 3} + \dfrac{r + 2}{r - 3} = 2$
28. $\dfrac{x + 1}{x - 2} + \dfrac{x - 3}{x + 2} = \dfrac{7x + 4}{x^2 - 4}$

Factor completely.

29. $x^2 - 3x - 18$
30. $y(y - 5) + 2(5 - y)$
31. $20z^2 + 3z - 2$
32. $a^2 + 9$
33. $3m^2 + 9m - 12$
34. $24n^2 + n - 10$
35. $2ab^3 - 18ab$
36. $16c^2 - 81$

Graph on a coordinate plane.

37. $3y - 2x \geq 9$
38. $4x + 2y = 3$
39. $5y - 3x = 10$
40. $15x - 5y > 20$

Determine an equation of the line that satisfies the given requirements.

41. passes through the point $(-1, -2)$ and is parallel to the y-axis

42. passes through the points $(-1, 0)$ and $(4, 10)$

43. has slope $\frac{2}{3}$ and y-intercept -5

44. passes through the point $(-2, 6)$ and has slope 3

Express in scientific notation.

45. 90,700,000,000,000

46. 0.0000000814

Solve.

47. Nancy invested part of her \$6500 savings at an annual interest rate of $6\frac{1}{2}\%$ and the rest at an annual interest rate of 8%. How much did she invest at each rate if her total income from these investments for one year is \$490?

48. Find two consecutive positive integers such that the sum of their squares is 85.

49. The difference of two numbers is 8 and their quotient is $\frac{5}{3}$. Find the two numbers.

50. One pipe can fill a tank in 4 h. A second pipe can fill the same tank in 6 h. How long will it take both pipes working together to fill the tank?

51. A suitcase with a mass of 4 kg causes a spring to stretch 14 cm. If the amount that the spring stretches varies directly with the mass, how much will a 2.5 kg suitcase stretch the spring?

52. The number of hours needed to enter a certain set of data into a computer is inversely proportional to the number of persons who are entering the data. If it would take 5 people 8 h to complete the job, how long would the job take if 12 people were entering the data?

53. The length of a certain rectangle is 12 m and the width is 5 m. If both the length and the width were increased by the same amount, the area would be increased by 60 m^2. Find the dimensions of the new rectangle.

54. The length of a square field is increased by 4 m and the width is increased by 2 m, creating a rectangular field that has an area of 195 m^2. Find the dimensions of the original field.

55. The volume of a pyramid varies jointly as the altitude of the pyramid and the area of its base. The volume of a pyramid with altitude 6 cm and base area 16 cm^2 is 32 cm^3. Find the volume of a pyramid with altitude 8 cm and base area 9 cm^2.

56. The number of persons needed to do a job varies directly as the amount of work to be done and inversely as the time in which the job must be done. If two typists can type 210 pages of manuscript in three days, how many typists will be needed to type 700 pages in two days?

PREPARING FOR COLLEGE ENTRANCE EXAMS

Strategy for Success: On a multiple-choice exam, you often are asked to determine the *best* answer to a question. For any given question, more than one answer may seem "right" to some degree, and you must be careful not to choose the first answer that seems reasonable. Be sure to evaluate all the possible choices before you determine which answer is "best."

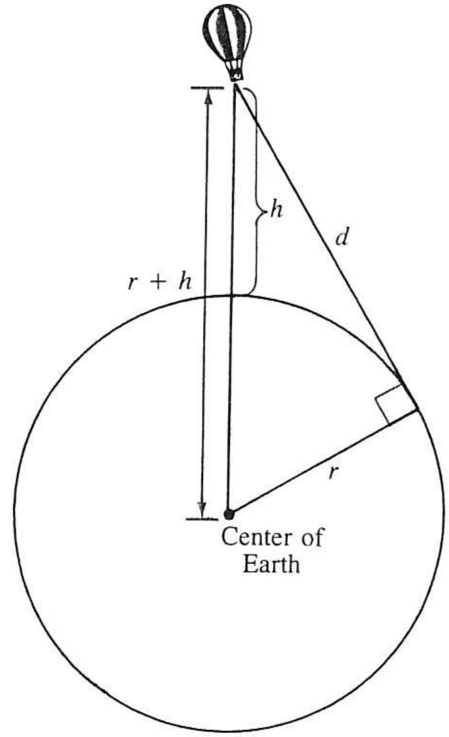
Decide which is the best of the choices and write the corresponding letter on your answer sheet.

1. What is the range of the function $\{(x, y): y = -\sqrt{x}\}$?
   (A) $\{y: y \geq 0\}$ **(B)** $\{y: y \leq 0\}$ **(C)** $\{y: y < 0\}$
   (D) all real numbers **(E)** none of these
2. A rectangle is three times as long as it is wide. One of its diagonals measures 30 cm. Find the width of the rectangle to the nearest tenth of a centimeter.
   (A) 9.5 cm **(B)** 28.5 cm **(C)** 15 cm **(D)** 11.2 cm **(E)** none of these
3. Simplify $\frac{4}{\sqrt[3]{3}}$.
   (A) $\frac{3\sqrt[3]{9}}{4}$ **(B)** 4 **(C)** $\frac{4\sqrt[3]{9}}{3}$ **(D)** $\frac{4\sqrt[3]{27}}{3}$ **(E)** none of these
4. Which of the following are factors of $4x^3 + 8x^2 - 7x - 5$?
   I. $2x + 1$ II. $2x - 1$ III. $2x + 5$
   (A) I only **(B)** II only **(C)** III only **(D)** I and III only **(E)** II and III only
5. For which value of p in the division $(2x^2 + 11x - p) \div (2x - 3)$ is the remainder 0?
   (A) 18 **(B)** 21 **(C)** 3 **(D)** 5 **(E)** 12
6. Which of the following is true if $b > c$ and $\frac{ab - ac}{b - c} < 0$?
   (A) $a > b$ **(B)** $a < b$ **(C)** $a = b$ **(D)** $a < 0$ **(E)** $b < 0$
7. The vertices of a triangle are at $(-4, 4)$, $(-2, -4)$, and $(6, -2)$. Which of the following statements is true?
   (A) The perimeter of the triangle is $4\sqrt{17} + 2\sqrt{34}$.
   (B) The triangle is isosceles.
   (C) The triangle is a right triangle.
   (D) The area of the triangle is 34 square units.
   (E) All of the above statements are true.
8. A reservoir can be filled in 6 days by pipe A running alone, or in 4 days by pipe B alone. How many days would be needed to fill the reservoir if both pipes were running?
   (A) 10 **(B)** 5 **(C)** $2\frac{2}{5}$ **(D)** $\frac{5}{12}$ **(E)** $3\frac{1}{2}$

APPLICATION

Raising Your Sights

Have you ever been up in a balloon? If so, you probably noticed that, the higher the balloon rose, the more of Earth's surface you could see. However, because the surface of Earth is curved and light travels in straight lines, from any given height you can see only as far as the *horizon*, where your line of sight just touches Earth's surface. Did you know that it is possible to calculate how far you can see?



The diagram at the right represents a balloon that is at height h above Earth's surface. The length of the line of sight from the balloon to the horizon is represented by d. Notice that, at the horizon, the line of sight forms a right angle with Earth's radius, r.

Now consider the line of sight d and the radius r as the two shorter sides of a right triangle. The distance from the balloon to the center of Earth, or $r + h$, is then the hypotenuse of the right triangle. Therefore, to find the length of the line of sight, you can use the Pythagorean Theorem as follows.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ r^2 + d^2 &= (r + h)^2 \\ r^2 + d^2 &= r^2 + 2rh + h^2 \\ d^2 &= 2rh + h^2 \\ d &= \sqrt{2rh + h^2} \end{aligned}$$

Suppose that you are in a balloon that is 1 km above Earth's surface. Earth's radius, r, is about 6380 km. To calculate how far you could see from this height, substitute these values into the above equation.

$$\begin{aligned} d &= \sqrt{2rh + h^2} \\ &= \sqrt{2(6380)(1) + 1^2} \\ d &= \sqrt{12{,}761} \approx 113 \end{aligned}$$

Therefore, from a height of 1 km you can see about 113 km to the horizon.

Notice that in the equation

$$d = \sqrt{2rh + h^2},$$

the value of h^2 is frequently much less than the value of $2rh$. This happens because the height that an object rises above Earth's surface, h, is frequently very small when compared to the radius of Earth, r. In such cases, therefore, the value of h^2 may be omitted to obtain the alternate formula

$$d \approx \sqrt{2rh}.$$

Using this formula in the previous example, notice that you would obtain

$$d \approx \sqrt{12{,}760},$$

which again rounds to an approximate distance to the horizon of 113 km.

Exercises

1. A balloon is at a height of 0.5 km. How far can the pilot see?
2. An airplane is flying at an altitude of 10 km. How far can the passengers see?
3. An airplane is flying at an altitude of 4000 m. How far can the passengers see?
4. An observer is in a fire tower that is 20 m high. How far can the observer see?
5. A lifeguard sitting in a tall chair can see 5 km out to sea. How tall is the chair?
6. The pilot of a balloon can see for 200 km. What is the altitude of the balloon?
7. A team of astronauts is 100,000 km above the surface of Earth.
   a. Use the formula $d = \sqrt{2rh + h^2}$ to calculate the distance they can see to the horizon.
   b. Use the formula $d \approx \sqrt{2rh}$ to calculate the distance they can see to the horizon.
   c. Explain why the answers to parts (a) and (b) are so different.